SPACES DETERMINED BY GENERALIZED METRIC SUBSPACES

YOSHIO TANAKA

Dedicated to Professor Akihiro Okuyama on his 60th birthday

INTRODUCTION

First, we shall give some definitions which will be used in this paper.

Let $X$ be a space. Let $d : X \times X \to \mathbb{R}$ be a non-negative, real valued function such that $d(x, y) = 0$ if and only if $x = y$. We shall consider the following conditions:

(a) $G \subset X$ is open if and only if for each $x \in G$, there exists $S_n(x) \subset G$, where $S_n(x) = \{ y \in X; d(x, y) < 1/n \} (n \in N)$.

(b) For $x \in X$ and $n \in N$, $S_n(x)$ is open in $X$.

(c) For $x \in X$ and $n \in N$, $\text{int } S_n(x) \ni x$.

Then $d$ is called an $o$-metric [16] if it satisfies (a). An $o$-metric $d$ is called a generalized metric [12] if it satisfies (b); equivalently, for each $x \in X$, $\{ S_n(x); n \in N \}$ is a base at $x$.

A space $X$ is called $o$-metric [16] if it has an o-metric $d$. Every o-metric space is a sequential space, hence a $k$-space.

We note that a space $X$ is weakly first countable ($= X$ satisfies the weak first axiom of countability in the sense of [1]) if and only if $X$ is o-metric; and that a space $X$ is first countable if and only if it has an o-metric satisfying (b) (or (c)); cf. [16].

Let $X$ be a space. Let $d : X \times X \to \mathbb{R}$ be a non-negative, real valued function. Let us consider following conditions as a generalization of metric functions.

(1) $d(x, y) = d(y, x)$. 

261
(2) \( d(x, z) \leq d(x, y) + d(y, z) \).

(3) \( d(x, z) \leq \max \{ d(x, y), d(y, z) \} \).

(4) For any compact set \( K \) and closed set \( F \) with \( K \cap F = \emptyset \),
\[
\inf \{ d(x, y); x \in K, y \in F \} > 0;
\]
A space \( X \) is called symmetric if it has an \( o \)-metric \( d \) satisfying (1), and such a function \( d \) is called symmetric for \( X \).

A space \( X \) is called semi-metric if it has an \( o \)-metric \( d \) satisfying (1) and (c).

A space \( X \) is called quasi-metric (= \( \Delta \)-metric in the sense of ([16]) if it has a generalized metric \( d \) satisfying (2). Here we can replace “generalized metric” by “\( o \)-metric”.

A space \( X \) is called non-archimedean quasi-metric (simply, \( n.a.- \) quasi-metric) if it has a generalized metric \( d \) satisfying (3). Here we can replace “generalized metric” by “\( o \)-metric”.

A space \( X \) is called \( \gamma \)-metric (\( = \gamma \)-space) if it has a generalized metric satisfying (4).

In this paper, we shall use “\( X \) is symmetric; \( (n.a.-) \) quasi-metric, etc” instead of “\( X \) is symmetrizable; \( (n.a.-) \) quasi-metrizable; etc”.

\( (n.a.-) \) quasi-metric spaces; \( \gamma \)-metric spaces are characterized by means of \( g \)-functions, interior-preserving covers, quasi-uniformities, or sequences of neighbornets, etc., and they are investigated or surveyed in [5], [6], [12], [16], etc.

Concerning symmetric, \( (n.a.-) \) quasi-metric, or \( \gamma \)-metric spaces, etc., the following diagram is known; see [6], for example. A space is \( Fréchet \) if whenever \( x \in \tilde{A} \), then there exists a sequence in \( A \) converging to the point \( x \). For the definition of semi-stratifiable spaces; see [3], and for (a), see [10]; and [4].

Diagram. For a space, the following implications hold.

(a) \( o \)-metric and semi-stratifiable \( \Rightarrow \) symmetric. But, symmetric \( \not\Rightarrow \) closed sets are \( G_\delta \)-sets.

(b) developable \( \Rightarrow \) semi-metric \( \Leftrightarrow \) \( Fréchet \) and symmetric \( \Leftrightarrow \) first countable and semi-stratifiable. But, semi-metric \( \not\Rightarrow \sigma \)-space.
(c) metacompact and developable $\Rightarrow$ n.a.-quasi-metric $\Rightarrow$ quasi-metric $\Rightarrow$ $\gamma$-metric $\Rightarrow$ first countable. But, n.a.-quasi-metric $\not\Rightarrow$ closed sets are $G_\delta$-sets.

(d) symmetric and $\gamma$-metric $\Leftrightarrow$ developable and quasi-metric. But, developable $\not\Rightarrow$ $\gamma$-metric.

Let $X$ be a space, and let $\mathcal{C}$ be a cover (not necessarily closed or open) of $X$. Then $X$ is determined by $\mathcal{C}$ [7] ($=X$ has the weak topology with respect to $\mathcal{C}$ in the usual sense), if $F \subset X$ is closed in $X$ if and only if $F \cap C$ is closed in $C$ for every $C \in \mathcal{C}$. Here, we can replace "closed" by "open". Every space is determined by an open cover. If a space $X$ is determined by a countable and increasing cover $\{X_n; n \in N\}$, then $X$ is called the inductive limit of $\{X_n; n \in N\}$, and denoted by $X = \lim_{\leftarrow} X_n$.

We recall that a space $X$ is sequential if $X$ is determined by the cover of all (compact) metric subspaces.

Let $X$ be a space, and let $\mathcal{F}$ be a closed cover of $X$. Then $X$ is dominated by $\mathcal{F}$ [14] ($=X$ has the weak topology with respect to $\mathcal{F}$ in the sense of [15]), if the union of any subcollection $\mathcal{F}'$ of $\mathcal{F}$ is closed in $X$, and the union is determined by $\mathcal{F}'$. Every space is dominated by a hereditarily closure-preserving closed cover. As is well-known, every CW-complex is dominated by a cover of compact metric subspaces.

We recall canonical quotient spaces $S_\omega$ and $S_2$, which is called the sequential fan and the Arens' space respectively.

$S_\omega$ is the quotient space obtained from the topological sum of countably many convergent sequences by identifying all the limit points. $S_2 = (N \times N) \cup N \cup \{\infty\}$ is the space with each point of $(N \times N)$ isolated. A basic neighborhood of $n \in N$ consists of all sets of the form $\{n\} \cup \{(m, n); m \geq k\}$. And $U$ is a neighborhood of $\infty$ if and only if $\infty \in U$ and $U$ is a neighborhood of all but finitely many $n \in N$.

The spaces $S_\omega$ and $S_2$ are dominated by an increasing countable cover of compact metric subspaces. But, $S_\omega$ nor $S_2$ is first
countable. Then $S_{\omega}$ is not semi-metric, not (n.a.-) quasi-metric, not $\gamma$-metric, and neither is $S_2$. Then the following question in [12; Question 3] is negative.

Let $X$ be a space dominated by a cover of quasi-metric; n.a.-quasi-metric; or $\gamma$-metric subsets. Then is $X$ so respectively?

In this paper, we give a characterization for the above space $X$ to be quasi-metric; n.a.-quasi-metric; or $\gamma$-metric respectively. We also give some analogous characterizations when spaces are determined by certain covers of these generalized subspaces, or semi-metric subspaces, etc.

We assume that all spaces are regular and $T_1$.

1. Spaces determined by countable covers.

For each $n \in N$, let $Y_n$ be homeomorphic to the product $X^n$ of a space $X$. First, we shall consider the inductive limit of $\{Y_n; n \in N\}$.

Lemma 1.1. Let $X$ be a sequential space, and let $x \in X$. For each $n \in N$, let $Y_n = X^n \times \{x\} \times \{x\} \times \ldots$. Let $Y = \lim Y_n$. If the point $x$ is not isolated in $X$, then $Y$ contains a closed copy of $S_\omega$, and a closed copy of $S_2$.

Proof: Since $X$ is sequential, there exists a sequence $\{x_n; n \in N\}$ in $X$ converging to $x$ with $x_n \neq x$. Let $p = (x, x, \ldots)$, let $p_m = (x_n, x_n, \ldots x_n, x, x, \ldots) \in Y_m$ for each $m, n \in N$. Let $S = \cup\{p_m; m, n \in N\} \cup \{p\}$. Since each $S \cap Y_n$ is closed in $Y_n$, $S$ is closed in $Y$. For each $m \in N$, let $k(m) \in N$, and let $F = \cup\{p_m; m \in N, n \leq k(m)\}$. Then each $F \cap Y_n$ is finite, hence closed in $Y_n$. Thus $F$ is closed in $Y$. This implies that $S$ is a copy of $S_\omega$. Then $Y$ contains a closed copy of $S_\omega$. Next, for each $m \in N$, let $q_m = (x_m, x, \ldots)$. And, for each $m, n \in N$, let $q_{m,n} = (x_m, x, \ldots x_n, x, \ldots)$, where $x_n$ is the $m$-th coordinate. Let $T = \{q_{m,n}; m, n \in N\} \cup \{q_m; m \in N\} \cup \{p\}$. Similarly $T$ is closed in $Y$, and $T$ is a copy of $S_2$. 
**Theorem 1.2.** Let $X$ be a space, and $x \in X$. For each $n \in N$, let $Y_n = X^n \times \{x\} \times \{x\} \times \ldots$. Let $Y = \lim_{n \to \infty} Y_n$. Then (1) and (2) below hold.

1. Suppose that $X$ is symmetric. Then the following are equivalent.
   - (a) $Y$ is symmetric.
   - (b) $Y$ is a sequential space which contains no closed copy of $S_\omega$.
   - (c) $Y$ is a sequential space, and the point $x$ is isolated in $X$.

2. Suppose that $X$ is metric; semi-metric; quasi-metric; n.a.-quasi-metric; $\gamma$-metric; or developable. Then the following are equivalent.
   - (a) $Y$ is so respectively.
   - (b) $Y$ contains no closed copy of $S_\omega$.
   - (c) $Y$ contains no closed copy of $S_2$.
   - (d) The point $x$ is isolated in $X$.

**Proof:** (1) For (a) $\Rightarrow$ (b), suppose that $Y$ contains a closed subset $\cup\{L_n; n \in N\} \cup \{p\}$, with $L_n \to p$, which is a copy $S_\omega$. Then each $L_n \cup \{p\}$ is closed, but $L_n$ is not closed in $Y$. Thus there exists a point $y_n \in L_n$ with $y_n \in S_n(p)$ for each $n \in N$. Then the sequence $\{y_n; n \in N\}$ converges to the point $\infty$. This is a contradiction. Hence $Y$ contains no closed copy of $S_\omega$. (b) $\Rightarrow$ (c) follows from Lemma 1.1. If (c) holds, then each $Y_n$ is closed and open in $Y$. Then $Y$ is the topological sum of $\{Y_n - Y_{n-1}; n \in N\}$. But each $Y_n$ is sequential with $X$ symmetric. Then each $Y_n$ is symmetric by [19; Theorem 4.1], hence so is each $Y_n - Y_{n-1}$. Thus $Y$ is symmetric. Hence (a) holds.

(2) (a) $\Rightarrow$ (b) or (c) is easy, because $Y$ is first countable. (b) or (c) $\Rightarrow$ (d) follows from Lemma 1.1. For (d) $\Rightarrow$ (a), $Y$ is the topological sum of $\{Y_n - Y_{n-1}; n \in N\}$. But each $Y_n$ is so respectively, then so is each $Y_n - Y_{n-1}$. Thus $Y$ is so respectively.
Theorem 1.3. Let $X$ be a non-discrete, sequential space. Suppose that $X$ is homogeneous; that is, for each $p, q \in X$, there exists a homeomorphism of $X$ onto $X$ taking $p$ to $q$. For each $n \in \mathbb{N}$, let us consider $X^n$ as a subspace of $X^{n+1}$; that is, as a subspace $X^n \times \{x\} \times \{x\} \times \ldots$ of $X^\omega$ for $x \in X$. For each $n \in \mathbb{N}$, let $Y_n = X^n$. Let $Y = \lim Y_n$. Then $Y$ contains a closed copy of $S_\omega$ and a closed copy of $S_2$; hence, $Y$ is not $o$-metric nor Fréchet.

Proof: Since $X$ is non-discrete and homogeneous, any point of $X$ is not isolated. Then $Y$ contains a closed copy of $S_\omega$ and a closed copy of $S_2$ by Lemma 1.1. Thus $Y$ is not Fréchet, for $Y$ contains a copy of $S_2$. Also, $Y$ is not $o$-metric, for $Y$ contains a closed copy of $S_\omega$.

Let $I; \mathbb{R}$ be the closed interval; the real line respectively. For all $n \in \mathbb{N}$, let $X_n = I^n$ (or $\mathbb{R}^n$). The previous theorem shows that $X = \lim X_n$ is not even $o$-metric nor Fréchet, hence $X$ is not symmetric, not quasi-metric, nor $\gamma$-metric, etc.

The following lemma is due to [5]. Unlike this, every space which is a countable union of open metric subsets is neither semi-metric nor symmetric; see, Example 1.9.

Lemma 1.4. Let $\{G_\alpha; \alpha\}$ be a $\sigma$-point-finite open cover of $X$. If the $G_\alpha$ are quasi-metric; n.a.-quasi-metric; or $\gamma$-metric, then so is $X$ respectively.

A space $X$ is strongly Fréchet [17], if whenever $\{A_n, n \in \mathbb{N}\}$ is a decreasing sequence in $X$ with $x \in \bar{A}_n$ for any $n \in \mathbb{N}$, then there exists a sequence $\{x_n; n \in \mathbb{N}\}$ in $X$ converging to the point $x$ with $x_n \in A_n$. The following lemma is due to [20].

Lemma 1.5. Let $X$ be a space determined by a countable cover $C$ such that each finite union of elements of $C$ is first countable. If $X$ contains no closed copy of $S_\omega$ and no $S_2$, then $X$ is strongly Fréchet.
Every space $\lim X_n$ with each $X_n$ compact metric need not be quasi-metric, nor $\gamma$-metric even if $X$ is symmetric (or Fréchet), as is seen by the space $S_2$ (or $S_\omega$). But we have the following theorem (cf. Theorem 1.2).

**Theorem 1.6.** Let $X = \lim X_n$. Suppose that the $X_n$ are quasi-metric; n.a.-quasi-metric; or $\gamma$-metric. Then the following are equivalent.

(a) $X$ is so respectively.

(b) $X$ is first countable.

(c) $X$ contains no closed copy of $S_\omega$ and no $S_2$.

**Proof:** (a) $\Rightarrow$ (b) is clear, and (b) $\Rightarrow$ (c) is obvious. So, we prove (c) $\Rightarrow$ (a). Since $X$ is strongly Fréchet by Lemma 1.5, for each $x \in X$, $x \in \text{int } X_m$ for some $m \in \mathbb{N}$. Indeed, suppose not. Then $x \in X - X_n$ for any $n \in \mathbb{N}$. Then there exists a sequence $K = \{p_n; n \in \mathbb{N}\}$ in $X$ converging to the point $x$ such that $p_n \in X - X_n$, and $p_n \neq x$ for any $n \in \mathbb{N}$. But, each $K \cap X_n$ is finite, hence closed in $X_n$. Thus $K$ is closed in $X$, hence $K \ni x$. This is a contradiction. Thus, $x \in \text{int } X_m$ for some $m \in \mathbb{N}$. This implies that $\{\text{int } X_n; n \in \mathbb{N}\}$ is a countable open cover of $X$. But, the int $X_n$ are quasi-metric; n.a.-quasi-metric; $\gamma$-metric respectively. Thus, $X$ is so respectively by Lemma 1.4.

A space $X$ is submetacompact (= $\theta$-refinable) if for each open cover $\mathcal{U}$ of $X$ there exists a sequence $\{\mathcal{U}_n; n \in \mathbb{N}\}$ of open refinements of $\mathcal{U}$ such that for each $x \in X$ there exists an open cover $\mathcal{U}_n$ which is finite at $x$. As is well-known, metacompact spaces, and subparacompact spaces are submetacompact.

Every semi-metric space is semi-stratifiable, hence submetacompact. But, every n.a.-quasi-metric space is not submetacompact; see, Example 1.9.

The following lemma is due to [18]. But, unlike this, every submetacompact and locally metric space is neither quasi-metric nor $\gamma$-metric; see, Example 2.5(2).
Lemma 1.7. Let $X$ be a submetacompact space.

(1) If $X$ is locally developable, then $X$ is developable.

(2) If $X$ is locally semi-metric, then $X$ is semi-metric.

Corollary 1.8. Let $X = \lim X_n$. Suppose that the $X_n$ are semi-metric; or developable. Then the following are equivalent.

(a) $X$ is semi-metric.

(b) $X$ is first countable, and every closed subset of $X$ is a $G_\sigma$-set.

(c) $X$ is first countable, and submetacompact.

Proof: Every semi-metric space is a submetacompact space in which every closed subset is a $G_\sigma$-set. Thus (a) $\Rightarrow$ (b) and (c) holds. Suppose that (b) holds. Since $X$ is first countable, $X$ is a countable union of open semi-metric subsets by the proof of Theorem 1.6. But each open subset of $X$ is an $F_\sigma$-set, then $X$ is a countable union of closed semi-stratifiable subsets. Thus $X$ is a semi-stratifiable space. Then $X$ is semi-metric by (b) in Diagram. Thus (b) $\Rightarrow$ (a) holds. Suppose that (c) holds. Since $X$ is first countable, $X$ is locally semi-metric by the proof of Theorem 1.6. But $X$ is submetacompact, then $X$ is semi-metric by (2) of Lemma 1.7. Thus (c) $\Rightarrow$ (a) holds. For the developable case, (b) implies that $X$ is a semi-stratifiable space which is locally developable. Since $X$ is submetacompact, $X$ is developable by (1) of Lemma 1.7. Thus (b) (or (c)) $\Rightarrow$ (a) also holds in this case.

Concerning symmetric spaces and semi-metric spaces, Theorem 1.6 does not hold by the following example. Hence, the additional condition of $X$ in (b) or (c) of Corollary 1.8 is essential.

Example 1.9. A n.a.-quasi-metric (hence first countable) space $X = \lim X_n$ such that the $X_n$ are semi-metric open subsets. But $X$ is not symmetric (nor submetacompact).
Proof: Let $X$ be the space $Z$ in Example 3.3 in [4], where $Z$ is not submetacompact and has a closed subset which is not a $G_{\delta}$-set. As is seen there, $Z$ has the $\sigma$-locally countable base $B = \cup \{B_n; \ n \in N\}$, which is also $\sigma$-disjoint. But, it follows that each member of $B$, which is the basic nbd $B(x_1, x_2, \ldots x_k)$ or $V_k(\alpha) = \{\alpha\} \cup \{UB_\alpha(n); \ n \geq k\}$ defined there, is clopen in $Z$, and metrizable (the $V_k(\alpha)$ has a $\sigma$-locally finite base, hence is metrizable). Let $G_n = \cup B_n$ for each $n \in N$. Then each $G_n$ has a locally finite closed cover $B_n$ in $G_n$, hence $G_n$ is metrizable. Thus $X$ has a countable open cover $\{G_n; \ n \in N\}$ of metric subsets. Hence $X$ is n.a.-quasi-metric by Lemma 1.4. For each $n \in N$, let $X_n = \cup \{G_m; \ m \leq n\}$. Then each $X_n$ is an open subset of $X$ which is developable, hence semi-metric. Then $X$ is determined by a countable, increasing open cover $\{X_n; \ n \in N\}$ of semi-metric subsets. But $X$ is a first countable space which is not semi-stratifiable. Then $X$ is not symmetric by (b) in Diagram.

2. SPACES DETERMINED BY POINT-FINITE COVERS.

The space $S_2$ is a symmetric space determined by a point-finite, countable cover of compact metric subspaces. But $S_2$ is neither semi-metric nor quasi-metric. Concerning spaces determined by point-finite covers of certain generalized metric subspaces, we have the following theorem.

Theorem 2.1. Let $X$ be a space determined by a point-finite cover $C = \{X_\alpha\}$.

(1) If the $X_\alpha$ are $\sigma$-metric; or symmetric, then so is $X$ respectively.

(2) Suppose that the $X_\alpha$ are semi-metric. Then $X$ is semi-metric if and only if $X$ is first countable (or Fréchet).

(3) Suppose that the $X_\alpha$ are metacompact developable. Then the following are equivalent.

(a) $X$ is (metacompact) developable.

(b) $X$ is (n.a.-) quasi-metric.

(c) $X$ is $\gamma$-metric.
(d) $X$ is semi-metric.
(e) $X$ is first countable.
(f) $X$ is Fréchet.

**Proof:** (1) Let the $X_\alpha$ be $o$-metric. Then, for $x \in X_\alpha$, one can associate a sequence $\{S_\alpha n(x); n \in N\}$ of subsets of $X_\alpha$ such that $x \in S_\alpha n(x) \subset S_\alpha n+1(x)$; and $U \subset X_\alpha$ is open in $X_\alpha$ if and only if for each $x \in U$ there exists $n \in N$ with $S_\alpha n(x) \subset U$. For $x \in X$, let $Q_n(x) = \bigcup \{S_\alpha n(x); x \in X_\alpha\}$. Since $X$ is determined by $\mathcal{C}$, for $x \in X$, the sequence $\{Q_n(x); n \in N\}$ of subsets of $X$ satisfies the above conditions with respect to $X$. For $x, y \in X$, let $d(x, y) = 1/n$, when $n = \text{Max} \{m; y \in Q_m(x)\}$. Then $d$ is $o$-metric for $X$. Hence $X$ is $o$-metric. When $X_\alpha$ are symmetric, similarly we show that $X$ is symmetric. (2) follows form (1) and (a) in Diagram. For (3), we shall prove only (f) $\Rightarrow$ (a). First, we prove that for each $x \in X$, $x \in \text{int} \text{St}(x, \mathcal{C})$.

To show this, suppose not. Then $x \in \overline{X - \text{St}(x, \mathcal{C})}$. Thus there exists a sequence $K = \{x_n; n \in N\}$ in $X - \text{St}(x, \mathcal{C})$ converging to the point $x$. Let $C \in \mathcal{C}$. If $C \not\ni x$, $K \cap C$ is closed in $C$. If $C \ni x$, $K \cap C = \emptyset$. Thus $K \cap C$ is closed in $C$. Then $K$ is closed in $X$. Hence $K \ni x$. This is a contradiction. Then $x \in \text{int} \text{St}(x, \mathcal{C})$ for each $x \in X$. Now, to show that $X$ is metacompact, let $U$ be an open cover of $X$. Since each $X_\alpha$ is metacompact, there exists a point-finite open refinement $U_\alpha$ of $\{U \cap X_\alpha; U \in U\}$ in $X_\alpha$. Let $\mathcal{V} = \bigcup \{U_\alpha; \alpha\}$. But $X$ is determined by a cover $\{X_\alpha\}$, and each $X_\alpha$ is determined by the open cover $U_\alpha$. Then $X$ is determined by $\mathcal{V}$. Thus, as is seen in the above, for each $x \in X$, $x \in \text{int} \text{St}(x, \mathcal{V})$. Then $X$ has a point-finite refinement $\mathcal{V}$ of $U$ such that $x \in \text{int} \text{St}(x, \mathcal{V})$ for each $x \in X$. Then $X$ is metacompact by [9; Theorem 2.2]. Finally, we show that $X$ is developable. Since $X$ is semi-metric by (2), by [8; Theorem 1] it suffices to show that $X$ has a point-countable base. Since each $X_\alpha$ is metacompact developable, it has a development $\{G_\alpha n; n \in N\}$ such that each $G_\alpha n$ is point-finite, and $G_\alpha n+1$ is a refinement of $G_\alpha n$. For
each $n \in N$, let $\mathcal{G}_n = \bigcup\{\mathcal{G}_\alpha; \alpha\}$, and let $\mathcal{G} = \bigcup\{\mathcal{G}_n; n \in N\}$. Then, $X$ is determined by a point-finite cover $\mathcal{G}_n(n \in N)$. Let $x \in U$ with $U$ open in $X$. Then there exists $n \in N$ such that $x \in \text{St}(x, \mathcal{G}_n) \subseteq U$. But, as is seen in the above, $x \in \text{int} \text{St}(x, \mathcal{G}_n)$. This shows that $X$ has a point-countable cover $\mathcal{G}$ such that for any $x \in X$ and any nbhd $U$ of $x$, there exists a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ with $x \in \text{int} \bigcup \mathcal{G}' \cup \mathcal{G}' \subseteq U$. But $X$ is Fréchet. Then $X$ has a point-countable base by [2; Theorem 6.2].

**Corollary 2.2.** Let $X$ be determined by a point-finite closed cover of developable subspaces. Then $X$ is developable if and only if $X$ is first countable (or Fréchet.)

**Proof:** For the “if” part, by (2) in Theorem 2.1, $X$ is semi-metric, hence submetacompact. On the other hand, in view of the proof of (f) $\Rightarrow$ (a) in Theorem 2.1(3), $X$ is locally developable. Hence $X$ is developable by (1) of Lemma 1.7.

In view of Theorem 2.1 and Corollary 2.2, we have the following question.

**Question 2.3.** Let $X$ be a first countable space determined by a point-finite cover $\{x_\alpha\}$. If the $X_\alpha$ are quasi-metric; n.a.-quasi-metric; $\gamma$-metric; or developable, then so is $X$ respectively?

**Remark 2.4.** In the previous question, if the $X_\alpha$ are metacompact and closed in $X$, then the question is affirmative. Indeed, the proof of (3) in Theorem 2.1 suggests that $X$ is a metacompact space, and each point has a nbhd which is quasi-metric; n.a.-quasi-metric; or $\gamma$-metric respectively. Hence, $X$ has a point-finite open cover of quasi-metric; n.a.-quasi-metric; or $\gamma$-metric subspaces respectively. Then so is $X$ respectively by Lemma 1.4.

Concerning the metrizability; or quasi-metrizability of a space determined by a point-finite; or point-countable cover of
Example 2.5. (1) There exists a n.a.-quasi-metric (hence first countable) space $X$ determined by a point-finite clopen cover of metric subspaces, but $X$ is not metric.

(2) There exists a semi-metric (hence, first countable) space $X$ determined by a point-countable clopen cover of metric subspaces, but $X$ is neither quasi-metric nor $\gamma$-metric.

Proof: (1) Let $X$ be an upper half plane. Let a basic nbd of $(x, y)$ with $y > 0$ be $\{(x, y)\}$, and let a basic nbd of $(r, 0)$ be $\{(x, y) ; y = |x - r| < 1/n\}$, $n \in N$. Then $X$ is metacompact and developable, hence n.a.-quasi-metric. For $(r, 0) \in X$, let $X_r = \{(x, y) ; y = |x - r|\}$. Then $X$ is determined by a point-finite clopen cover $\{X_r; (r, 0) \in X\}$ of metric subspaces. But, by the R. Baire’s Category theorem, $X$ is not normal, hence not metric.

(2) Let $X$ be the developable space $Y$ constructed in [13; Example 2], where $Y$ is not quasi-metric. That is; let $A = R \times \{0\}$, and $B = \{(x, y) ; x, y$ are rationals with $y > 0\}$. For each $p \in A$ and $n \in N$, let $T(p, 1/n)$ denote the set of all points in $B$ that belong to the interior of the isosceles right triangle above $A$ having vertex $p$ and hypothenuse of length $2/n$ parallel to $A$. For each $q \in B$ and $n \in N$, let $C(q, 1/n)$ denote the intersection with $B$ of the circle of radius $1/n$ and center $q$. Let $U$ be the collection of all countable infinite subsets of $A$. Let $Y = AU(B \times U)$. Let a basic nbd of $p \in A$ be $V_n(p) = \{p\} \cup T(p, 1/n) \times U(p)$, where $U(p) = \{\alpha \in U ; p \in \alpha\}$, and let a basic nbd of $(q, \alpha) \in B \times U$ be $V_n(q, \alpha) = C(q, 1/n) \times \{\alpha\}$. Then the basic nbds $V_n(p)$ and $V_n(q, \alpha)$ are metric (indeed, the $V_n(p)$ has a $\sigma$-locally finite base, hence it is metric). Obviously $B = \{V_n(p), V_n(q, \alpha) ; p \in A, (q, \alpha) \in B \times U, n \in N\}$ is a point-countable base for $Y$. But, we can assume that each element of $B$ is clopen in $Y$ ($Y$ is zero-dimensional). Therefore, $Y$ has a point-countable base consisting of clopen and metric subspaces. But $Y$ is developable, hence symmetric. Thus $Y$ is
not \( \gamma \)-metric by (d) in Diagram.

3. SPACES DOMINATED BY COVERS.

The following lemma is due to [5] (for the symmetric case, see [19]; and for the smi-metric or developable case, see [18]).

**Lemma 3.1.** Let \( \{F_\alpha; \alpha\} \) be a locally finite closed cover of a space \( X \). If the \( F_\alpha \) are symmetric; semi-metric; quasi-metric; n.a.-quasi-metric; \( \gamma \)-metric; or developable, then so is \( X \) respectively.

Let \( X \) be a space dominated by a cover \( \{X_\alpha; \alpha < \lambda\} \). For each \( \alpha < \lambda \), let \( L_0 = X_0 \), \( L_\alpha = X_\alpha - \cup\{X_\beta; \beta < \alpha\} \), and let \( F_\alpha = \bar{L}_\alpha \). Then we have

**Lemma 3.2.** Let \( X \) be a space dominated by a cover \( \{X_\alpha; \alpha < \lambda\} \). Then (1) and (2) hold. If the \( X_\alpha \) are Fréchet, then (3) ~ (5) hold.

(1) \( X \) is determined by \( \{F_\alpha; \alpha < \lambda\} \).

(2) Let \( x \in X \). For each \( \alpha < \lambda \), let \( A_\alpha \) be any subset of \( L_\alpha \) such that \( A_\alpha \cup \{x\} \) is closed in \( X \). Then \( S = \cup\{A_\alpha; \alpha < \lambda\} \cup \{x\} \) is closed in \( X \).

(3) If \( X \) contains no closed copy of \( S_\omega \), then \( \{F_\alpha; \alpha < \lambda\} \) is point-finite in \( X \).

(4) If \( X \) contains no closed copy of \( S_2 \), then \( \{F_\alpha; \alpha < \lambda\} \) is hereditarily closure-preserving in \( X \).

(5) If \( X \) contains no closed copy of \( S_\omega \) and no \( S_2 \), then \( \{F_\alpha; \alpha < \lambda\} \) is locally finite in \( X \).

**Proof:** (1) and (2) are due to [21; Lemma 2.5]. For (3), suppose that \( \{F_\alpha; \alpha < \lambda\} \) is not point-finite. Since each \( F_\alpha \) is Fréchet, it follows from (2) that \( X \) contains a closed copy of \( S_\omega \). This is a contradiction. Then \( \{F_\alpha; \alpha < \lambda\} \) is point-finite. For (4), suppose that \( \{F_\alpha; \alpha < \lambda\} \) is not hereditarily closure-preserving. Then for each \( \alpha < \lambda \), there exists a closed subset \( C_\alpha \) of \( F_\alpha \) such that \( A = \cup\{C_\alpha; \alpha < \lambda\} \) is not closed in \( X \). By (1), \( A \cap F_\alpha \) is not closed in some Fréchet space \( F_\alpha \). Thus there exist a point \( x \notin A \), a sequence \( \{x_m; m \in \mathbb{N}\} \) in \( A \), and
an infinite subset \( \{a(m); \ m \in N\} \) of \( \{a; \ a < \lambda\} \) such that \( x_m \to x, \ x_m \in C_{\alpha(m)} \). But each \( F_{\alpha(m)} \) is Fréchet. Then for each \( m \in N \), there exists a sequence \( \{x_{m_n}; \ n \in N\} \) in \( L_{\alpha(m)} \) such that \( x_{m_n} \to x_m \). Let \( T = \{x\} \cup \{x_m; \ m \in N\} \cup \{x_{m_n}; m, n \in N\} \). Then, it follows from (2) that \( T \) is a closed copy of \( S_2 \). Thus \( X \) contains a closed copy of \( S_2 \). This is a contradiction. Then \( \{F_{\alpha}; \ a < \lambda\} \) is hereditarily closure-preserving. (5) follows form (1) and (2).

**Theorem 3.3** Let \( X \) be a space dominated by \( \{X_\alpha\} \). Then (1) and (2) below hold.

1. Let each \( X_\alpha \) be first countable. Then \( X \) is an o-metric if and only if \( X \) contains no closed copy of \( S_\omega \).
2. Let each \( X_\alpha \) be Fréchet. Then \( X \) is Fréchet if and only if \( X \) contains no closed copy of \( S_2 \).

**Proof:** (1) We prove only the "if" part. By (1) and (2) in Lemma 3.2, \( X \) is determined by a point-finite cover \( \{F_{\alpha}; \ a < \lambda\} \). But each \( F_{\alpha} \) is an o-metric. Then \( X \) is an o-metric by Theorem 2.1.

(2) For the "if" part by Lemma 3.2(4), \( X \) has a hereditarily closure-preserving cover \( \{F_{\alpha}; \ a < \lambda\} \). Since each \( F_{\alpha} \) is Fréchet, so is \( X \). The "only if" part follows from the easy fact that any Fréchet space contains no copy of \( S_2 \).

**Theorem 3.4.** Let \( X \) be a space dominated by \( \{X_\alpha\} \).

1. Suppose that the \( X_\alpha \) are semi-metric. Then the following are equivalent.
   a. \( X \) is symmetric.
   b. \( X \) is o-metric.
   c. \( X \) contains no closed copy of \( S_\omega \).

2. Suppose that the \( X_\alpha \) are metric; semi-metric; quasi-metric; n.a.-quasi-metric; \( \gamma \)-metric; or developable. Then the following are equivalent.
   a. \( X \) is so respectively.
   b. \( X \) is first countable.
   c. \( X \) contains no closed copy of \( S_\omega \) and no \( S_2 \).
Proof: (1) holds in view of (1) in Theorem 2.1 & 3.3 and Lemma 3.2. (2) follows from Lemma 3.1 and Lemma 3.2(5). (For the metric case, (a) ⇔ (c) is due to [22; Theorem 1.5].)

In view of (1) in Theorem 3.3 and 3.4, we have a question below. If (2) is affirmative, then so is (1). When the $X_\alpha$ are semi-stratifiable, (1) is affirmative. Indeed, by [18; Theorem 4.5], $X$ is semi-stratifiable. While, $X$ is o-metric. Then $X$ is symmetric by (a) in Diagram.

Question 3.5. Let $X$ be a space dominated by $\{X_\alpha\}$. Suppose that the $X_\alpha$ are symmetric.

(1) If $X$ is o-metric, then $X$ is symmetric?
(2) If $X$ contains no closed copy of $S_\omega$, then $X$ is symmetric?

REFERENCES


Tokyo Gakugei University
Koganei-shi, Tokyo, 184 Japan