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# SYMMETRIC PRODUCTS AND HIGHER DIMENSIONAL DUNCE HATS

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## 1. Introduction and Preliminaries

An interesting construction in topology is the *nth symmetric product* of a metric space X, denoted X(n), which is defined to be the hyperspace of n (n$\geq$ 1) or fewer points of X, metrized with the Hausdorff metric. An attraction of these spaces is that $S^1(2) \cong$ Mobius Band and $S^1(3) \cong S^3$. In this paper we will see that the Dunce Hat also comes up in a natural way and this leads to a symmetric product definition of what we call "higher dimensional Dunce Hats".

This notion of symmetric product was introduced by Borsuk and Ulam in [BU] where they proved, for $I = [0,1]$ and $n = 1,2,3$, that $I(n) \cong I^n$, and for $n \geq 4$, that $I(n)$ is not homeomorphic to any subset of $\Re^n$. In [M], Molski proved that if $M$ is a 2-manifold, then $M(2)$ is a 4-manifold, and for $n \geq 3$ that neither $I^2(n)$ nor $I^n(2)$ is homeomorphic with any subset of $\Re^{2n}$. In [Bo], Bott corrected Borsuk's [B] statement that $S^1(3) \cong S^1 \times S^2$ by showing that actually $S^1(3) \cong S^3$.

The techniques used to obtain the above mentioned results generally were to parameterize the points of $I(n)$ in clever ways to form homeomorphisms between $I(n)$ and known spaces. In [S], Schori expanded on [BU] and [M] and among other results identified a naturally occuring subspace $D^{n-2}$ of $I(n)$ such that $I(n) \cong$ cone$(D^{n-2}) \times I$. In this paper we study the $D^k s$ with a parameterization that clearly identifies them as polyhedra and then we determine their homotopy type. The spaces $I(n)$ and $D^{n-2}$, while not manifolds, are very natural objects and have some interesting properties. For example, $D^2$ is the topological Dunce Hat and has been of interest in the study of the Poincaré Conjecture. In fact, we show that each $D^{2n}, n \geq 1$, is contractible but not collapsible, the two salient properties of the Dunce Hat. Consequently, we call the spaces $D^{2n}$, $n \geq 2$, *higher dimensional Dunce Hats.* Our contractibility proof is based on a version of the Homotopy Addition Theorem.

The topological Dunce Hat $D$ is discussed in detail by E.C. Zeeman in [Z]. The space $D$ is remarkable because it is the simplest example of a polyhedron that is contractible, in the sense of homotopy, but not collapsible, in the sense of Whitehead. The point of [Z] was to analyze the Dunce Hat, and the manifolds of which it is a spine because of some intimate relations to the Poincaré Conjecture. The point of this connection is that being contractible yet not collapsible had been identified as a primary source of difficulty in the study of manifolds of dimension $\geq 3$. It is well known, see [Z], that the statement "Any contractible 2-complex cross an interval is collapsible" implies the Poincaré Conjecture. (See Remark 3.7 in this paper.)

We will define our Dunce Hats as symmetric products.

**Definitions 1.1.** For a compact metric space $X$, if $X^n$ is the $n$-fold cartesian product of $X$, define the *$n$-fold symmetric product* of $X$,

$$X(n) = X^n / \sim,$$

where $\sim$ is the equivalence relation on $X^n$ defined by

$$(x_1, \dots, x_n) \sim (y_1, \dots, y_n) \text{ if and only if}$$
$$\{x_1, \dots, x_n\} = \{y_1, \dots, y_n\}$$

That is, two points in $X^n$ are equivalent if the sets consisting of their coordinates are equal. If $n = 2$, this means that

$(x,y) \sim (y,x)$, but if $n \geq 3$, then not only do we have identification under permutations of coordinates, but we also have the extra identifications as illustrated by the case for $n = 3$ where $(a,a,b) \sim (a,b,b)$.

The $n$-fold symmetric product of $X$, $X(n)$, is well known (see [S]) to be homeomorphic to the space of (closed) subsets of $X$ consisting of $n$ or fewer points topologized with the Hausdorff metric,

$$D(A,B) = \inf\{\epsilon > 0 : A \subset U(B,\epsilon) \text{ and } B \subset U(A,\epsilon)\}$$

where $A$ and $B$ are closed subsets of $X$ and $U(C,\epsilon)$ is the open $\epsilon$-ball about $C \subset X$. In fact, for the rest of this paper we will think of $X(n)$ as the "hyperspace" of $n$ or fewer points of $X$.

For the closed unit interval $I = [0,1]$ and for $n \geq 2$, we let

$$I_0^1(n) = \{A \in I(n) : 0,1 \in A\}.$$

We remark that the space $I_0^1(3)$ is homeomorphic to the 1-sphere $S^1$ as can be seen by noting that the generic point of $I_0^1(3)$ is $x = \{0,b,1\}$, and as $b \in I$ moves from 0 to 1, the point $x$ moves from the base point $* = \{0,1\}$ around a circle and back to $*$. In the next section we will see that $I_0^1(4)$ is the classical topological Dunce Hat.

**Definition 1.2.** We use Whitehead's notion of *simplicially collapsible* and say that a (compact) polyhedron P is *collapsible* if there exists a complex $K$ such that $P = |K|$ and $K$ is simplicially collapsible.

**Definition 1.3.** Let $X$ and $Y$ be disjoint topological spaces, $A$ be a closed subset of $X$, and $f : A \to Y$ be a continuous map. We will use the notation $X \cup_f Y$ for the *adjunction space determined by f*, where $X$ is attached to $Y$ by $f$.

We will be constructing CW-complexes by inductively attaching $n$-cells $B^n$ to a space $Y$ by a map $f : Bd\ B^n \to Y$.

**Remark 1.4.** If $A$ is a closed subset of the compact metric space $Y$ and

$$f : (B^n, Bd\ B^n) \to (Y, A)$$

is a *relative homeomorphism* (that is, $f$ is a continuous surjection, $f|B^n - Bd\ B^n$ is one-to-one, and $f|Bd\ B^n : Bd\ B^n \to A$ is a surjection), then $Y \cong B^n \cup_{\bar{f}} A$ where $\bar{f} = f|Bd\ B^n$.

## 2. Topological Dunce Hats.

The traditional topological Dunce Hat $D$ is obtained from a 2-simplex, $\Delta^2$, say $ABC$, by identifying the sides $AB = AC = BC$.

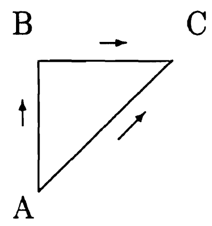


Fig. 1

We state and prove the following theorem to motivate the next section where these ideas are generalized.

**Theorem 2.1.** *The space $I_0^1(4)$ is the Dunce Hat.*

*Proof:* Let $\sigma^2 = \{(a,b) \in I^2 : 0 \le a \le b \le 1\}$ and define $q : \sigma^2 \to I_0^1(4)$ by $q(a,b) = \{0, a, b, 1\}$. The edge of $\sigma^2$ labeled $AB$ in Figure 1 is the set $\{(0,b) : b \in I\}$ and is mapped onto $I_0^1(3) \cong S^1$ by $q$ with positive orientation. Likewise, the edges $BC = \{(b,1) : b \in I\}$ and $AC = \{(b,b) : b \in I\}$ are mapped onto $I_0^1(3)$ by $q$ with positive orientations. Consequently, under the map $q$ the edges $AB, BC$, and $AC$ are identified as indicated in Figure 1. Furthermore, the map $q$ restricted to the interior of $\sigma^2$ is one-to-one and hence the diagram in Figure 1 faithfully represents the space $I_0^1(4)$. $\square$

The contractibility of the Dunce Hat is well-known and is a consequence of a basic result in homotopy theory, since $q : Bd\,\sigma^2 \to I_0^1(3) \cong S^1$ has degree 1, hence is a homotopy equivalence and induces a homotopy equivalence $q : \sigma^2 \to I_0^1(4)$ (see Theorem 3.2 below). Also, in [Z] Zeeman gives a picture proof that $D \times I$ is collapsible, implying that $D$ is contractible.

The symmetric product representation of a Dunce Hat as presented above gives a natural way for defining higher dimensional analogues.

**Definition 2.2.** For each integer $n \geq 0$, let

$$D^n = I_0^1(n+2) = \{C \in I(n+2) : 0 \in C \text{ and } 1 \in C\}.$$

(We remark the $D^0$ is a point, and $D^1$ is homeomorphic to a circle $S^1$.)

Furthermore, let

$$\sigma^n = \{(a_1, a_2, \ldots, a_n) : 0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq 1\}$$

and define $q_n : \sigma^n \to D^n$ by

$$q_n(a_1, a_2, \ldots, a_n) = \{0, a_1, a_2, \ldots, a_n, 1\}$$

The following theorem is a main result of this paper and is our justification for calling $D^{2n}, n \geq 2$, *higher-dimensional Dunce Hats.*

**Theorem 2.3.** *The spaces $D^{2n}$, $n \geq 1$, are contractible and not collapsible.*

The proof will be given in Section 3. The following material is needed for that proof.

**Lemma 2.4.** *The map $q_n : (\sigma^n, Bd\sigma^n) \to (D^n, D^{n-1})$, $n \geq 1$, as a map of pairs, is a relative homeomorphism and hence $D^n \cong \sigma^n \cup_{\bar{q}_n} D^{n-1}$, the CW-complex obtained by attaching $\sigma^n$ to $D^{n-1}$ with the map $\bar{q}_n = q_n | Bd\sigma^n$.*

*Proof:* The $n+1$ conditions: $a_1 = 0, a_1 = a_2, \ldots, a_{n-1} = a_n, a_n = 1$, individually determine the $n+1$ faces of $\sigma^n$ and the image of any one of these faces under $q_n$ is the space $D^{n-1}$ and hence $q_n$ is the required map of pairs. Furthermore, $q_n|\sigma^n - Bd\sigma^n$ is clearly one-to-one and hence $q_n$ is a relative homeomorphism and the attaching space condition then follows from Remark 1.4. □

Thus we have a sequence of CW-complexes

$$D^0 \subset D^1 \subset \cdots \subset D^{n-1} \subset D^n$$

where $D^n$ is formed by attaching the $n$-cell $\sigma^n$ to $D^{n-1}$ with the map $\hat{q}_n : Bd\sigma^n \to D^{n-1}$.

The use of rectangular coordinates in the description of

$$\sigma^n = \{(a_1, a_2, \ldots, a_n) : 0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq 1\}$$

has been very convenient for the construction of $D^n$. We now switch to barycentric coordinates for an $n$-simplex in the rest of the paper.

Let $\Delta^n =< v_0, v_1, \ldots, v_n >$ be a standard $n$-simplex where each point $x \in \Delta^n$ is uniquely determined as $x = \lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_n v_n$ where $\lambda_0 + \lambda_1 + \cdots + \lambda_n = 1$ and each $\lambda_i \geq 0$. Then $\alpha_n : \Delta^n \to \sigma^n$ defined by $\alpha_n(x) = (\lambda_0, \lambda_0 + \lambda_1, \ldots, \lambda_0 + \cdots + \lambda_{n-1})$ is a homeomorphism and $p_n : \Delta^n \to D^n$ defined by

$$p_n(x) = \{0, \lambda_0, \lambda_0 + \lambda_1, \ldots, \lambda_0 + \cdots + \lambda_{n-1}, 1\}$$

is *topologically equivalent* to $q_n : \sigma^n \to D^n$. That is, the following diagram

$$\begin{array}{ccc} \Delta^n & \xrightarrow{p_n} & D^n \\ \downarrow \alpha_n & & \downarrow \mathrm{id} \\ \sigma^n & \xrightarrow{q_n} & D^n \end{array}$$

commutes ($p_n = q_n \circ \alpha_n$) where the vertical arrows are homeomorphisms. We also have the attachment result corresponding to Lemma 2.4. That is, for each $n \geq 1$,

$$D^n \cong \Delta^n \cup_{\bar{p}_n} D^{n-1}$$

where $\bar{p}_n = p_n | Bd\Delta^n$.

We remark that in [T], Thomas gives a different but equivalent definition of the spaces $D^n$ but neither states nor proves any properties of the $D^n$.

Let $d_i^n : \Delta^n \rightarrow Bd\Delta^{n+1}, i = 0, \ldots, n+1$, be the *face map* which takes $\Delta^n$ simplicially onto the $n$-face of $\Delta^{n+1} = \langle v_0, v_1, \ldots, v_{n+1} \rangle$ opposite the vertex $v_i$. The following lemma follows from the definitions.

**Lemma 2.5.** *The following diagram commutes, where* $\bar{p}_{n+1} : Bd\Delta^{n+1} \rightarrow D^n$ *is the restriction of* $p_{n+1}$ *to* $Bd\Delta^{n+1}$. *That is, for all allowable indices,* $p_n = \bar{p}_{n+1} \circ d_i^n$.

$$\begin{array}{ccc} \Delta^n & \xrightarrow{p_n} & D^n \\ \downarrow d_i^n & \nearrow & \bar{p}_{n+1} \\ Bd\Delta^{n+1} & & \end{array}$$

*Proof:* The proof follows directly from the definitions and can probably be best illustrated for the case that $n = 2$ and $i = 1$. If $x = \lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2 \in \Delta^2$, then $d_1^2(x) = \lambda_0 v_0 + 0 v_1 + \lambda_1 v_2 + \lambda_2 v_3 \in Bd\Delta^3$ and $\bar{p}_3 \circ d_1^2(x) = \{0, \lambda_0, \lambda_0, \lambda_0 + \lambda_1, 1\} = \{0, \lambda_0, \lambda_0 + \lambda_1, 1\} = p_2(x)$. □

## 3. Proof of Main Results.

We will use the Homotopy Addition Theorem to prove our main theorem. There are several versions of the the Homotopy Addition Theorem and we will use a particularly useful one presented in [Hu, p. 165]. Let

$$f : (Bd\Delta^{n+1}, (\Delta^{n+1})^{(n-1)}) \to (X, x_0)$$

be any map into a pointed space $(X, x_0)$. Then $f$ represents and element $[f]$ of $\pi_n(X, x_0)$. On the other hand, we have the face map

$$d_i^n : \Delta^n \to Bd\Delta^{n+1}$$

for $i = 0, 1, \ldots, n+1$, and hence the composition

$$f \circ d_i^n : (\Delta^n, Bd\Delta^n) \to (X, x_0)$$

represents an element, $[f \circ d_i^n]$, of $\pi_n(X, x_0)$ for every $i = 0, 1, \ldots, n+1$.

**Theorem 3.1. (Homotopy Addition Therorem).** (See [H, p. 165]) *For any map*

$$f : (Bd\Delta^{n+1}, (\Delta^{n+1})^{(n-1)}) \to (X, x_0)$$

*and $n \geq 2$, we always have*

$$[f] = \sum_{i=0}^{n+1} (-1)^i [f d_i^n],$$

*and for $n = 1$, $[f] = [f d_2^1] \cdot [f d_0^1] \cdot [f d_1^1]^{-1}$.*

We need two more basic results from homotopy theory which follow from Whitehead [Wh, Corollary 5.12 in Chapter 1].

**Theoren 3.2.** *Let $A \subseteq X$ and $B$ be CW-Complexes and let $h : A \to B$ be a homotopy equivalence. Then the induced map $\hat{h} : X \to X \cup_h B$ is a homotopy equivalence.*

**Theorem 3.3.** *For a CW pair $(X, A)$, if $A$ is a contractible closed subset of $X$, then the quotient map $q : X \to X/A$ defined by*

$$q(x) = \begin{cases} * & \text{if } x \varepsilon A \\ x & \text{if } x \varepsilon X \backslash A \end{cases}$$

*is a homotopy equivalence.*

We are finally ready to prove part of our main result.

**Theorem 3.4.** *For each $n \geq 0$, the space $D^{2n}$ is contractible and $D^{2n+1}$ is of the same homotopy type as $S^{2n+1}$.*

*Proof:* We have already proved that $D^0$ and $D^2$ are contractible and that $D^1$ is homeomorphic to $S^1$. Let $n > 2$ be odd and assume that $D^{n-1}$ is contractible. Now, $D^n \cong \Delta^n \cup_{\bar{p}_n} D^{n-1}$ and since $D^{n-1}$ is contractible, by Theorem 3.3, the quotient map

$$q : D^n \to D^n/D^{n-1}$$

is a homotopy equivalence. Furthermore, $D^n/D^{n-1}$ is homeomorphic to $S^n$ since it is equivalent to $\Delta^n/Bd\Delta^n$. This verifies that $D^n \simeq S^n$, for $n$ odd.

We now apply the Homotopy Addition Theorem to the map of pairs

$$q \circ \bar{p}_{n+1} : (Bd\Delta^{n+1}, (\Delta^{n+1})^{(n-1)}) \to (S, *).$$

Lemma 2.5 yields $\bar{p}_{n+1} \circ d_i^n = p_n$, and combining this with the Homotopy Addition Theorem, we have

$$[q \circ \bar{p}_{n+1}] = \sum_{i=0}^{n+1} (-1)^i [q \circ \bar{p}_{n+1} \circ d_i^n] = \sum_{i=0}^{n+1} (-1)^i [q \circ p_n].$$

Since n is odd, it follows that $[q \circ \bar{p}_{n+1}] = [q \circ p_n]$. Hence, $[q \circ \bar{p}_{n+1}]$, as an element of $\pi_n(S^n, *)$, is represented by $q \circ p_n : (\Delta^n, Bd\Delta^n) \to (S^n, *)$, which represents the identity element of $\pi_n(S^n)$. Consequently, $q \circ \bar{p}_{n+1} : Bd\Delta^{n+1} \longrightarrow S^n$ is homotopic to a homeomorphism and is therefore a homotopy equivalence.

By hypothesis, $D^{n-1}$ is contractible and consequently $q : D^n \longrightarrow D^n/D^{n-1}$ is a homotopy equivalence by Theorem 3.3. Since both $q \circ \bar{p}_{n+1}$ and $q$ are homotopy equivalences, it follows that $\bar{p}_{n+1} : Bd\Delta^n \longrightarrow D^n$ is a homotopy equivalence. It now follows from Theorem 3.2 that

$$D^{n+1} = \Delta^{n+1} \cup_{\bar{p}_{n+1}} D^n$$

and $\Delta^{n+1}$ are of the same homotopy type and hence $D^{n+1}$ is contractible since $\Delta^{n+1}$ is.

**Theorem 3.5.** *For each $n \geq 1, D^n$ is not collapsible.*

*Proof:* The space $D^0$ is a point and is trivially collapsible. To see that $D^n, n \geq 1$, is triangulable, let $K(\Delta^n)$ be the complex consisting of $\Delta^n$ and all of its faces and consider its second barycentric subdivision $sd^2(K(\Delta^n))$. Then $D^n$ is homeomorphic to the polyhedron $|sd^2(K(\Delta^n))|/p_n$. This polyhedron has no free faces and consequently no place to start the collapsing process. To see that $D^n$ is not collapsible (for any triangulation), we proceed inductively. First, we acknowledge that $D^2$ is not collapsible, and assume, as an inductive hypothsis, that $D^n$ is not collapsible. Now, $D^{n+1}$ is obtained by starting with $\Delta^{n+1}$ and attaching each of its $n$-faces, $\Delta^n$, to $D^n$ by $p_n$. Thus, each open set in $D^{n+1}$ is $(n+1)$-dimensional yet has no possibility for a triangulation with a free $n$-face and consequently no place to start the collapsing process. Consequently, the $D^n, n \geq 1$, are not collapsible. □

**Remark 3.6.** Our sequence of higher dimensional Dunce Hats, $D^{2n}, n \geq 2$, are not unique in having the property that they are contractible and not collapsible. Suspensions of $D^2$ give examples in all higher dimensions of contractible polyhedra that are not collapsible. In this connection, we point out that $S(S(D^2))$ and $D^4$ are not homeomorphic since their sets of points at which they fail to be locally Euclidean are the 3-sphere and $D^3$, respectively.

**Remark 3.7.** As stated earlier an affirmative answer to the Poincare Conjecturé can be achieved by showing that every contractible space cross an interval is collapsible. In this connection it would be interesting to determine whether a higher dimensional Dunce Hat cross an interval is collapsible.

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