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More On The Property Of A Space Being Lindelöf In Another *

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Abstract

Answering a question of D.V.Rachin, A.Dow and J.Vermeer constructed examples of X and Z such that X is Lindelöf in Z but there is no Lindelöf Y with $X \subset Y \subset Z$. We solve similar problem with n-star-Lindelöf instead of Lindelöf.

A subspace X of a space Z is called to be Lindelöf in Z provided every open cover of Z contains a countable subfamily that covers X.

Answering a question of D.V.Ranchin [Ran], A.Dow and J.Vermeer [DV] gave examples of pairs $X \subset Z$ such that X is Lindelöf in Z and there is no Lindelöf space Y such that $X \subset Y \subset Z$. In fact, the problem of Ranchin was a special case of the following general problem (Problem 1 from [AG], where it was also called the Generalized Ranchin Problem):

(*) Suppose X is relatively $\mathcal{P}$ in Z. Is there a space Y, $X \subset Y \subset Z$, which has property $\mathcal{P}$?

*The paper was written when the second author was visiting University of Messina. He uses opportunity to express his gratitude to his colleagues from Messina for kind hospitality

So, Dow and Vermeer solved this problem to the negative for $\mathcal{P}$=Lindelöf.

In this paper we do the same for $\mathcal{P}$=n-star-Lindelöf for $n = 1$ and 3. Also we consider the following modification of problem (*):

(**) Let $\mathcal{Q}$ be certain property weaker than $\mathcal{P}$. Suppose X is relatively $\mathcal{P}$ in Z. Is there a space Y, $X \subset Y \subset Z$, which has property $\mathcal{Q}$?

A space X is n-star-Lindelöf if for every open cover $\mathcal{U}$ of X there exists a countable subset $F \subset X$ such that $St^n(F, \mathcal{U}) = X$ (see [vDRRT], where it was called strongly n-star-Lindelöf). It is clear that

$$\text{Lindelöf} \Rightarrow \text{1-star-Lindelöf} \Rightarrow \text{2-star-Lindelöf} \Rightarrow \text{3-star-Lindelöf} \Rightarrow \ldots$$

It turns out that the first three arrows can not be reversed, while after 3-star-Lindelöf in the class of regular spaces in fact there is $\Leftrightarrow$ instead of $\Rightarrow$ ([Mat1], see also [vDRRT]). The same is true for n-star-compactness (to obtain the definition of n-star-compactness one must just replace "countable subset F" by "finite subset F" in the definition of n-star-Lindelöfness. Moreover, for $n \geq 3$, n-star-compactness is equivalent in the class of regular spaces to feebly compactness, also called DFCC property: every discrete family of nonempty open sets is finite. For Tychonoff spaces, DFCC is equivalent to pseudocompactness. Likewise, for $n \geq 3$, n-star-Lindelöfness is equivalent in the class of regular spaces to DCCC, the discrete countable chain condition: every discrete family of nonempty open sets is at most countable.

Now we pass to relative version of these properties. Like it is usually the case with relativization, there is more than one way to do this:

Definition 1

(1) $X \subset Z$ is n-star-Lindelöf (n-star-compact) in Z provided for every open cover $\mathcal{U}$ of Z there exists a countable (resp., finite) subset $F \subset X$ such that $St^n(F,\mathcal{U}) \supset X$.

(2) $X \subset Z$ is n-star-Lindelöf (n-star-compact) in Z provided for every open cover $\mathcal{U}$ of Z there exists a countable (resp., finite) subset $F \subset Z$ such that $St^n(F,\mathcal{U}) \supset X$.

Of course, definition (1) is stronger.

Below we present two examples. The first concerns 1-star-Lindelöfness in the sense of Definition (1) while the second concerns 3-star-Lindelöfness and 3-star-compactness. From Lemma 1 (see below) the difference between (1) and (2) in the second example is not important.

Definition 2 *$X \subset Z$ is DCCC (DFCC) in Z if every discrete family of open sets in Z that intersect X is countable (finite).*

Lemma 1 [Mat1] *If $X \subset Z$ and Z is a regular space, then the following conditions are equivalent:*

(a) X is DCCC (DFCC) in Z,

(b) X is 3-star-Lindelöf (3-star-compact) in Z in sense (1),

(c) X is 3-star-Lindelöf (3-star-compact) in Z in sense (2).

Example 1 *A Tychonoff space Ψ and its subspace $\mathcal{D}$ such that $\mathcal{D}$ is 1-star-Lindelöf in Ψ and there does not exist an 1-star-Lindelöf space Y such that $\mathcal{D} \subset Y \subset \Psi$.*

Our space Ψ is a modification of a well-known Mrówka space Ψ ([Mro], see also [Eng, 3.6.I(a)]): we just take " countable" instead of "finite" and "of cardinality ω_1" instead of "countable". Let $\mathcal{D}$ be a set of cardinality ω_1. We divide $\mathcal{D}$ into ω_1-many parts: $\mathcal{D} = \cup\mathcal{E}_0$, where $\mathcal{E}_0 = \{\mathcal{D}_\alpha : \alpha < \omega_1\}$, $|\mathcal{D}_\alpha| = \omega_1$

for each α and $\mathcal{D}_\alpha \cap \mathcal{D}_\beta = \emptyset$, for $\alpha \neq \beta$. Denote $\mathcal{E}$ the family of all subsets of $\mathcal{D}$ of cardinality ω_1. Then $\mathcal{E}_0 \subset \mathcal{E}$. By Zorn's Lemma, there exists a subfamily $\mathcal{R} \subset \mathcal{E}$ such that $\mathcal{R} \supset \mathcal{E}_0$ and $\mathcal{R}$ is maximal with respect to the following property

($\dagger$) for every two distinct elements $A, B \in \mathcal{R}$, $|A \cap B| \leq \omega$.

Then $|\mathcal{R}| \geq \omega_1$. Put $\Psi = \mathcal{D} \cup \mathcal{R}$. We define the topology on Ψ so that points of $\mathcal{D}$ are isolated in Ψ and a basic neighbourhood of a point $r \in \mathcal{R}$ takes the form $O_{r,C} = \{r\} \cup (r \setminus C)$, where C is arbitrary (at most) countable subset of $\mathcal{D}$. Then Ψ is Hausdorff and zero-dimensional, hence it is Tychonoff. We claim that $\mathcal{D}$ is 1-star-Lindelöf in Ψ. Suppose the contrary: let $\mathcal{U}$ be an open cover of Ψ such that for every countable $F \subset \mathcal{D}$ we have $St(F, \mathcal{U}) \not\supset \mathcal{D}$. Then one can choose by induction points $x_\alpha \in \mathcal{D}$, $\alpha < \omega_1$, so that $x_\alpha \notin St(\{x_\beta : \beta < \alpha\}, \mathcal{U})$ ($\dagger\dagger$). Then, by ($\dagger\dagger$), no element of $\mathcal{U}$ contains more than one point x_α. Denote $A = \{x_\alpha : \alpha < \omega_1\}$. By maximality of $\mathcal{R}$, there exists an $r \in \mathcal{R}$ such that $|r \cap A| = \omega_1$. Let U be any element of $\mathcal{U}$ that contains point r. By the definition of neighbourhood base at r, U contains all points of r but at most countably many, hence it contains uncountably many points of A. This contradicts ($\dagger\dagger$).

Now, let $\mathcal{D} \subset Y \subset \Psi$. We claim that Y is not 1-star-Lindelöf. Denote $\mathcal{R}_Y = Y \cap \mathcal{R}$. Then $Y = \mathcal{D} \cup \mathcal{R}_Y$. There are two possibilities: $|\mathcal{R}_Y| \leq \omega$ and $|\mathcal{R}_Y| \geq \omega_1$.

Case 1: $|\mathcal{R}_Y| \leq \omega$. Since $|\mathcal{R}| \geq \omega_1$, we can pick a point $r^* \in \mathcal{R} \setminus \mathcal{R}_Y$. For each $r \in \mathcal{R}_Y$, the set $C_r = r \cap r^*$ is at most countable, hence so is the set $C = \cup\{C_r : r \in \mathcal{R}_Y\}$. Therefore, the set $r^* \setminus C = r^* \setminus \cup\mathcal{R}_Y$ is uncountable and hence the set $\mathcal{D}_0 = \mathcal{D} \setminus \cup\mathcal{R}_Y$ is uncountable. The family $\mathcal{U} = \{r \cup \{r\} : r \in \mathcal{R}_Y\} \cup \{\{x\} : x \in \mathcal{D}\}$ is an open cover of the space Y. But for every at most countable subset $A \subset Y$, $St(A, \mathcal{U}) \not\supset Y$ since for uncountably many points x of Y (that is, for all points $x \in \mathcal{D}_0$), $\{x\}$ is the only element of $\mathcal{U}$ which contains x.

Case 2: $|\mathcal{R}_Y| \geq \omega_1$. We enumerate $\mathcal{D}$ on type ω_1: $\mathcal{D} = \{a_\alpha : \alpha < \omega_1\}$ and choose points $r_\alpha \in \mathcal{R}_Y$, $\alpha < \omega_1$ so that $r_\alpha \neq r_{\alpha'}$, if $\alpha \neq \alpha'$. For each $\alpha < \omega_1$ we denote $C_\alpha = \{a_\gamma : \gamma \leq \alpha\}$. Also, we denote $\mathcal{R}_1 = \{r_\alpha : \alpha < \omega_1\}$. Then the family of sets $\mathcal{U} = \{O_{r_\alpha C_\alpha} : \alpha < \omega_1\} \cup \{O_{r\emptyset} : r \in \mathcal{R}_Y \setminus \mathcal{R}_1\} \cup \{\{x\} : x \in \mathcal{D}\}$ is an open cover of Y. Let $A \subset Y$ be a countable subset. Denote $A_\mathcal{D} = A \cap \mathcal{D}$ and $A_\mathcal{R} = A \cap \mathcal{R}_Y$. Then $St(A,\mathcal{U}) = St(A_\mathcal{D},\mathcal{U}) \cup St(A_\mathcal{R},\mathcal{U})$. Since $A_\mathcal{D}$ is countable, $A_\mathcal{D} \subset C_{\alpha^*}$ for some $\alpha^* < \omega_1$. Hence for $\alpha > \alpha^*$, $O_{r_\alpha C_\alpha} \cap A_\mathcal{D} = \emptyset$ and $r_\alpha \notin St(A_\mathcal{D},\mathcal{U})$ since $O_{r_\alpha C_\alpha}$ is the only element of $\mathcal{U}$ that contains r_α. Therefore, the set $\mathcal{R}_Y \setminus St(A_\mathcal{D},\mathcal{U})$ is uncountable. But the set $\mathcal{R}_Y \cap St(A_\mathcal{R},\mathcal{U})$ is countable. Hence the set $\mathcal{R}_Y \setminus St(A,\mathcal{U})$ is uncountable and thus nonempty.

Remark It is easy to check that the space Ψ is 2-star-Lindelöf (this fact follows from the simple observation: If a space Z contains a dense subspace X which is 1-star-Lindelöf in Z, then Z is 2-star-Lindelöf). Hence the following question is open

Question Does there exist a space Z and a subspace $X \subset Z$ such that X is 1-star-Lindelöf in Z and there does not exists a 2-star-Lindelöf Y such that $X \subset Y \subset Z$? (Of course, in this case X is not assumed to be dense in Z).

Example 2 *A Tychonoff space Z which contains a closed nowhere dense subspace X which is 3-star-compact in Z (hence 3-star-Lindelöf in Z) and such that there is no 3-star-Lindelöf space Y such that $X \subset Y \subset Z$.*

In the construction of X and Z we use, in fact, one regular step from the ω_1-many step construction of the second author [Mat2]. As X, we take the discrete space of cardinality ω_1. We denote $\mathcal{A}$ the set of all countable sequences of distinct points of X:

$$\mathcal{A} = \{a \in {}^\omega X : a(m) \neq a(n) \text{ if } m \neq n\}.$$

$\mathcal{J} = \mathcal{A} \times^{\omega} \omega$ is the index set for the enumeration of what we shall call "pages"; also $\mathcal{J}$ is a part of the space Z we are going to construct. We put $Z = X \cup P \cup \mathcal{J}$, where $P = \cup\{P_j : j \in \mathcal{J}\}$ and for each $j \in \mathcal{J}$, $P_j = X \times \omega \times \{j\}$ is the j-th page. We define a topology on Z as follows: points of P are isolated. The basic neighbourhood of a point $x \in X$ takes the form

$$O_\kappa(x) = \{x\} \cup \{< x, m, j >\in P : m \geq \kappa, j \in \mathcal{J}\},$$

where $\kappa \in \omega$. To define a basic neighbourhood of a point $j =< a, f >\in \mathcal{J}$ we choose an untrafilter ξ on ω and put $U_F(j) = \{j\} \cup \{< a(n), f(n), j >\in P_j : n \in F\}$, where $F \in \xi$. Clearly, the space $X \cup P$ is metrizable; for each $j =< a, f >\in \mathcal{J}$, the set $S_j = \{< a(n), f(n), j >: n \in \omega\}$ is clopen in $X \cup P$; in the space Z the points of this set converge to point j according to the type of ultrafilter ξ. Therefore, Z is homeomorphic to a subspace of the Čech-Stone compactification of $X \cup P$ and thus Z is a Tychonoff space. We claim that X is 3-star-compact in Z. By Lemma 1, we are to show that X is DFCC in Z. Suppose the contrary: let $\eta = \{U_n : n \in \omega\}$ be a disjoint sequence of open sets in Z each of which intersects X. We verify that it is not discrete in Z. For each $n \in \omega$ we choose a point $a(n) \in X \cap U_n$ and an integer κ_n so that $a(n) \in O_{\kappa_n}(a(n)) \subset U_n$. Then $a =< a(n) : n \in \omega >\in \mathcal{A}$. Denote f the function from ω to ω for which $f(n) = \kappa_n$ for each n. Then $j =< a, f >$ is a point each neighbourhood of which intersects infinitely many sets $O_{\kappa_n}(a(n))$, and hence infinitely many sets U_n. Indeed, the set S_j is contained in the union of the sets U_n: one point of S_j in each U_n, and $j \in \overline{S_j}$. Therefore η is not discrete in Z.

Now, let $X \subset Y \subset Z$. We claim that Y contains an uncountable clopen discrete subspace. Clearly, this implies that Y is not 3-star-Lindelöf. For a subset $T \subset P$, let us denote $\pi(T)$ the projection of T to X: $\pi(T) = \{x \in X :< x, m, j >\in T$ for some $m \in \omega$ and $j \in \mathcal{J}\}$. Also, for a subset $T \subset P$ and a point $x \in X$ we denote $T(x) = \{p \in T : \pi(\{p\}) = x\}$. Put $Y_P = Y \cap P$ and $Y_{\mathcal{J}} = Y \cap \mathcal{J}$. Then $Y = X \cup Y_P \cup Y_{\mathcal{J}}$. We have two cases:

Case 1: $|\pi(Y_P)| \leq \omega$. Then the set $X \setminus \pi(Y_P)$ is uncountable, clopen and discrete in Y.

Case 2: $|\pi(Y_P)| = \omega_1$. For each $x \in \pi(Y_P)$ we choose one point $t_x \in Y_P(x)$ and denote $T = \{t_x : x \in \pi(Y_P)\}$. Then $\pi(T) = \pi(Y_P)$ (and therefore $|\pi(T)| = |T| = \omega_1$) and $|T(x)| \leq 1$ for each $x \in X$. Hence T has no limit points in X. We claim that there exists a subset $T_0 \subset T$, $|T_0| = \omega_1$ which has no limit points also in $\mathcal{J}$. Since T consists of isolated points of Z, this will imply that T_0 is clopen and discrete in Y (and of cardinality ω_1). Indeed, to have a limit point in $\mathcal{J}$, say point j, a subset of P must have infinite intersection with S_j. But the sets S_j, $j \in \mathcal{J}$, clearly have the following properties: 1) $|S_j| = \omega, \forall j \in \mathcal{J}$ and 2) $S_j \cap S_{j'} = \emptyset$, whenever $j \neq j'$. Using 1) and 2) one can easily choose a subset $T_0 \subset T$ of cardinality ω_1 which has at most one point in each S_j and thus has no limit points in $\mathcal{J}$.

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