PAIRS OF INDECOMPOSABLE CONTINUA WHOSE PRODUCT IS MUTUALLY APOSYNDETIC

Alejandro Illanes

Abstract

In this paper we prove that if $p$ and $q$ are relative prime positive integers and $S_p$, $S_q$ are the respective $p$-adic and $q$-adic solenoids, the their topological product $S_p \times S_q$ is mutually aposyndetic. This answers a question by Charles L. Hagopian.

1991 Mathematics Subject Classification: Primary 54F20; Secondary 54F25.

Keywords and phrases: Aposyndesis, Mutual aposyndesis, Products, Solenoids.
**Introduction**

A *continuum* is a compact connected metric space. A *map* is a continuous function. A continuum $X$ is said to be *mutually aposyndetic* provided that for any two distinct points $x$ and $y$ in $X$ there exist two disjoint subcontinua $L$ and $M$ of $X$ such that $x \in \text{int}_X(L)$ and $y \in \text{int}_X(M)$. A continuum $X$ is said to be *strictly non-mutually aposyndetic* if each pair of subcontinua of $X$ which have interiors intersect. Clearly, a nondegenerate mutually aposyndetic continuum is not strictly non-mutually aposyndetic.

The concept of mutual aposyndesis was introduced by Charles L. Hagopian in [1] where he proved that, the product of two chainable continua is strictly non-mutually aposyndetic if and only if each factor is indecomposable. He also asked the question ([1, p. 622]): Is the topological product of two indecomposable compact metric continua strictly non-mutually aposyndetic?

In this paper we answer Hagopian's question in the negative by showing that the product $S_p \times S_q$ is mutually aposyndetic, where for an integer $m \geq 2$, $S_m$ is the $m$-adic solenoid and $p$ and $q$ are relative prime.

For a discussion on the relationship between aposyndesis and products we refer the interested reader to the paper by Leland E. Rogers ([3]).

1. **$S_p \times S_q$ is mutually aposyndetic.**

For each $p = 2, 3, \ldots$, let $f^p : S^1 \to S^1$ be given by $f^p(z) = z^p$ for each $z \in S^1$ (where $S^1$ is the unit circle in the plane, and $z^p$ denotes the $p$th power of $z$ using complex multiplication). For a given $p$, let

$$ S_p = \lim\inf\{X_n, f_n\}_{n=1}^\infty, $$

where each $X_n = S^1$ and each $f_n = f^p$. 
As usual $S_p$ is called the \textit{$p$-adic solenoid}.

**Theorem** If $p, q \geq 2$ are relative prime integers, then $S_p \times S_q$ is mutually aposyndetic.

**Proof:** We consider $S_p$ with the usual group structure, where the product of two elements $u = (u_1, u_2, \ldots)$ and $v = (v_1, v_2, \ldots)$ in $S_p$ is defined by $u * v = (u_1v_1, u_2v_2, \ldots)$ and $u_n v_n$ is the product of $u_n$ and $v_n$ as complex numbers.

We consider the exponential map $e : E^1 \rightarrow S^1$ given by $e(t) = (\cos(t), \sin(t))$. We also consider in $S^1$ the metric $D$ defined by $D(z, w) =$ the length of the shortest subarc of $S^1$ which joins $z$ and $w$. Given $z \in S^1$ and $\epsilon > 0$, define $N(\epsilon, z) = \{w \in S^1 : D(z, w) \leq \epsilon\}$. Define $g_p : E^1 \rightarrow S_p$ by:

$$g_p(t) = (e(t), e(-\frac{t}{p^2}), e(-\frac{t}{p^3}), \ldots)$$

Given two points $z, w \in S^1$, define:

$$T(z, w) = \{(a * g_p(t), b * g_q(t)) \in S_p \times S_q : t \in E^1, a \in \rho_1^{-1}(z) \text{ and } b \in r_1^{-1}(w)\},$$

where, for each $n$, $\rho_n$ (resp., $r_n$) is the $n$-th projection from the solenoid $S_p$ (resp., $S_q$) into $S^1$.

We will prove that:

$$T(z, w) \text{ is a subcontinuum of } S_p \times S_q \ldots (1)$$

In order to prove that $T(z, w)$ is compact it is enough to show that $T(z, w)$ is the image $A$ of the compact set $\rho_1^{-1}(z) \times r_1^{-1}(w) \times [0, 2\pi]$ under the continuous function $F((a, b, t)) = (a * g_p(t), b * g_q(t))$.

It is clear that $A \subset T(z, w)$. For proving the other inclusion, take an element $\alpha = (a * g_p(t), b * g_q(t)) \in T(z, w)$, with
Let \( t \in E^1 \), \( \rho_1(a) = z \) and \( r_1(b) = w \). Let \( k \) be an integer and let \( s \in [0, 2\pi) \) be such that \( t = s + 2k\pi \). Then \( g_p(t) = g_p(s + 2k\pi) = g_p(s) * g_p(2k\pi) \) and \( g_q(t) = g_q(s + 2k\pi) = g_q(s) * g_q(2k\pi) \), Since \( \rho_1(a * g_p(2k\pi)) = z \) and \( r_1(b * g_q(2k\pi)) = w \), we conclude that \( \alpha \in \mathcal{A} \). This completes the proof of the compactness of \( T(z, w) \).

Now, we will prove that \( T(z, w) \) is connected.

Let \( s_0, t_0 \in [0, 2\pi) \) be real numbers such that \( e(s_0) = z \) and \( e(t_0) = w \). Then the set \( G = \{(g_p(s_0) * g_p(t), g_q(t_0) * g_q(t)) \in T(z, w) : t \in E^1 \} \) is a connected subset of \( T(z, w) \). Then, in order to show that \( T(z, w) \) is connected, it will be enough to prove that \( G \) is dense in \( T(z, w) \).

Take an element \( \alpha = (a * g_p(t), b * g_q(t)) \in T(z, w) \), with \( t \in E^1 \), \( a = (a_1, a_2, \ldots) \in \rho_1^{-1}(z) \) and \( b = (b_1, b_2, \ldots) \in r_1^{-1}(w) \) and take a basic open subset \( W = [(U_1 \times \ldots \times U_m \times S^1 \times \ldots) \cap S_p] \times [(V_1 \times \ldots \times V_m \times S^1 \times \ldots) \cap S_q] \) of \( S_p \times S_q \) containing the point \( \alpha \), where \( m \geq 2 \) and \( a_n e\left(\frac{t}{p^{n-1}}\right) \in U_n \) and \( b_n e\left(\frac{t}{q^{n-1}}\right) \in V_n \) for each \( 1 \leq n \leq m \).

Since \( a \in S_p, a_{m+1}^p = a_1 = z \), so \( a_{m+1} \) is a \( p^m \)-th root of \( z \). On the other hand, \( e\left(\frac{s_0}{p^m}\right) \) is another \( p^m \)-th root of \( z \). Then there is a \( p^n \)-th root \( x \) of 1 such that \( a_{m+1} = e\left(\frac{s_0}{p^m}\right)x \). Thus there exists \( i \in \{1, \ldots, p^m\} \), such that \( a_{m+1} = e\left(\frac{s_0}{p^m}\right)e\left(\frac{2\pi i}{p^m}\right) \). Similarly, there exists \( j \in \{1, \ldots, q^m\} \) such that \( b_{m+1} = e\left(\frac{t_0}{q^m}\right)e\left(\frac{2\pi j}{q^m}\right) \).

Since \( p^m \) and \( q^m \) are relative prime, there exists integers \( i_1 \) and \( j_1 \) such that \( i - j = i_1 p^m + j_1 q^m \). Let \( k = i - i_1 p^m = j + j_1 q^m \). Define \( \beta = (g_p(s_0) * g_p(t + 2\pi k), g_q(t_0) * g_q(t + 2\pi k)) \in G \).

For each \( 1 \leq n \leq m \),

\[
a_n e\left(\frac{t}{p^{n-1}}\right) = a_{m+1}^{p^{m-n+1}} e\left(\frac{t}{p^{n-1}}\right) = e\left(\frac{s_0}{p^m}\right) e\left(\frac{2\pi i p^{m-n+1}}{p^m}\right) e\left(\frac{t}{p^{n-1}}\right) = e\left(\frac{s_0}{p^{n-1}}\right) e\left(\frac{t + 2\pi i}{p^{n-1}}\right).
\]
On the other hand,
\[
e^\left(\frac{s_0}{p^{n-1}}\right) e^{\left(\frac{t + 2\pi k}{p^{n-1}}\right)} = e^\left(\frac{s_0}{p^{n-1}}\right) e^{\left(\frac{t + 2\pi i p^m}{p^{n-1}}\right)} = a_n e^{\left(\frac{t}{p^{n-1}}\right)}.
\]

Thus \(e^\left(\frac{s_0}{p^{n-1}}\right) e^{\left(\frac{t + 2\pi k}{p^{n-1}}\right)} = a_n e^{\left(\frac{t}{p^{n-1}}\right)}\).

Similarly, for each \(1 \leq n \leq m\), \(e^\left(\frac{t_0}{q^{n-1}}\right) e^{\left(\frac{t + 2\pi k}{q^{n-1}}\right)} = b_n e^{\left(\frac{t}{q^{n-1}}\right)}\).

This proves that \(\beta \in W \cap G\). Thus \(G\) is dense in \(T(z, w)\).

Therefore, \(T(z, w)\) is connected.

Hence, \(T(z, w)\) is a subcontinuum of \(S_p \times S_q\).

Now, we will show that there is a homeomorphism \(h : S_p \rightarrow S_p\) such that, for each \(b \in S_p - \rho_1^{-1}(1), \rho_1(b) \neq \rho_1(h(b))\).

Fix a homeomorphism \(\gamma : S^1 \rightarrow S^1\) such that \(\gamma(1) = 1, z \neq \gamma(z)\) for each \(z \in S^1 - \{1\}\) and \(D(z, \gamma(z)) < \pi\) for every \(z \in S^1\). Consider a continuous fold \(\delta : S^1 - \{-1\} \rightarrow S^1 - \{-1\}\) of the \(p\)-th root function.

Define \(h : S_p \rightarrow S_p\) by:
\[
h(b) = (\gamma(b_1), b_2 \delta(\frac{\gamma(b_1)}{b_1}), b_3 \delta(\delta(\frac{\gamma(b_1)}{b_1})), b_4 \delta(\delta(\delta(\frac{\gamma(b_1)}{b_1}))),...),
\]

where \(b_n = \rho_n(b)\).

Clearly, \(h\) has the desired properties.

We are ready to prove that \(S_p \times S_q\) is mutually aposyndetic. Let \((a, b)\) and \((c, d)\) be two distinct points of \(S_p \times S_q\).

Since \(S_p\) and \(S_q\) are topological groups, applying a translation if necessary, we may assume that \(a = (1, 1, ...) \in S_p\) and \(c = (1, 1, ...) \in S_p\). Since \(b \neq a\) or \(c \neq d\), we may also assume that \(c \neq d\). Then there exists \(n \geq 1\) such that \(c_n \neq d_n\). Since \(S_q\) is homeomorphic to \:\{(u_n, u_{n+1}, ...) \in S_q : (u_1, u_2, ...) \in S_q\}, we may assume that \(c_1 \neq d_1\), that is \(d_1 \neq 1\). Finally, if \(b_1 = d_1\), applying the homeomorphism constructed in the paragraph
above, we may assume that $b_1 \neq d_1$. Let $\epsilon = D(d_1, b_1)/3$.

Define

$$L = [\rho_1^{-1}(N(\epsilon, 1)) \times r_1^{-1}(N(\epsilon, b_1))] \cup T(1, b_1)$$

and

$$M = [\rho_1^{-1}(N(\epsilon, 1)) \times r_1^{-1}(N(\epsilon, d_1))] \cup T(1, d_1).$$

Clearly $L$ and $M$ are closed subsets of $S_p \times S_q$, $(a, b) \in Int(L)$ and $(c, d) \in Int(M)$.

Since $N(\epsilon, b_1) \cap N(\epsilon, d_1) = \emptyset$, we have $[\rho_1^{-1}(N(\epsilon, 1)) \times r_1^{-1}(N(\epsilon, b_1))]$ does not intersect $[\rho_1^{-1}(N(\epsilon, 1)) \times r_1^{-1}(N(\epsilon, d_1))]$.

If there is a point $(u \ast g_p(s), v \ast g_q(s))$ in $T(1, b_1) \cap [\rho_1^{-1}(N(\epsilon, 1)) \times r_1^{-1}(N(\epsilon, d_1))]$, with $s \in E^1$, $\rho_1(u) = 1$ and $r_1(v) = b_1$, then $e(s) = \rho_1(u \ast g_p(s)) \in N(\epsilon, 1)$ and $b_1 e(s) = r_1(v \ast g_q(s)) \in N(\epsilon, d_1)$. Since $D(e(s), 1) \leq \epsilon$, then $D(b_1 e(s), b_1) \leq \epsilon$. Thus $b_1 e(s) \in N(\epsilon, b_1) \cap N(\epsilon, d_1)$, which contradicts the choice of $\epsilon$. Therefore, $T(1, b_1)$ does not intersect $[\rho_1^{-1}(N(\epsilon, 1)) \times r_1^{-1}(N(\epsilon, d_1))]$.

Similarly, $T(1, d_1)$ does not intersect $[\rho_1^{-1}(N(\epsilon, 1)) \times r_1^{-1}(N(\epsilon, b_1))]$.

Finally, if there is a point $(u \ast g_p(s), v \ast g_q(s)) = (x \ast g_p(t), y \ast g_q(t))$ in $T(1, b_1) \cap T(1, d_1)$, where $s, t \in E^1$, $\rho_1(u) = 1 = \rho_1(x)$, $r_1(v) = b_1$ and $r_1(y) = d_1$, then $e(s) = \rho_1(u \ast g_p(s)) = \rho_1(x \ast g_p(t)) = e(t)$. Thus $b_1 e(s) = r_1(v \ast g_q(s)) = r_1(y \ast g_q(t)) = d_1 e(t)$. This implies that $b_1 = d_1$. This contradiction proves that $T(1, b_1) \cap T(1, d_1) = \emptyset$.

Therefore, $L \cap M = \emptyset$.

In order to prove that $L$ is connected, take any point $(u, v)$ in $\rho_1^{-1}(N(\epsilon, 1)) \times r_1^{-1}(N(\epsilon, b_1))$. Then $D(u_1, 1) \leq \epsilon$ and $D(v_1, b_1) \leq \epsilon$, where $u_1 = \rho_1(u)$ and $v_1 = r_1(v)$. Let $\lambda, \eta : [0, 1] \to S^1$ be maps such that $\lambda(0) = u_1, \eta(0) = v_1, \lambda(1) = 1,$
\[ \eta(1) = b_1 \text{ and } D(\lambda(t), 1), D(\lambda(t), u_1), D(\eta(t), b_1), \text{ and } D(\eta(t), v_1) \leq \epsilon \text{ for every } t \in [0, 1]. \]

Consider the continuous fold

\[ \delta_p : S^1 - \{-1\} \to S^1 - \{-1\} \text{ (resp., } \delta_q : S^1 - \{-1\} \to S^1 - \{-1\}) \]

of the \( p \)-th root (resp., \( q \)-th root) function such that \( \delta_p(1) = 1 \) (resp., \( \delta_q(1) = 1 \)).

Define \( \sigma : [0, 1] \to S_p \times S_q \) by:

\[
\sigma(t) = 
[ (\lambda(t), u_2 \delta_p(\frac{\lambda(t)}{u_1})), u_3 \delta_p(\frac{\lambda(t)}{u_1})), \ldots), 
(\eta(t), v_2 \delta_q(\frac{\eta(t)}{v_1})), v_3 \delta_q(\frac{\eta(t)}{v_1})), \ldots)].
\]

Then \( \sigma \) is continuous, \( \sigma(t) \in \rho^{-1}_1(N(\epsilon, 1)) \times \rho^{-1}_1(N(\epsilon, b_1)) \subset L \) for every \( t \in [0, 1], \sigma(0) = (u, v) \) and \( \sigma(1) \in T(1, b_1) \).

Hence \( (u, v) \) can be connected with \( T(1, b_1) \) by a connected subset of \( L \). Since \( T(1, b_1) \) is connected, we conclude that \( L \) is connected.

Similarly, \( M \) is connected.

Therefore, \( S_p \times S_q \) is mutually aposyndetic.

**Questions**

**QUESTION 1.** (C. L. Hagopian) Are there two tree-like indecomposable continua \( X \) and \( Y \) such that \( X \times Y \) is mutually aposyndetic?

**QUESTION 2.** ([2, p. 87]) If \( M \) is an indecomposable plane continuum, must the product \( M \times M \) be strictly non-mutually aposyndetic?

**QUESTION 3.** Is there an indecomposable continuum \( X \) such that \( X \times X \) is mutually aposyndetic?
References


Instituto de Matemáticas
UNAM, Cd. Universitaria
México, 04510
D.F., MEXICO

e-mail address: illanes@gauss.matem.unam.mx