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# THE DYNAMICS OF HOMEOMORPHISMS OF HEREDITARILY DECOMPOSABLE $\theta^*$-CONTINUA

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ABSTRACT. Let $M$ be a Suslinean $\theta^*$-continuum and $F$ a homeomorphism of $M$. We show that for each $x \in R(F)$(the set of all recurrent points of $F$) the $\omega$-limit set $\omega(x, F)$ contains only one minimal set of $F$ and $F$ has zero topological entropy. Furthermore we also give some results concerning homeomorphisms of hereditarily decomposable $\theta^*$-continua.

## 1. INTRODUCTION

One of the considerable studies in the theory of dynamical systems is how to recognize chaos. The topological entropy, which was introduced by R. L. Adler, A. G. Konheim and M. H. McAndrew[1] in 1965, is an effective method to measure chaoticity. In this paper we consider the theory on one-dimensional continua.

Let $M$ be a hereditarily decomposable chainable continuum and $F$ a homeomorphism of $M$. In [13, Theorem 3.2] Xiangdong Ye has shown that for each $x \in R(F)$ either $\omega(x, F)$ is

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a periodic orbit of $F$ or $(\omega(x,F),F)$ is semi-conjugate to the adding machine, furthermore when $M$ is a Suslinean chainable continuum $\omega(x,F)$ contains the unique minimal set of $F$ and the topological entropy $h(F) = 0$. This is a partial answer to the problem of Marcy Barge, namely : Does every homeomorphism of a heredetarily decomposable chainable continuum have zero topological entropy? Moreover it also supports the question of the third author [6] : Does every homeomorphism of a Suslinean continuum have zero topological entropy?

Our aim of this paper is to study the dynamics of homeomorphisms of hereditarily decomposable $\theta^*$-continua and to prove that the results above on a Suslinean chainable continuum are true for a Suslinean $\theta^*$-continuum. The strategy of the proofs essentially comes from combining the technique of [7] with the proof of [13, Theorem 3.2].

## 2. Definitions and preliminaries

In the first half of this section, we introduce some necessary definitions from the theories of continua and dynamical systems.

A *continuum* is a nonempty connected compact metric space. A *subcontinuum* is a continuum which is a subset of a space. A continuum is said to be *decomposable* if it can be written as the union of two proper subcontinua. A continuum is *hereditarily decomposable* if each nondegenerate subcontinuum is decomposable. A continuum is *Suslinean* if each collection of its disjoint nondegenerate subcontinua is at most countable. It is known that each Suslinean continuum is hereditarily decomposable.

Let $X$ and $Y$ be metric spaces and $\epsilon > 0$. Then $f_\epsilon : X \longrightarrow Y$ is called an *$\epsilon$-map* if $f_\epsilon$ is continuous and the diameter $f_\epsilon^{-1}(f_\epsilon(x)) < \epsilon$ for all $x \in X$. Let $M$ and $P$ be continua. Then $M$ is said to be *$P$-like* if for each $\epsilon > 0$, there is an $\epsilon$-map $f_\epsilon$ from $M$ onto $P$.

Let $n$ be a natural number. A *$\theta_n$-continuum* is a continuum $M$ such that no subcontinuum of $M$ separates it into more than

$n$ components. If no subcontinuum of $M$ separates it into an infinite number of components then $M$ is called a *$\theta$-continuum.* Furtheremore $M$ is *$\theta^*$-continuum* if for any subcontinuum $N$ of $M$, $N$ is a $\theta$-continuum. Note that each chainable (= arc-like) continuum is a $\theta_2$-continuum and each circle-like continuum is a $\theta_1$-continuum. In general, if a finite graph $G$ is a $\theta_n$-continuum for some natural number $n$ and a continuum $X$ is $G$-like, then each subcontinuum of $X$ is a $\theta_n$-continuum, and hence $X$ is a $\theta^*$-continuum.

Let $(X, d)$ be a compact metric space and $f$ a continuous map of $X$. We define $f^0 = id$ and inductively $f^n = f \circ f^{n-1}$ for a natural number $n$. A point $x \in X$ is a *periodic point of $f$ with period $n$* if $f^n(x) = x$ and $f^i(x) \neq x$ for $1 \leq i \leq n-1$. A point $x \in X$ is a *recurrent point of $f$* if for each $\epsilon > 0$ there is some natural number $n$ such that $d(x, f^n(x)) < \epsilon$. A point $x \in X$ is a *nonwandering point* of $f$ if for each neighborhood $V$ of $x$ there is some natural number $n$ such that $f^n(V) \cap V \neq \emptyset$. The set of periodic points, recurrent points and nonwandering points of $f$ will be denoted by $P(f)$, $R(f)$ and $\Omega(f)$ respectively. Note that $P(f^n) = P(f)$ and $R(f^n) = R(f)$ for each natural number $n$. The *orbit* $O(x, f)$ of an $x \in X$ is the set $\{f^n(x) \mid n = 0, 1, 2, \cdots\}$. A nonempty closed subset $A$ of $X$ is called a *minimal set* of $f$ if the orbit of each point of $A$ is dense in $A$. For each $x \in X$, the *$\omega$-limit set* $\omega(x, f)$ of $x$ is the set of all limit points of $O(x, f)$. It is clear that $P(f) \subset R(f) \subset \bigcup_{x \in X} \omega(x, f) \subset \Omega(f)$.

Let $X_i$ be a compact metric space and $f_i$ a continuous map of $X_i$ for $i = 1, 2$. We say that $(X_1, f_1)$ is *semi-conjugate* (or *conjugate*, respectively) to $(X_2, f_2)$ if there is a continuous map (or homeomorphism) $\phi$ from $X_1$ onto $X_2$ such that $\phi \circ f_1 = f_2 \circ \phi$. We call $\phi$ a *semi-conjugacy* (or *conjugacy*).

Let $h(f)$ be the topological entropy of a continuous map $f$ of a compact metric space $X$ and $h_\mu(f)$ the measure theoretic entoropy of a measure-preserving map $f$ of a probability space $(X, \mu)$(see [12, p.87, p.166, p.169]). We use the following result in a proof of our main theorem (see [13]):

$$h(f) = sup\{h(f\mid_{\omega(x,f)}) \mid x \in R(f)\}.$$

Let $\Sigma_g = \prod_{i=1}^{\infty} Y_{m_i}$, where $Y_{m_i} = \{0, 1, \cdots, m_i - 1\}$ ($m_i$ is some natural number, $i \geq 1$). For $\alpha = (\alpha_1, \alpha_2, \cdots)$, $\beta = (\beta_1, \beta_2, \cdots) \in \Sigma_g$, $\alpha + \beta = (\gamma_1, \gamma_2, \cdots)$ is defined by : if $\alpha_1 + \beta_1 < m_1$ then $\gamma_1 = \alpha_1 + \beta_1$ ; if $\alpha_1 + \beta_1 \geq m_1$ then $\gamma_1 = \alpha_1 + \beta_1 - m_1$ and we carry 1 to the next position. Inductively continue this procedure. Let $\delta_g : \Sigma_g \longrightarrow \Sigma_g$ be defined by $\delta_g(\gamma) = \gamma + (1, 0, 0, \cdots)$. We shall call $(\Sigma_g, \delta_g)$ a *generalized adding machine*. In particular when $m_i = 2$ for each $i \geq 1$, i.e. $\Sigma = \prod_{i=1}^{\infty}\{0, 1\}$, we shall call $(\Sigma, \delta)$ an *adding machine*. It is known that $\delta_g$ is a minimal homeomorphism of $\Sigma_g$.

The following lemma depends on X. Ye.

**Lemma 2.1.** *The topological entropy of the homeomorphism $\delta_g$ of a generalized adding machine is zero.*

**Proof:** Let $x = (x_i)$, $y = (y_i) \in \Sigma_g$, where $x_i, y_i \in Y_{m_i}$. Define a metric function $d$ on $\Sigma_g$ as follows : $d(x, y) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$, where $d_i(x_i, y_i) = 0$ if $x_i = y_i$ and $d_i(x_i, y_i) = 1$ if $x_i \neq y_i$. For any $\epsilon > 0$ there is some natural number $i_0$ such that $d(x, y) < \epsilon$ if $x_i = y_i$ for each $i \leq i_0$. Put $S = \{(z_1, z_2, \cdots, z_{i_0}, 0, 0, \cdots) \mid z_i \in Y_{m_i}, 1 \leq i \leq i_0\}$. Then for any $x = (x_1, x_2, \cdots) \in \Sigma_g$, $x' = (x_1, x_2, \cdots, x_{i_0}, 0, 0, \cdots)$ is an element of $S$. By the assumption $d(\delta_g^i(x), \delta_g^i(x')) < \epsilon$ for each $i = 1, 2, \cdots$. This implies $h(\delta_g) = 0$. $\square$

The next lemma is a simple generalization of Lemma 3.1 in [13]. For completeness we give the proof.

**Lemma 2.2.** *Let $X$, $Y$ be compact metric spaces and $F$, $G$ homeomorphisms of $X$, $Y$ respectively. Assume that $(X, F)$ is semi-conjugate to $(Y, G)$, $\phi$ is a semi-conjugacy and $A = \{y \in Y \mid Card(\phi^{-1}(y)) \geq 2\}$ is countable. If $Y$ is an uncountable minimal set of $G$ and $h(G) = 0$ then :*

(1) *$F$ has the unique minimal set.*
(2) *$h(F) = 0$.*

**Proof:** (1) This proof is a similar one to Lemma 3.1 (1) in [13].

(2) Note that if $D$ is an open set of $X$ then

$$\phi(D) = \{Y \backslash \phi(X \backslash D)\} \cup \{\phi(D) \cap \phi(X \backslash D) \cap A\}. \qquad (*)$$

Let $O(A,G) = \bigcup_{a \in A} O(a,G)$, $X_1 = X \backslash \phi^{-1}(O(A,G))$ and $Y_1 = Y \backslash O(A,G)$. Then $F_1 = F|_{X_1}$ is a map of $X_1$ and $G_1 = G|_{Y_1}$ is a map of $Y_1$. By $(*)$, $\phi_1 = \phi|_{X_1} : X_1 \longrightarrow Y_1$ is a conjugacy.

Let $\mu$ be an invariant measure for $F$. Then for any $a \in Y$, $\mu(\bigcup_{i=1}^{\infty} F^i \circ \phi^{-1}(a)) = \sum_{i=1}^{\infty} \mu(F^i \circ \phi^{-1}(a))$. Thus $\mu(\phi^{-1}(a)) = 0$. As $A$ is countable, $\mu(\phi^{-1} O(A,G)) = 0$. Therefore $h_\mu(F) = h_\mu(F|_{X_1})$. Define a measure $\nu_1$ on $Y_1$ by $\nu_1(B) = \mu(\phi_1^{-1}(B))$ for $B \subset Y_1$. Also define a measure $\nu$ on $Y$ by $\nu|_{Y_1} = \nu_1$ and $\nu(Y \backslash Y_1) = 0$. By the Variational Principle, $h_\nu(G) \leq h(G) = 0$. Therefore $h_\mu(F) = h_\mu(F|_{X_1}) = h_{\nu_1}(G|_{Y_1}) = h_\nu(G) = 0$. Then we get that $h(F) = 0$. $\square$

In the remainder of this section, we mention important properties of continua which we target. We introduce the aposyndetic set functions $T$ and $K$. Let $M$ be a continuum and $H$ a subset of $M$. Put

$$\begin{aligned} T(H) = \{x \in M \mid & \text{ if } Q \text{ is a subcontinuum of } M \text{ such that} \\ & x \in Int(Q) \text{ then } H \cap Q \neq \emptyset\} \text{ and} \\ K(H) = \{x \in M \mid & \text{ if } Q \text{ is a subcontinuum of } M \text{ such that} \\ & H \subset Int(Q) \text{ then } \{x\} \cap Q \neq \emptyset\}. \end{aligned}$$

When $A$ is a subcontinuum of $M$, $T(A)$ is a continuum. If $M$ is a $\theta$-continuum and $H$ is a subcontinuum of $M$ then $T(H) = K(H)$ [11, p.106]. The following is very useful to study the dynamics on $\theta_n$-continua.

**Theorem 2.3.** **[5, Theorem 1]** *Let $X$ be a $\theta_n$-continuum. Then $X$ admits a monotone upper semicontinuous decomposition $\mathbf{D}$ such that the elements of $\mathbf{D}$ have void interior and the quotient space $X/\mathbf{D}$ is a finite graph if and only if $Int[T(H)] = \emptyset$ for every subcontinuum $H$ with void interior. Furthermore $\mathbf{D} = \{T^{n(n+1)}(x) \mid x \in X\}$. Moreover if $F : X \longrightarrow X$ is a homeomorphism of $X$, then there is the unique homeomorphism $G : X/\mathbf{D} \longrightarrow X/\mathbf{D}$ such that $g \circ F = G \circ g$, where $g : X \longrightarrow X/\mathbf{D}$ is the decomposition map.*

In the remainder of this paper decompositions used will be of the kind described in the above theorem.

**Theorem 2.4.** **[10, Theorem 9]** *If $M$ is a hereditarily decomposable continuum, then $Int[K(H)] = \emptyset$ for every subcontinuum $H$ of $M$ with void interior.*

**Theorem 2.5.** **[4, Theorem 3]** *If $X$ is a $\theta$-continuum such that $Int[T(H)] = Int[K(H)] = \emptyset$ for each subcontinuum $H$ of $X$ with void interior, then $X$ is a $\theta_n$-continuum for some natural number $n$.*

By the above theorems if $M$ is a hereditarily decomposable $\theta^*$-continuum, then each subcontinuum of $M$ is a $\theta_n$-continuum for some natural number $n$. By Theorem 2.3, $M$ admits the monotone upper semicontinuous decomposition $\mathbf{D}$ which has the properties in Theorem 2.3.

**Lemma 2.6.** *Let $F$ be a homeomorphism of a hereditarily decomposable $\theta_n$-continuum $M$, $g$ the decomposition map from $M$ onto the quotient space $M/\mathbf{D}$ and $G$ the homeomorphism of $M/\mathbf{D}$ ($\mathbf{D} = \{T^{n(n+1)}(x) \mid x \in M\}$) such that $g \circ F = G \circ g$. Then if $P(G) \neq \emptyset$, there is some natural number $m$ such that $F^m(\mathbf{D}(x)) = \mathbf{D}(x)$ for each $x \in \Omega(F^m)$ .*

**Proof:** First we assume that $M/\mathbf{D}$ is the unit circle $S^1$ and $t \in P(G)$. There is a natural number $m$ such that $G^m(t) = t$ and $G^m$ preserves the orientation of $S^1$. Then $\Omega(G^m) = P(G^m) = P_1(G^m)$, where $P_1(G^m)$ is the set of fixed points of $G^m$. Therefore $F^m(\mathbf{D}(x)) = g^{-1} \circ G^m \circ g(\mathbf{D}(x)) = g^{-1} \circ G^m \circ g(x) = g^{-1} \circ g(x) = \mathbf{D}(x)$ for each $x \in \Omega(F^m)$. When $M/\mathbf{D}$ is not the unit circle, there are finitely many branch points $t_1, t_2, \cdots, t_k$ of $M/\mathbf{D}$. Then there is a natural number $m$ such that $G^m$ is identity on the set $\{t_1, t_2, \cdots, t_k\}$, $G^m$ maps each component of $S^1 \backslash \{t_1, t_2, \cdots, t_k\}$ onto itself and $G^m$ preserves the orientation of each component. By a similar way we can have that $F^m(\mathbf{D}(x)) = \mathbf{D}(x)$ for each $x \in \Omega(F^m)$. $\square$

## 3. Entropy of homeomorphisms of Suslinean $\theta^*$-continua

To prove the next proposition, we introduce the following notion.

By transfinite induction, we shall define $\mathbf{D}_\alpha$ for each ordinal number $\alpha < \omega_1$ as follows. Let $\mathbf{D}_0$ be $\{M\}$ . If $\alpha = \beta + 1$ then $\mathbf{D}_\alpha$ will consist of degenerate elements of $\mathbf{D}_\beta$ and the elements of the decompositions as in Theorem 2.3 of nondegenerate elements of $\mathbf{D}_\beta$. For a limit ordinal number $\alpha$ define $\mathbf{D}_\alpha$ to be the set consisting of the intersections $\bigcap_{\beta<\alpha} D_\beta$, where $D_\beta \in \mathbf{D}_\beta$. For every $x \in M$ denote by $\mathbf{D}_\alpha(x)$ the element of $\mathbf{D}_\alpha$ containing $x$. For each $x \in M$ there is a countable ordinal number $\tau = \tau_x$ such that $\mathbf{D}_\tau(x) = x$.

**Proposition 3.1.** *Let $M$ be a hereditarily decomposable $\theta^*$-continuum and $F$ be a homeomorphism of $M$. Then for each $x \in R(F)$, one of the following cases holds :*

(a) *$\omega(x, F)$ is a periodic orbit of $F$.*
(b) *$(\omega(x, F), F)$ is semi-conjugate to a generalized adding machine.*
(c) *There is a natural number $m$, a continuous map $g$ from $\omega(x, F^m)$ onto the unit circle $S^1$ and a minimal homeomorphism $G$ of $S^1$ such that $g \circ F^m = G \circ g$.*

$$\begin{tikzcd} \omega(x,F^m) \arrow[r, "F^m"] \arrow[d, "g"'] & \omega(x,F^m) \arrow[d, "g"] \\ S^1 \arrow[r, "G"'] & S^1 \end{tikzcd}$$

(d) *There is a natural number $m$, a continuous map $h$ from $\omega(x, F^m)$ onto a Cantor set $C$ and a minimal homeomorphism $H$ of $C$ such that $h \circ F^m = H \circ h$.*

$$\begin{tikzcd} \omega(x,F^m) \arrow[r, "F^m"] \arrow[d, "h"'] & \omega(x,F^m) \arrow[d, "h"] \\ C \arrow[r, "H"'] & C \end{tikzcd}$$

**Proof:** Let $x \in R(F)$. If $\omega(x, F)$ is finite then $\omega(x, F)$ is a periodic orbit of $F$.

Let $\omega(x, F)$ be infinite. By Theorem 2.3 either $F(\mathbf{D}_1(x)) = \mathbf{D}_1(x)$ or $\mathbf{D}_1(x) \cap \mathbf{D}_1(F(x)) = \emptyset$, where $\mathbf{D}_1$ is the monotone upper semicontinuous decomposition with the properties in Theorem 2.3 and $\mathbf{D}_1(x)$ is the element of $\mathbf{D}_1$ containing $x$. As $O(x, F)$ is infinite and $\mathbf{D}_\tau(x) = \{x\}$ for some countable ordinal number $\tau$, there is a countable ordinal number $\alpha_0 = min\{\alpha \mid \mathbf{D}_\alpha(x) \cap \mathbf{D}_\alpha(F(x)) = \emptyset\}$. Note that $\alpha_0$ is not a limit ordinal number.

As $\mathbf{D}_{\alpha_0 - 1}(x)$ is a hereditarily decomposable $\theta_n$-continuum, $\mathbf{D}_{\alpha_0-1}(x)/\mathbf{D}_{\alpha_0}$ is a finite graph. Let $g_0$ be the decomposition map from $\mathbf{D}_{\alpha_0-1}(x)$ onto $\mathbf{D}_{\alpha_0-1}(x)/\mathbf{D}_{\alpha_0}$ and $G_0$ the homeomorphism of $\mathbf{D}_{\alpha_0-1}(x)/\mathbf{D}_{\alpha_0}$ such that $g_0 \circ F|_{\mathbf{D}_{\alpha_0-1}(x)} = G_0 \circ g_0$(see Theorem 2.3).

If $P(G_0) = \emptyset$ then $\mathbf{D}_{\alpha_0-1}(x)/\mathbf{D}_{\alpha_0}$ is the unit circle $S^1$. Then by [2] the nonwandering set $\Omega(G_0)$ is the unique minimal set of $G_0$ such that $\Omega(G_0) = S^1$ or $\Omega(G_0)$ is homeomorphic to a Cantor set.

By the minimality of $\Omega(G_0)$ and $g_0(\omega(x, F)) \subset \Omega(G_0)$, we get that $g_0(\omega(x, F)) = \Omega(G_0)$. Thus when $\Omega(G_0) = S^1$, $G_0$ is a minimal homeomorphism of $S^1$ and $g_0$ is a continuous map from $\omega(x, F)$ onto $S^1$ such that $g_0 \circ F|_{\omega(x,F)} = G_0 \circ g_0$. This implies the case of $(c)$.

Assume that $\Omega(G_0)$ is not $S^1$. Then there are a Cantor set $C$ which is homeomorphic to $\Omega(G_0)$, a continuous map $h_0$ from $\omega(x, F)$ onto $C$ and a minimal homeomorphism $H_0$ of $C$ with $h_0 \circ F|_{\omega(x,F)} = H_0 \circ h_0$. This is the case of $(d)$.

If $P(G_0) \neq \emptyset$ then there is a natural number $m_0$ such that $F^{m_0}(\mathbf{D}_{\alpha_0}(x)) = \mathbf{D}_{\alpha_0}(x)$ and $\mathbf{D}_{\alpha_0}(F^i(x)) \cap \mathbf{D}_{\alpha_0}(F^j(x)) = \emptyset$ $(0 \leq i \neq j \leq m_0 - 1)$ (see Lemma 2.6). Let $\mathbf{D}_{\alpha_0}(F^i(x))$ be $M_i$ for each $i = 0, 1, \cdots, m_0 - 1$. Note that $M_i$ is a hereditarily decomposable $\theta_{n_0}$-continuum for some natural number $n_0$ and $F^{m_0}|_{M_i}$ is a homeomorphism of $M_i$. Then there is a countable ordinal number $\alpha_1 = min\{\alpha \mid \mathbf{D}_\alpha(x) \cap \mathbf{D}_\alpha(F^{m_0}(x)) = \emptyset\}$. Let $g_1$ be the decomposition map from $\mathbf{D}_{\alpha_1-1}(x)$ onto

$\mathbf{D}_{\alpha_1-1}(x)/\mathbf{D}_{\alpha_1}$ and $G_1$ the homeomorphism of $\mathbf{D}_{\alpha_1-1}(x)/\mathbf{D}_{\alpha_1}$ such that $g_1 \circ F^{m_0}|_{\mathbf{D}_{\alpha_1-1}(x)} = G_1 \circ g_1$.

If $P(G_1) = \emptyset$ then $\mathbf{D}_{\alpha_1-1}(x)/\mathbf{D}_{\alpha_1} = S^1$. This implies the case $(c)$ or $(d)$ holds.

Let $P(G_1) \neq \emptyset$. As $R(F^n) = R(F)$ for each natural number $n$, $x \in R(F) = R(F^{m_0})$. There is a natural number $m_1$ such that $F^{m_0 m_1}(\mathbf{D}_{\alpha_1}(x)) = \mathbf{D}_{\alpha_1}(x)$ and $\mathbf{D}_{\alpha_1}(F^i(x)) \cap \mathbf{D}_{\alpha_1}(F^j(x)) = \emptyset$ $(0 \leq i \neq j \leq m_0 m_1 - 1)$. Let $\mathbf{D}_{\alpha_1}(F^{i_0+m_0 i_1}(x))$ be $M_{i_0 i_1}$ $(0 \leq i_0 \leq m_0 - 1, 0 \leq i_1 \leq m_1 - 1)$. Note that $M_{i_0 i_1}$ is a hereditarily decomposition $\theta_{n_1}$-continuum for some natural number $n_1$ and $F^{m_0 m_1}|_{M_{i_0 i_1}}$ is a homeomorphism of $M_{i_0 i_1}$. Then there is $\alpha_2 = min\{\alpha \mid \mathbf{D}_\alpha(x) \cap \mathbf{D}_\alpha(F^{m_0 m_1}(x)) = \emptyset\}$.

Let $g_2$ be the decomposition map from $\mathbf{D}_{\alpha_2-1}(x)$ onto $\mathbf{D}_{\alpha_2-1}(x)/\mathbf{D}_{\alpha_2}$ and $G_2$ the homeomorphism of $\mathbf{D}_{\alpha_2-1}(x)/\mathbf{D}_{\alpha_2}$ such that $g_2 \circ F^{m_0 m_1}|_{\mathbf{D}_{\alpha_2-1}(x)} = G_2 \circ g_2$.

If $P(G_2) = \emptyset$ then by the same procedure as the above $g_2$ is a continuous map from $\omega(x, F^{m_0 m_1})$ onto $S^1$ and $G_2$ is a minimal homeomorphism of $S^1$ such that $g_2 \circ F^{m_0 m_1} = G_2 \circ g_2$ or there are a continuous map $h_2$ from $\omega(x, F^{m_0 m_1})$ onto a Cantor set $C_2$ and a minimal homeomorphism $H_2$ of $C_2$ such that $h_2 \circ F^{m_0 m_1} = H_2 \circ h_2$, that is to say, $(c)$ or $(d)$ holds.

If $P(G_2) \neq \emptyset$ then there is a natural number $m_2$ such that $F^{m_0 m_1 m_2}(\mathbf{D}_{\alpha_2}(x)) = \mathbf{D}_{\alpha_2}(x)$ and $\mathbf{D}_{\alpha_2}(F^i(x)) \cap \mathbf{D}_{\alpha_2}(F^j(x)) = \emptyset$ $(0 \leq i \neq j \leq m_0 m_1 m_2 - 1)$.

We may get hereditarily decomposable $\theta_{n_2}$-continua $M_{i_0 i_1 i_2}$ for some natural number $n_2$ and $F^{m_0 m_1 m_2}|_{M_{i_0 i_1 i_2}}$ is a homeomorphism of $M_{i_0 i_1 i_2}$ $(0 \leq i_0 \leq m_0, 0 \leq i_1 \leq m_1, 0 \leq i_2 \leq m_2)$.

Continue this procedure. If $P(G_j) = \emptyset$, where $G_j$ is the homeomorphism of $\mathbf{D}_{\alpha_j-1}(x)/\mathbf{D}_{\alpha_j}$, then $(c)$ or $(d)$ holds.

We may assume that $P(G_j) \neq \emptyset$ for $j = 1, 2, \cdots$. Then we get hereditarily decomposable $\theta_{n_j}$-continua $M_{i_0 i_1 \cdots i_j}$ such that $M_{i_0 i_1 \cdots i_j} \subset M_{i_0 i_1 \cdots i_{j-1}}$ and $F^{m_{i_0 i_1 \cdots i_j}}|_{M_{i_0 i_1 \cdots i_j}}$ is a homeomorphism of $M_{i_0 i_1 \cdots i_j}$ for each $j$.

Let $M_a = \bigcap_{j=0}^{\infty} M_{i_0 i_1 \cdots i_j}$ for each $a = (i_0, i_1, \cdots) \in \Sigma_g = \prod_{i=0}^{\infty} Y_{m_i}$, where $Y_{m_i} = \{0, 1, \cdots, m_i - 1\}$. Note that $M_a$ is a subcontinuum of $M$ for each $a \in \Sigma_g$.

Define $\phi : \bigcup_{a \in \Sigma_g} M_a \longrightarrow \Sigma_g$ by $\phi(M_a) = a$. Since $\omega(x, F) \subset \bigcup_{a \in \Sigma_g} M_a$ and $\omega(x, F) \cap M_a \neq \emptyset$ for each $a \in \Sigma_g$, $(\omega(x, F), F)$ is semi-conjugate to $(\Sigma_g, \delta_g)$. This is the case of $(b)$. This ends the proof. □

The next is our main theorem.

**Theorem 3.2.** *Let $F$ be a homeomorphism of a Suslinean $\theta^*$-continuum $M$. Then :*

(1) *$\omega(x, F)$ has the unique minimal set for each $x \in R(F)$.*
(2) *$h(F) = 0$.*

**Proof:** As $h(F) = sup\{h(F\mid_{\omega(x,F)}) \mid x \in R(F)\}$ (see Section 2), we need only prove $h(F\mid_{\omega(x,F)}) = 0$ for $x \in R(F)$. We shall use Proposition 3.1 and its proof.

Let $x \in R(F)$. In the case of $(a)$, it is clear that $\omega(x, F)$ is the unique minimal set of $F$ and $h(F\mid_{\omega(x,F)}) = 0$.

In the case of $(b)$, $(\omega(x, F), F)$ is semi-conjugate to a generalized adding machine $(\Sigma_g, \delta_g)$. As $M$ is Suslinean, $A = \{a \in \Sigma_g \mid Card(\phi^{-1}(a)) \geq 2\}$ is countable, where $\phi$ is a continuous map from $\omega(x, F)$ onto $\Sigma_g$ with $\phi \circ F = \delta_g \circ \phi$. By Lemma 2.2 $\omega(x, F)$ contains only one minimal set of $F$ and $h(F|_{\omega(x,F)}) = 0$.

In the case of $(c)$, there are a natural number $m$, a continuous map $g_i$ from $\omega(F^i(x), F^m)$ onto $S^1$ and a minimal homeomorphism $G_i$ of $S^1$ such that $g_i \circ F^m = G_i \circ g_i$ for each $i = 0, 1, \cdots, m-1$. Note that $h(G_i) = 0$(see [1]). By Lemma 2.2 $\omega(F^i(x), F^m)$ contains only one minimal set of $F^m$ and $h(F^m\mid_{\omega(F^i(x),F^m)}) = 0$. As $\omega(x, F) = \bigcup_{i=0}^{m-1} \omega(F^i(x), F^m)$, $\omega(x, F)$ contains only one minimal set of $F$. Moreover $mh(F\mid_{\omega(x,F)}) = h(F^m\mid_{\omega(x,F)}) \leq max\{h(F^m\mid_{\omega(F^i(x),F^m)}) \mid i = 0, 1, \cdots, m-1\} = 0$. Hence $h(F\mid_{\omega(x,F)}) = 0$.

In the case of $(d)$ we can see $\omega(x, F)$ contains only one minimal set of $F$ and $h(F\,|_{\omega(x,F)}) = 0$ by a similar way to the case of $(c)$.

Therefore $h(F) = sup\{h(F\,|_{\omega(x,F)}) \mid x \in R(F)\} = 0$. □

**Note.** After the authors finished their work on this paper they learned from Xiangdong Ye that Jie Lühskip, Jincheng Xiong and Xiangdong Ye have obtained the same result as Theorem 3.2.

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