THE SAMUEL COMPACTIFICATION OF A QUASI-UNIFORM FRAME

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Abstract

This paper presents a new description of the completion of a quasi-uniform frame. The completion is used to extend the notion of the Samuel compactification to quasi-uniform frames.

1. Introduction

In [10, Theorem 3.4] it was proved that each quasi-uniform frame has a unique completion. The proof of that Theorem used the fact that each uniform frame has a unique completion. In this note we construct the completion of a quasi-uniform frame without assuming the existence of the completion of a uniform frame. As a consequence, we obtain the completion of a uniform frame as a special case of the completion of a quasi-uniform frame. This is analogous to the presentation of the bicompletion of a quasi-uniform space given in [8, p. 60, ff]. We utilize a frame constructed by B. Banaschewski and A. Pultr [3] to carry the completion of the given quasi-uniform frame.

In the final section we show that the category of totally bounded quasi-uniform frames forms a coreflective subcategory of the category of quasi-uniform frames. We extend the notion of Samuel compactification to quasi-uniform frames.

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2. Preliminaries

A frame \((L, \leq)\) is a complete lattice that satisfies the frame distributive law: for any \(x \in L\) and any \(A \subseteq L\),

\[ x \land \bigvee A = \bigvee \{ x \land a : a \in A \}. \]

A function between frames is a join homomorphism provided that it preserves arbitrary joins. A join homomorphism that preserves finite meets is called a frame homomorphism. Let \(f\) and \(g\) be order preserving functions from \(L\) to \(L\). We write \(f \leq g\) provided that \(f(a) \leq g(a)\) for each \(a \in L\). This obviously defines a partial order on the set of all order preserving functions from \(L\) to \(L\). In any frame \(L\), we use 0 to denote \(\bigvee \varnothing\) and \(e\) to denote \(\bigvee L\). A function \(f\) between frames with the property that \(f(a) = 0\) implies that \(a = 0\) is said to be dense. Let \(L\) and \(M\) be frames and let \(h : L \to M\) be a frame homomorphism. The function \(h_* : M \to L\) defined by \(h_*(m) = \bigvee \{ l \in L : h(l) \leq m \}\), \(m \in M\), is the right adjoint of \(h\). If \(h\) maps onto \(M\), then \(hh_* = id_M\). For further information on frames the reader is referred to [11].

Let \(L\) be a frame and let \(a, b \in L\). The function \(a \# b : L \to L\) is defined in [7] by:

\[ a \# b(x) = \begin{cases} b & \text{if } a \land x \neq 0 \\ 0 & \text{otherwise.} \end{cases} \]

The following proposition is proved in [10] and will be used in section 3.

**Proposition 2.1.** Let \(L\) and \(M\) be frames and let \(h : L \to M\) be a dense frame homomorphism from \(L\) onto \(M\). If \(a \# b \leq \bigvee \{ x_\alpha \# y_\alpha : \alpha \in A \}\) then \(h(a) \# h(b) \leq \bigvee \{ h(x_\alpha) \# h(y_\alpha) : \alpha \in A \}\).

The following definitions and notations are taken from [4], [5], [6], [7], and [10]. Let \(L\) be a frame, let \(x \in L\) and let \(u : L \to L\). Then \(x\) is \(u\)-small provided that \(x \# x \leq u\). The set of all
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u-small elements of \( L \) is denoted by \( S_u \). If \( u \) is an order preserving function and \( \bigvee S_u = e \), then \( u \) is said to be a \( \Delta \)-map. A quasi-uniform base on \( L \) is a nonempty collection \( B \) of \( \Delta \)-maps that satisfy:

1. For each \( u \in B \) there exists \( v \in B \) such that \( v \circ v \leq u \), and

2. For \( u, v \in B \) there is a join homomorphism \( w \) and a \( z \in B \) such that \( z \leq w \leq u \wedge v \).

The frame quasi-uniformity \( U \) for which \( B \) is a base is the collection of all order preserving functions \( w : L \to L \) for which there exists a \( u \in B \) with \( u \leq w \). The frame of \( U \), \( Fr(U) \), is the subframe of \( L \) to which \( a \) belongs provided that \( a = \bigvee \{ b \in L : u(b) \leq a \text{ for some } u \in U \} \). Let \( u : L \to L \) be a function. Then \( u \) is symmetric provided that for each \( x, y \in L \), \( u(x) \wedge y = 0 \) if and only if \( u(y) \wedge x = 0 \). A quasi-uniformity \( U \) on a frame \( L \) is a uniformity \([4]\) provided that \( U \) has a base of symmetric entourages and \( Fr(U) = L \). The pair \( (L, U) \) is called a uniform frame. If \( (L, U) \) is a uniform frame, then \( \{ S_u : u \in U \} \) is a base for a covering uniformity which is denoted by \( \mu_U \)[4].

Let \( U \) be a quasi-uniformity on a frame \( L \) and let \( u \in U \). Then \( u^* \) is defined by \( u^*(x) = \bigvee \{ c \in S_u : c \wedge x \neq 0 \} = S_u x \). The uniformity \( U^* = \{ u^* : u \in U \} \) is the coarsest frame uniformity containing \( U \). Equivalently, \([4]\) \( \{ S_u : u \in U \} \) is a base for \( U^* \) when considered as a covering uniformity. We shall use the fact that for any \( x \in L \), and any \( u \in U \), \( x \) is \( u^* \)-small if and only if \( x \) is \( u \)-small \([7, \text{Proposition 2.1(1)}]\). If \( U \) is a quasi-uniformity on \( L \) and \( Fr(U^*) = L \), then \( (L, U) \) is said to be a quasi-uniform frame \([5]\). Let \( (L, U) \) be a quasi-uniform frame and let \( a, b \in L \). We write \( a \triangleleft b \) if \( u^*(a) \leq b \) for some \( u^* \in U^* \).

Let \( L \) and \( M \) be frames and let \( U \) and \( V \) be (quasi-) uniformities on \( L \) and \( M \) respectively. Let \( f : L \to M \) be a frame homomorphism, and for \( u \in U \), let \( u_f = \bigvee \{ f(x) : x \uparrow y \leq u \} \). Then \( f \) is a (quasi-) uniform frame homomorphism if and only if
$u_f \in V$ whenever $u \in U$. Let $(L, U)$ and $(M, V)$ be (quasi-) uniform frames, and let $f$ be a (quasi-) uniform frame homorphism of $L$ onto $M$. Then $f$ is a (quasi-) uniform surjection provided that $\{u_f : u \in U\}$ is a base for $V$. There is an equivalent formulation of the concept of a quasi-uniform frame homomorphism that will be used in §4: for each $u \in U$ there exists $v \in V$ such that $v \circ f \leq f \circ u$.

The following proposition is proved in [10] and will be used in describing the completion of a quasi-uniform frame in §3. Let $L$ be a frame and let $u : L \to L$. Then $u^- = \bigvee \{a \# b : a \# b \leq u\}$.

**Proposition 2.2.** Let $(L, U)$ be a quasi-uniform frame. If $u, v \in U$ and $v^3 \leq u$ then $v \leq u^- \leq u$. Consequently, $\{u^- : u \in U\}$ is a base for $U$.

**Proposition 2.3.** [10, Proposition 2.6] Let $g : L \to M$ be a dense frame homomorphism of $L$ onto $M$. Let $U$ be a quasi-uniformity on $M$. Then $(u_g)_g = u^-$ for $u \in U$.

**Proposition 2.4.** [10, Proposition 2.3] Let $f : (M, V) \to (L, U)$ be a quasi-uniform surjection. Then $f^* : (M, V^*) \to (L, U^*)$ is a uniform surjection where $f^*(x) = f(x)$ for $x \in M$.

The reader is invited to compare the following proposition with Proposition 2.2 of [10].

**Proposition 2.5.** Let $(L, U)$ and $(M, V)$ be uniform frames and let $f : (L, U) \to (M, V)$ be a uniform surjection. Then $f : (L, \mu_U) \to (M, \mu_V)$ is a uniform surjection, where $\mu_U(\mu_V)$ is the covering uniformity associated with $U(V)$.

**Definition.** A quasi-uniform frame $(L, U)$ is complete if every dense quasi-uniform surjection $(M, V) \to (L, U)$ is an isomorphism.

This is equivalent to the statement that $(L, U^*)$ is a complete uniform frame [10, Proposition 3.3].
Definition. Let \((M, V)\) and \((L, U)\) be quasi-uniform frames with \((M, V)\) complete. Then \((M, V)\) is a completion of \((L, U)\) provided that there is a dense quasi-uniform surjection \((M, V) \to (L, U)\).

In [10, Theorem 3.4] it was proved that each quasi-uniform frame has a unique completion.

3. The Completion of a Quasi-Uniform Frame

Let \((L, U)\) be a quasi-uniform frame. The frame \(CL\) described below will carry the completion of \((L, U)\). The description is due to B. Banaschewski and A. Pultr [3, §4]. Recall that a nonempty subset \(A\) of \(L\) is called a downset provided that if \(x \leq y\) and \(y \in A\) then \(x \in A\). Let \(DL\) be the frame of all downsets of \(L\) ordered by inclusion. For any \(a \in L\), let \(k(a) = \{x \in L : x \triangleleft a\}\).

Banaschewski and Pultr utilized the quotient of \(DL\) determined by the prenucleus \(l_0 : DL \to DL\), where

\[
l_0(A) = \{a \in L : k(a) \subseteq A\} \cup \{a \in L : a \land S_u \subseteq A \text{ for some } u \in U\}.
\]

This quotient is denoted by \(CL\). (Recall that \(A \in CL\) provided that \(A = l_0(A)\).) B. Banaschewski and A. Pultr [2] show that \(CL\), with set inclusion as the partial order, is a frame.

In \(CL\), meet is intersection and for \(C_\alpha \in CL\),

\[
\bigwedge_{\alpha} C_\alpha = l\left(\bigcup_{\alpha} C_\alpha\right),
\]

where \(l\) is the nucleus on \(DL\) defined by \(l(A) = \bigwedge\{B \in DL : l_0(B) = B, A \subset B\}\) for \(A \in DL\) [3].

Let \(CL \to L\) be the join map. Then the right adjoint \(r : L \to CL\) of the join is given by \(r(a) = \downarrow a\), where \(\downarrow a = \{x \in L : x \leq a\}\) for \(a \in L\). We shall make use of the following formulas which are proved in [3]. For any \(a \in L\) and any \(V \in CL\),

\[
r(a) = \bigvee\{r(x) : x \triangleleft a\}
\]
and
\[ V = \bigvee \{ r(a) : a \in V \}. \]

**Lemma 3.1.** Let \((L, U)\) be a quasi-uniform frame, and let \(r\) be the right adjoint of the frame homomorphism \(CL \to L\) given by the join map. Then \(\{ u_r : u \in U \}\) is a base for a quasi-uniformity \(CU\) on \(CL\) and \((CL, CU)\) is a quasi-uniform frame.

**Proof.** Clearly each \(u_r\) for \(u \in U\) is nondecreasing. To show that \(u_r\) is a \(\Delta\)-map we must show that the \(u_r\)-smalls cover \(CL\).

We show that \(\bigvee S_{u_r} = \downarrow e\). By definition, \(\bigvee S_{u_r} = l \left( \bigcup \{ c : c \in S_{u_r} \} \right)\). It follows from the definition of \(u_r\) that if \(x\) is \(u\)-small, then \(r(x)\) is \(u_r\)-small. Since \(r(x) = \downarrow x\), it suffices to show that \(\bigvee \downarrow x = \downarrow e\). Note that \(e \land S_u = S_u \subseteq \bigcup_{x \in S_u} \downarrow x\). By the definition of \(l_0\), we have \(e \in l_0 \left( \bigcup_{x \in S_u} \downarrow x \right)\). But \(l_0 \left( \bigcup_{x \in S_u} \downarrow x \right) \subseteq l \left( \bigcup_{x \in S_u} \downarrow x \right)\). Consequently \(\downarrow e = l \left( \bigcup_{x \in S_u} (\downarrow x) \right) = \bigvee \downarrow x\).

This shows that each \(u_r\) is a \(\Delta\)-map. Let \(v \in U\) and choose \(u \in U\) such that \(u^2 \leq v\). It can be verified that \(u^2 \leq v_r\). Also, if \(u, v, w \in U\) are such that \(u \leq v \land w\), then \(u_r \leq v_r \land w_r\). This proves that \(\{ u_r : u \in U \}\) is a base for a frame quasi-uniformity which we denote by \(CU\).

For \(V, W \in CL\), we write \(W \triangleleft V\) whenever \(S_{u_r} W \subseteq V\) for some \(u \in U\). Let \(V \in CL\). We show that \(V = \bigvee \{ W \in CL : W \triangleleft V \}\). We have previously commented that \(V = \bigvee \{ r(a) : a \in V \}\). Consequently, \(V = \bigvee \{ r(x) : x \triangleleft a, a \in V \\leq \bigvee \{ r(x) : r(x) \triangleleft r(a) \leq V, a \in V \}\). Since each \(r(x) \in CL\), we have that \(V \leq \bigvee \{ W \in CL : W \triangleleft V \\leq V\). Therefore \((CL, CU)\) is a quasi-uniform frame.

The following lemma is based upon the proof of [3, Proposition 8].
Lemma 3.2. Let \((M, V)\) and \((L, U)\) be quasi-uniform frames. Let \(h : (M, V) \to (L, U)\) be a dense quasi-uniform surjection. Then there exists a dense quasi-uniform surjection \(g : (CL, CU) \to (M, V)\) such that \(hg : CL \to L\) is the join map.

Proof. Let \(h : (M, V) \to (L, U)\) be a dense quasi-uniform surjection. Define \(f : DL \to M\) by \(f(A) = \bigvee h^*(A)\). In the proof of Proposition 8 [3], it is established that: (1) \(f\) is a frame homomorphism, (2) \(f\) determines a frame homomorphism \(g : CL \to M\) such that \(f = gl\), (3) \(hg : CL \to L\) is the join map, (4) \(g\) is dense and onto, (5) \(h_*(x) = f(\downarrow x)\), and (6) \(l(\downarrow x) = \downarrow x\).

We now show that \(g\) is a quasi-uniform frame homomorphism. Let \(u_r \in CU\), where \(u \in U\). Since \(h\) is a quasi-uniform surjection, we may assume that \(u = v_h\) for some \(v \in V\). We show that \(v^- \leq (u_r)_g\). Suppose that \(a^{\#}b \leq v\). Then \(h(a)^{\#}h(b) \leq u\) and consequently \(rh(a)^{\#}rh(b) \leq u_r\). From this we have that \(grh(a)^{\#}grh(b) \leq (u_r)_g\). In the proof of Proposition 8 [3] it is established that \(gr = h_*\). Let \(x \in M\). Then \(x \leq h_*h(x) = grh(x)\). Consequently

\[
v^- = \bigvee \{a^{\#}b : a^{\#}b \leq v\} \leq \bigvee \{grh(a)^{\#}grh(b) : a^{\#}b \leq v\}
\leq (u_r)_g.
\]

We now show that \(g : (CL, CU) \to (M, V)\) is a quasi-uniform surjection. By [10, Proposition 3.1] it suffices to show that \(\{v_{g_*} : v \in V\}\) is a base for \(CU\). Let \(v \in V\). By [10, Proposition 3.1], we may assume that \(v = u_{h_*}\). Then

\[
v_{g_*} = (u_{h_*})_{g_*} = \bigvee \{g_*^*(a) : a^{\#}b \leq u_{h_*}\}
\geq \bigvee \{g_*h_*(x) : x^{\#}y \leq u\}
= \bigvee \{g_*f(\downarrow x) : x^{\#}y \leq u\}
= \bigvee \{g_*gl(\downarrow x) : x^{\#}y \leq u\}
= \bigvee \{g_*g(\downarrow x) : x^{\#}y \leq u\}
\geq \{\downarrow x^{\#}y : x^{\#}y \leq u\}
= \bigvee \{r(x)^{\#}r(y) : x^{\#}y \leq u\} = u_r.
\]

Therefore \(v_{g_*} \in CU\). It remains to show that \(\{v_{g_*} : v \in V\}\) is a
base for \( C \). Let \( A \in CL \), and let \( v \in V \). Then

\[
v_g(A) = \bigvee \{ g_*(b) : a \leq b \leq v, A \land g_*(a) \neq 0 \} = \bigvee \{ g_*(b) : a \leq b \leq v, a \land g(A) \neq 0 \}
\]

If \( a \land g(A) \neq 0 \) and \( a \leq b \leq v \), then \( b \leq v(g(A)) \) and hence \( g_*(b) \leq g_*v_g(A) \). Consequently \( v_g(A) \leq g_*v_g(A) \), or \( v_g \leq g_*v_g \).

The proof will be completed by showing that \( \{ g_*v_g : v \in V \} \) is a base for \( C \). Let \( u, w \in C \) be such that \( w \leq u \). In the proof of Theorem 2.7 [10] it is proved that \( g_*w_g \leq u \). Since \( w_g \in V \), this completes the proof. 

It follows from the definition of a complete quasi-uniform frame that the function \( g \) in the statement of Lemma 3.2 is an isomorphism whenever \( M \) is complete. It also follows from Lemma 3.2 that if \( (L, U) \) has a completion, then the completion is unique up to isomorphism.

**Theorem 3.3.** Let \( (L, U) \) be a quasi-uniform frame. Then \( (CL, CU) \) is the completion of \( (L, U) \).

**Proof.** Let \( (M, V) \) be a quasi-uniform frame, and let \( f : (M, V) \to (CL, CU) \) be a dense quasi-uniform surjection. Then \( \bigvee \circ f : (M, V) \to (L, U) \) is a dense surjection. By Lemma 3.2 there exists a dense surjection \( g : (CL, CU) \to (M, V) \) such that \( \bigvee \circ f \circ g = \bigvee \). Dense maps are monic for maps between regular frames. Consequently \( f \circ g = id_{CL} \). Also \( (\bigvee \circ f) \circ g \circ f = \bigvee \circ f \). But \( \bigvee \circ f \) is dense and hence monic. Therefore \( g \circ f = id_M \). We have that \( f \) is an isomorphism and consequently \( (CL, CU) \) is complete. It follows from Lemma 3.2 that \( (CL, CU) \) is a completion of \( (L, U) \). By the remarks following Lemma 3.2 we have that each quasi-uniform frame has a unique completion. \( \square \)

**Corollary 3.4.** Each uniform frame has a unique completion.

**Proof** Let \( (L, U) \) be a uniform frame. Let \( (CL, CU) \) be the quasi-uniform completion of \( (L, U) \). Then \( (CL, (CU)^*) \) is a
complete uniform frame [10, Proposition 3.3]. Let 
\( \phi_L : (CL, CU) \to (L, U) \) be a dense quasi-uniform surjection. Then, by [10, Proposition 2.3], \( \phi^*_L : (CL, (CU)^*) \to (L, U) \) is a dense uniform surjection. It follows that \( (CL, (CU)^*) \) is a completion of \( (L, U) \). But the quasi-uniform completion is unique and consequently \( (CL, CU) \) is isomorphic to \( (CL, (CU)^*) \). It follows that \( CU \) is a frame uniformity. Furthermore, any uniform completion of \( (L, U) \) is a quasi-uniform completion of \( (L, U) \). Therefore the uniform completion is unique. \( \square \)

**Corollary 3.5.** Let \( (L, U) \) be a quasi-uniform frame and let \( (CL, CU) \) be the completion of \( (L, U) \). Then \( (CU)^* = C(U^*) \).

**Proof.** \( (CL, (CU)^*) \) is a complete uniform frame. If \( h : (CL, CU) \to (L, U) \) is a dense quasi-uniform surjection, then \( h : (CL, (CU)^*) \to (L, U^*) \) is a dense uniform surjection. By definition, \( (CL, CU^*) \) is the completion of \( (L, U^*) \) which implies that \( CU^* = (CU)^* \). \( \square \)

**Corollary 3.6.** Let \( (L, U) \) be a uniform frame. Then the quasi-uniform completion, \( (CL, CU) \), is the uniform completion of \( (L, U) \).

### 4. The Totally Bounded Coreflection

Recall [5] that a quasi-uniform frame \( (L, U) \) is totally bounded provided that for each \( u \in U \) there is a finite cover of \( L \) by \( u \)-small elements. The following theorem is taken from [5, Theorem 2.3].

**Theorem 4.1.** Let \( L \) be a frame and let \( \triangleleft \) be a quasi-proximity inclusion on \( L \). For \( a, b \in L \) define

\[
u_{a,b}(x) = \begin{cases} 
0 & \text{if } x = 0 \\
b & \text{if } x \leq a, \ x \neq 0 \\
1 & \text{otherwise}
\end{cases}
\]
Then \( \{ u_{a,b} : a \lhd b \} \) is a subbase for the unique totally bounded frame quasi-uniformity \( U_\prec \) on \( L \) that determines \( \lhd \). Furthermore \( U_\prec \) is the coarsest frame quasi-uniformity that determines \( \lhd \).

**Proposition 4.2.** The category of totally bounded quasi-uniform frames forms a coreflective subcategory of the category of quasi-uniform frames.

**Proof.** Let \((L, U)\) be a quasi-uniform frame. By [5, Proposition 2.1] there is an associated quasi-proximity inclusion \( \prec \) on \( L \) defined by \( a \lhd b \) if and only if \( u(a) \leq b \) for some \( u \in U \). We show that \((L, U_\prec)\) gives the totally bounded coreflection. Clearly the identity map \((L, U_\prec) \to (L, U)\) is a quasi-uniform frame homomorphism. Let \( f : (M, V) \to (L, U) \) be a quasi-uniform frame homomorphism where \( V \) is totally bounded. Consider the diagram

\[
\begin{array}{ccc}
(L, U_\prec) & \xrightarrow{h} & (L, U) \\
\downarrow i & & \\
(M, V) & \xrightarrow{f} & (L, U)
\end{array}
\]

where \( h(x) = f(x) \) for \( x \in M \). Clearly \( h \) is a frame homomorphism. So it remains to show that \( h \) is a quasi-uniform frame homomorphism. This follows from the fact that given \( a, b \in M \) with \( v(a) \leq b \) for some \( v \in V \), we have that \( f(a) \lhd f(b) \) and

\[
u_{f(a), f(b)} \circ h \leq h \circ u_{a,b}.
\]

Obviously \( h \) is unique. This completes the proof. \( \square \)

**Theorem 4.3.** A quasi-uniform frame \((L, U)\) is compact if and only if it is totally bounded and complete.

**Proof.** Suppose that \((L, U)\) is totally bounded and complete. Then \((L, U^*)\) is totally bounded, and by [10, Proposition 3.3], \((L, U^*)\) is complete. Hence \( L = CL \), and by [1, Proposition 3], \( L \) is compact.

Suppose that \((L, U)\) is a quasi-uniform frame and that \( L \) is compact. Then if \( u \in U \), \( S_u \) has a finite subcover and therefore
\( \mathbf{U} \) is totally bounded. Since \( L \) is compact, it is paracompact, and consequently the fine uniformity is complete [2]. Since \( L \) is compact, the fine uniformity is the unique admissible uniformity. Since \( \mathbf{U}^* \) is admissible, it is the fine uniformity. Hence \((L, \mathbf{U}^*)\) is complete. By [10, Proposition 3.3], \((L, \mathbf{U})\) is complete. \( \square \)

**Corollary 4.4.** Let \((L, \mathbf{U})\) be a totally bounded quasi-uniform frame. Then \((L, \mathbf{U})\) is complete if and only if \( L \) is compact.

**Theorem 4.5.** The completion of a totally bounded quasi-uniform frame is compact.

**Proof.** Let \((L, \mathbf{U})\) be a totally bounded quasi-uniform frame. Then \( \mathbf{U}^* \) is totally bounded. By [1, Proposition 3], \((CL, CU^*)\) is compact. The result now follows from \((CL, (CU)^*) = (CL, CU^*)\).

Since the completion of a totally bounded quasi-uniform frame is compact [10, Corollary 4.9], it is natural to extend the notion of the Samuel compactification of a uniform frame to quasi-uniform frames. The Samuel compactification of a quasi-uniform frame \((L, \mathbf{U})\) is defined to be the completion of the totally bounded coreflection of \((L, \mathbf{U})\).

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