SOME EXAMPLES OF MI-SPACES AND OF SI-SPACES

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ABSTRACT. An $MI$-space is a crowded space in which every dense subset is open, and an $SI$-space is a crowded space in which every nonempty subset is irresolvable. E. Hewitt [He] showed that every $MI$-space is $SI$. Here, we present a Tychonoff $SI$-space with dispersion character $\alpha \geq \omega$ which is not $MI$, for every infinite cardinal $\alpha$. We also show that if a Tychonoff space contains a non-closed discrete subset, then there is a maximal Tychonoff extension of the topology which is not $MI$. This provides a general method to construct maximal Tychonoff spaces which are not $MI$.

0. INTRODUCTION

The topological spaces considered in this paper will be Tychonoff without isolated points (crowded). For a set $X$, the family of all Tychonoff topologies on $X$ is denoted by $TY(X)$. The dispersion character of a nonempty space $X$ is the cardinal

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number defined by \( \Delta(X) = \min\{|U| : \emptyset \neq U \subseteq X \text{ and } U \text{ is open} \} \), and the nowhere density number of a space \( X \) is \( nwd(X) = \min\{|A| : A \text{ is not nowhere dense in } X \} \). The dispersion character was introduced in [He] and the nowhere density number in [CG2]. For every space \( X \), we have that \( nwd(X) \leq \Delta(X) \leq |X| \). If \( P \) is a topological property, then a space \( (X, \tau) \) is said to be \( P \)-maximal if for every topology \( \tau' \) on \( X \) finer than \( \tau \) we have that either \( (X, \tau') \) has an isolated point or \( (X, \tau') \) does not have property \( P \). Using Zorn's Lemma, one may show that every (crowded) topology extends to a maximal (crowded) topology, and that every Tychonoff topology extends to a maximal Tychonoff topology (for details and related results see [He], [vD] and [CG1]).

A space is irreolvable if it does not contain two disjoint dense subsets, and a space that is not irreolvable is called resolvable. E. Hewitt [He] introduced and studied these two classes of spaces. Many of the familiar spaces fail to be irresolvable (for instance, \( k \)-spaces [Py] and countably compact spaces [CG1]), and several known examples of irresolvable spaces are constructed via maximal topologies (see [He]). In [He], Hewitt called a space \( X \) an \( MI \)-space (respectively, \( SI \)-space) if \( X \) is crowded and every dense subset is open (respectively, every nonempty subspace is irresolvable). An \( MI \)-space is simply called \( MI \) and an \( SI \)-space is simply called \( SI \). The \( MI \)-spaces, not necessary being crowded, were also introduced by Bourbaki [Bo] with the name submaximal spaces.

The following relationships were established by Hewitt [He].
1. [He, Th. 23] Every \( MI \)-space is an \( SI \)-space.
2. [He, Th. 26] Every maximal Tychonoff space is an \( SI \)-space.

In this paper, we give the following examples:
1. A Tychonoff \( SI \)-space of dispersion character \( \alpha \) which is not \( MI \), for every cardinal number \( \alpha \geq \omega \).
2. A maximal Tychonoff space which is not \( MI \).
3. A Tychonoff $M_1$-space which is not maximal Tychonoff.

The first listed example is new. Although it is not explicitly stated in his paper, one may check that the van Douwen's example [vD, Ex. 1.9] serves as the second example. Here, we construct such an example in a general way. The third example is almost trivial and it is known to some topologists, but the authors did not find it in the literature. For the sake of completeness, we believe that this paper is the right place to write this example. We shall prove general results (Theorems 1.5, 1.8 and 1.9) and the three examples will follow directly from them. One may see that the second example is also an example of a maximal Tychonoff space which is not maximal, as is van Douwen's example mentioned above.

1. The Examples

The following two lemmas are taken from [Fe].

**Lemma 1.1.** For every cardinal $\alpha \geq \omega$, there exists an $SI$-space $X$ such that $\text{nwd}(X) = \Delta(X) = |X| = \alpha$.

**Lemma 1.2.** Let $\{\tau_i : i \in I\}$ be a chain of topologies on a set $X$ (ordered by the usual set-theoretical inclusion), and let $\tau$ be the topology on $X$ generated by the chain, that is, $\tau$ has $\bigcup_{i \in I} \tau_i$ as a base. If $N$ is nowhere dense in $(X, \tau_i)$ for every $i \in I$, then $N$ is nowhere dense in $(X, \tau)$.

It follows from Lemma 1.2 that if $\{\tau_i : i \in I\}$ is a chain of topologies on $X$ and $\tau$ is the topology generated by this chain, then

$$\text{nwd}(X, \tau) = \min\{\text{nwd}(X, \tau_i) : i \in I\}.$$ 

The method of extending a Tychonoff topology to another Tychonoff topology by using a function (not necessarily continuous) $f : X \to \mathbb{R}$ was considered in [CG1]. The next lemma is the main tool in this technique.

**Notation.** For a space $X = (X, \tau)$ and a function $f : X \to \mathbb{R}$, the symbol $\tau_f$ denotes the topology on $X$ induced by $\tau$ and...
$f$, that is, the topology in which $\tau \cup \{f^{-1}(V) : V \text{ open in } \mathbb{R}\}$ is a subbase.

**Lemma 1.3.** [CG1] Let $X$ be a set and let $f : X \to \mathbb{R}$ be a function.

1. If $\tau \in TY(X)$, then $\tau_f \in TY(X)$; and
2. if every $W \in \tau$ and every open $V \subseteq \mathbb{R}$ satisfy either $W \cap f^{-1}(V) = \emptyset$ or $|W \cap f^{-1}(V)| \geq \omega$, then $(X, \tau_f)$ is crowded.

**Lemma 1.4.** If $X$ is a topological space such that $X = A \cup B$, where $A$ and $B$ are both $SI$ subspaces, then $X$ is also an $SI$-space.

**Proof:** Suppose that $R$ is a nonempty resolvable subset of $X$. Then, $R \cap (X - B)$ is an open resolvable subset of $R$. Since $R \cap (X - B) \subseteq A$ and $A$ is $SI$, $R \cap (X - B) = \emptyset$ and then $R \subseteq B$, which contradicts the assumption that $B$ is an $SI$ subspace. $\square$

The first example follows directly from the next theorem.

**Theorem 1.5.** For every infinite cardinal $\alpha$, there is an $SI$-space $X$ which is not an $MI$-space, and $nwd(X) = \Delta(X) = |X| = \alpha$.

**Proof:** Fix $\alpha \geq \omega$. Our first claim is the following.

**Claim 1.** There is a space $Y$ with $nwd(Y) = \Delta(Y) = |Y| = \alpha$ that contains a nowhere dense, closed and crowded subset $N$ which is an $SI$ subspace of $Y$.

**Proof of Claim 1.** By Lemma 1.1 there is an $SI$-space $Z$ such that $nwd(Z) = \Delta(Z) = |Z| = \alpha$. If $Y = Q \times Z$, where $Q$ is the space of all rational numbers, then $nwd(Y) = \Delta(Y) = |Y| = \alpha$, and $N = \{0\} \times Z$ is a nowhere dense, closed and crowded subset of $Y$ which is $SI$. This shows Claim 1.

Fix the space $(Y, \sigma)$ and an $SI$ subspace $N$ of $Y$ as in Claim 1. Now, we consider the family $\mathcal{C}$ of all $\tau \in TY(Y)$ such that

1. $\sigma \subseteq \tau$;
2. $nwd(Y, \tau) = \Delta(Y, \tau) = |Y| = \alpha$;
3. $N$ is nowhere dense in $(Y, \tau)$; and
4. $(N, \tau|_N)$ is crowded.

Claim 2. $C$ has a maximal element in the usual set-theoretical inclusion order.

Proof of Claim 2. We have that $\sigma \in C$. Let $\mathcal{A}$ be a chain in $C$ and let $\tau$ be the topology on $Y$ generated by $\bigcup \mathcal{A}$. It is evident that $\sigma \subsetneq \tau$. By Lemma 1.2, $N$ is nowhere dense in $(Y, \tau)$ and $nwd(Y, \tau) = \min\{nwd(Y, \tau_i) : i \in I\} = \alpha$. Since $nwd(Y, \tau) \leq \Delta(Y, \tau) \leq |Y| = \alpha$, we must have $nwd(Y, \tau) = \Delta(Y, \tau) = |Y| = \alpha$. To prove that $(N, \tau|_N)$ is crowded, we let $V \in \tau$ such that $\emptyset \neq V \cap N$. Then, there is $\tau' \in \mathcal{A}$ and $W \in \tau'$ such that $\emptyset \neq W \cap N \subseteq V \cap N$. Since $(N, \tau'|_N)$ is crowded, $\omega \leq |W \cap N| \leq |V \cap N|$. So, $\sigma \in C$. By Zorn's Lemma, $C$ has a maximal element.

In what follows, we fix a maximal element $\tau$ of $C$.

Claim 3. Every nonempty open subset of $(Y - N, \tau|_{Y-N})$ is irresolvable.

Proof of Claim 3. Suppose the contrary. Then, there is a nonempty open resolvable subset $E$ of $(Y - N, \tau|_{Y-N})$. Since $N$ is closed in $(Y, \tau)$ (in fact, it is $\sigma$-closed), by the regularity of $Y$, we may find $U \in \tau$ such that $\emptyset \neq U \subseteq \overline{U} \subseteq E \subseteq Y - N$. Put $F = \overline{U}$ and equip $F$ with the subspace topology from $(Y, \tau)$. Then, $F$ is resolvable and hence $F = D_0 \cup D_1$, where $D_0$ and $D_1$ are disjoint and dense in $F$. Notice that $nwd(F) = \Delta(F) = |F| = \alpha$, hence $|D_0| = |D_1| = \alpha$. Let us consider the topology $\tau_f$ on $Y$ induced by the function $f : Y \to \mathbb{R}$ defined by $f(x) = 0$ for any $x \in D_0$; $f(x) = 1$ for any $x \in D_1$ and $f(x) = 2$ for any $x \in Y - F$. We have that the family $\tau \cup \{W \cap D_i : i = 0, 1, W \in \tau\}$ is a base for the topology $\tau_f$. By Lemma 1.3, $\tau_f \in T(Y)$. To get a contradiction, it suffices to show that $\tau_f \in C$ since $\tau_f \neq \tau$. Indeed, it is evident that $\sigma \subsetneq \tau_f$. Assume that $A \subseteq Y$ has size $< \alpha$ and there is $V \in \tau_f$ such that $\emptyset \neq V \subseteq \overline{A}^{\tau_f}$. Since $A$ is nowhere dense in $(Y, \tau)$ and $\overline{A}^{\tau_f} \subseteq \overline{A}^\tau$, $V = W \cap D_i$ for some $W \in \tau$ and some
$i \in \{0,1\}$. Since $D_i$ is dense in $(F,\tau|_F)$ and $F = \overline{U}$, $\emptyset \neq U \cap W \subseteq \overline{U} \cap W \cap D_i^T \subseteq \overline{W} \cap D_i^T \subseteq \overline{A}$, which means that $A$ is not nowhere dense in $(Y,\tau)$ and $|A| < \alpha$, a contradiction. This shows that $nwd(Y,\tau_f) = \Delta(Y,\tau_f) = |Y| = \alpha$. If $N$ is not nowhere dense in $(Y,\tau)$, then there is $V \in \tau_f$ such that $\emptyset \neq V \subseteq \overline{N}^\tau \subseteq \overline{N}^\tau = N$. On the other hand, since $N \subseteq Y - F$ and $\tau_f|_{Y-F} = \tau|_{Y-F}$, we must have that $V \in \tau$, which contradicts the hypothesis $N$ is nowhere dense in $(Y,\tau)$. Thus, $N$ is nowhere dense in $(Y,\tau_f)$. Now, we shall verify that $(N,\tau_f|_N)$ is crowded. Given $y \in N$ and a neighborhood $V_y \in \tau_f$ of $y$, there is $W \in \tau$ such that $y \in W \cap (Y - F) \subseteq V_y$. Since $W \cap (Y - F) \in \tau$, $\omega \leq |W \cap (Y - F) \cap N| \leq |V_y \cap N|$. So, $(N,\tau_f|_N)$ does not have any isolated points. Thus, we have proved that $\tau_f \in C$. But this is impossible since $D_0, D_1 \in \tau_f - \tau$. Therefore, every nonempty open subset of $(Y - N,\tau|_{Y-N})$ is irresolvable.

By Theorem 3.7(b) of [FM], the space $(Y - N,\tau|_{Y-N})$ contains an open dense $SI$ subset $S$. Our space will be the subspace $X = S \cup N$ of $(Y,\tau)$. Since $N$ is closed and nowhere dense in $(Y,\tau)$, $S \in \tau$ and $S$ is dense in $(X,\tau|_X)$. To show that $nwd(X) = \Delta(X) = |X| = \alpha$, we fix $A \subseteq X$ with $|A| < \alpha$. Then, $A \cap N$ is nowhere dense in $(X,\tau|_X)$. Since $nwd(Y,\tau) = \alpha$ and $S \in \tau$, $A \cap S$ is also nowhere dense in $(X,\tau|_X)$. Hence, $A$ is a nowhere dense subset of $(X,\tau|_X)$. This proves the equality $nwd(X,\tau|_X) = \Delta(X,\tau|_X) = |X| = \alpha$. It remains to show the following.

**Claim 4.** $(X,\tau|_X)$ is $SI$-space but not $MI$.

**Proof of Claim 4.** It is evident that $(N,\tau|_N)$ is an $SI$-space, since $(N,\sigma|_N)$ is so and $\sigma \subseteq \tau$. Thus, $(X,\tau|_X)$ is the union of two $SI$ subspaces. Since $N$ is closed, by Lemma 1.4, $(X,\tau|_X)$ is $SI$. Finally, we shall verify that $(X,\tau|_X)$ cannot be $MI$. Pick $y \in N$. Since $N$ is nowhere dense in $(X,\tau|_X)$, $(X - N) \cup \{y\}$ is dense in $(X,\tau)$. But, $(X - N) \cup \{y\}$ cannot be open since $(N,\tau|_N)$ has no isolated points. Therefore, $(X,\tau|_X)$ is not an $MI$-space. \qed
A maximal Tychonoff space which is not maximal was given in [vD, Ex. 1.9]. We shall give a condition on a space \((X, \tau)\) which guarantees that \(\tau\) has a maximal Tychonoff extension that fails to be \(MI\). Thus spaces with such maximal Tychonoff topologies are not maximal. Some basic facts from [vD] will be useful in this task. E. K. van Douwen [vD] called a crowded space \(X\) ultradisconnected if every nonempty proper subset \(A \subset X\) is clopen iff both \(A\) and \(X - A\) are crowded. He used this concept to characterize the maximal regular spaces. He showed [vD, Th. 1.8] that a space \(X\) is maximal regular iff \(X\) is regular and ultradisconnected. He also proved that every maximal regular space is extremally disconnected and hence zero-dimensional. Hence we have the following lemma.

**Lemma 1.6.** 1. Every maximal regular extension of a Hausdorff topology is a maximal Tychonoff topology.  
2. Every maximal Tychonoff topology is maximal regular.

The following characterization of an ultradisconnected \(MI\) space can be deduced directly from Fact 1.15, Theorem 2.2 of [vD], and Theorem 24 of [He].

**Lemma 1.7.** For an ultradisconnected space \(X\), the following are equivalent.

1. \(X\) is \(MI\).  
2. Every discrete subset of \(X\) is closed.

**Theorem 1.8.** If \((X, \sigma)\) is a Tychonoff space that contains a non-closed discrete subset, then there is a maximal Tychonoff extension of \(\sigma\) which is not \(MI\).

**Proof:** Suppose that \((X, \sigma)\) is a Tychonoff space that contains a discrete subset \(N \subset X\) and a point \(x \in X\) such that \(x \in \overline{N^\sigma} - N\). Let \(D\) be the family of all \(\tau \in TY(X)\) such that

1. \(\sigma \subseteq \tau\); and  
2. \(x \in \overline{N^\tau} - N\).

First, we observe that if \(\sigma \subseteq \tau\) and \((X, \tau)\) is crowded, then \(\text{int}_{(X, \tau)}N = \emptyset\), since \(N\) is discrete in \((X, \sigma)\). It is clear that
\( \sigma \in \mathcal{D} \) and if \( \mathcal{A} \) is a chain in \( \mathcal{D} \), then the topology on \( X \) generated by the chain is again an element of \( \mathcal{D} \). By Zorn's Lemma, \( \mathcal{D} \) has a maximal element. Let us still use \( \tau \) to denote this maximal element of \( \mathcal{D} \). We will show that \( \tau \) is a maximal regular extension of \( \sigma \). Suppose the contrary. By van Douwen's characterization, \((X, \tau)\) is not ultradisconnected, that is, there is \( A \subseteq X \) such that \( A \) is not \( \tau \)-clopen and both \( A \) and \( X - A \) are \( \tau \)-crowded. Without loss of the generality, we may assume that \( x \in A \). Since \( x \in \overline{N'} \setminus N \), it is enough to consider the following two cases:

Case 1. \( V \cap (A - \{x\}) \cap N \neq \emptyset \) for every neighborhood \( V \in \tau \) of \( x \).

Define \( f : X \to \mathbb{R} \) by \( f(x) = 0 \) for any \( x \in A \), and \( f(x) = 1 \) for any \( x \in X - A \). By Lemma 1.3, \( \tau_f \) is a Tychonoff extension of \( \tau \) which does not have isolated points, since both \( A \) and \( X - A \) are \( \tau \)-crowded. A base for \( \tau_f \) is \( \tau \cup \{V \cap A : V \in \tau\} \cup \{V \cap (X - A) : V \in \tau\} \). It follows from our hypothesis that \( x \in \overline{N'}_{\tau_f} \). That is, \( \tau_f \in \mathcal{D} \). But this happens only when \( \tau_f = \tau \), which is impossible since \( A, X - A \in \tau_f \) and neither \( A \in \tau \) nor \( X - A \in \tau \).

Case 2. \( V \cap (X - A) \cap N \neq \emptyset \) for every neighborhood \( V \in \tau \) of \( x \).

Define \( B = A - \{x\} \). We shall prove that both \( B \) and \( X - B \) are \( \tau \)-crowded. Since \((X, \tau)\) is a Hausdorff space, \( A \) is \( \tau \)-crowded implies that \( B = A - \{x\} \) is also \( \tau \)-crowded, and since \( x \) is an accumulation point of \( X - A \), \( X - B = (X - A) \cup \{x\} \) is \( \tau \)-crowded as well. So, \( B \) and \( X - B \) are \( \tau \)-crowded. Define \( g : X \to \mathbb{R} \) by \( g(x) = 0 \) for any \( x \in B \), and \( f(x) = 1 \) for any \( x \in X - B \). In virtue of Lemma 1.3, \( \tau_g \) is a crowded Tychonoff extension of \( \tau \), because of \( B \) and \( X - B \) are both \( \tau \)-crowded. By the hypothesis, we obtain that \( x \in \overline{N'}_{\tau_g} \). Thus, \( \tau_g \in \mathcal{D} \). Since \( \tau \subseteq \tau_g \) and \( \tau \) is a maximal element of \( \mathcal{D} \), \( \tau = \tau_g \). But, this is a contradiction, since \( A \) is \( \tau \)-crowded, \( X - B \in \tau_g = \tau \) and \( A \cap (X - B) = \{x\} \).

Therefore, \((X, \tau)\) is maximal regular and hence maximal Tychonoff. Since \( x \in \overline{N'} \setminus N \) and \( N \) is discrete in \((X, \tau)\) (in fact,
it is discrete in every finer extension of \( \sigma \), by Lemma 1.7, 
\((X, \tau)\) cannot be an \( MI \)-space.

The above theorem provides us a general method to construct maximal Tychonoff, non-\( MI \) spaces. We observe that E. K. van Douwen's example [vD, Ex. 1.9] mentioned above has a countable non-closed, discrete subset and, therefore, is not an \( MI \)-space.

In the last example, we require the adjunction spaces (for the definition and basic properties of adjunction spaces, we refer the reader to the books [Du] and [En]). We mainly consider the following special type of adjunction spaces.

Let \((X_1, \tau_1)\) and \((X_2, \tau_2)\) be two disjoint spaces, \( p \in X_1 \) and \( f : \{p\} \to X_2 \) be a function. Then, \( X_1 \cup_f X_2 \) will denote the adjunction space determined by \( X_1, X_2 \) and \( f \). \( X_1 \cup_f X_2 \) is just the space obtained by identifying \( p \) with \( f(p) \). The space \( X_1 \cup_f X_2 \) is homeomorphic to the following one.

Suppose that \( X_1 \cap X_2 = \{p\} \). On the set \( Z = X_1 \cup X_2 \), we define a topology \( \tau \) on \( Z \) as follows: \( \tau_0 |_{X_0 - \{p\}} \cup \tau_1 |_{X_1 - \{p\}} \subseteq \tau \) and the open neighborhoods of \( p \) are \( \{U \cup V : U \in \tau_0, V \in \tau_1, p \in U \cap V\} \).

It is easy to prove that \((Z, \tau)\) is homeomorphic to \( X_1 \cup_f X_2 \). For simplicity, we work on \((Z, \tau)\) rather than \( X_1 \cup_f X_2 \) and keep the notation \( X_1 \cup_f X_2 \) for this space. We remark that \( X_1 - \{p\} \) and \( X_2 - \{p\} \) are open in \( X_1 \cup_f X_2 \), \( (X_1, \tau |_{X_1}) \) and \( (X_2, \tau |_{X_2}) \) are closed in \( X_1 \cup_f X_2 \), and \((X_1, \tau |_{X_1})\) and \((X_2, \tau |_{X_2})\) are homeomorphic to \((X_1, \tau_1)\) and \((X_2, \tau_2)\), respectively.

**Theorem 1.9.** Let \((X_1, \tau_1)\) and \((X_2, \tau_2)\) be two disjoint spaces, \( p \in X_1 \) and \( f : \{p\} \to X_2 \) a function. Let \( X_1 \cup_f X_2 \) be the adjunction space determined by \( X_1, X_2 \) and \( f \).

1. If \((X_1, \tau_1)\) and \((X_2, \tau_2)\) are Tychonoff, then \( X_1 \cup_f X_2 \) is also Tychonoff.
2. If \((X_1, \tau_1)\) and \((X_2, \tau_2)\) are \( MI \)-spaces, then \( X_1 \cup_f X_2 \) is also \( MI \).
3. \( \Delta(X_1 \cup_f X_2) = \min\{\Delta(X_1), \Delta(X_2)\} \).
4. $X_1 \cup_f X_2$ is never extremally disconnected.
5. $X_1 \cup_f X_2$ is never maximal Tychonoff.

**Proof:** 1. It is easy to verify that $X_1 \cup_f X_2$ is Hausdorff. Let $V$ be an open subset of $X_1 \cup_f X_2$ with $p \in V$. Since $V \cap X_1$ and $V \cap X_2$ are open in $X_1$ and $X_2$ respectively, there are two continuous functions $f_1 : X_1 \to [0, 1]$ and $f_2 : X_2 \to [0, 1]$ such that $f_1(p) = f_2(p) = 0$, $f_1(X_1 - V) = 1$ and $f_2(X_2 - V) = 1$. Define $F : X_1 \cup_f X_2 \to [0, 1]$ by $F(z) = f_1(z)$ if $z \in X_1$, and $F(z) = f_2(z)$ if $z \in X_2$. Then, $F$ is continuous, $F(p) = 0$ and $F(X_1 \cup_f X_2 - V) = 1$. Now, let $V$ be an open subset of $X_1 \cup_f X_2$ and let $q \in V \cap (X_1 - \{p\})$. We may find $W \in \tau_1$ such that $q \in W \subseteq V \cap (X_1 - \{p\})$. Since $X_1$ is completely regular there is a continuous function $g : X_1 \to [0, 1]$ such that $g(q) = 0$ and $g(X_1 - W) = 1$. Define $G : X_1 \cup_f X_2 \to [0, 1]$ by $G(z) = g(z)$ if $z \in X_1$, and $G(z) = 1$ if $z \in X_2$. Then $G$ is continuous, $G(q) = 0$ and $G(X_1 \cup_f X_2 - V) = 1$. A similar argument shows that there are continuous functions to separate points in $X_2 - \{p\}$ and closed subsets of $X_1 \cup_f X_2$. Therefore, $X_1 \cup_f X_2$ is Tychonoff.

2. Suppose that $D$ is a dense subset of $(Z, \tau)$. We will show that $D$ is an open subset of $Z$. Thus $(Z, \tau)$ is an M I-space.

Case 1. $p \notin D$.

Since $X_1 - \{p\}$ and $X_2 - \{p\}$ are open in $Z$, we have $D \cap X_1 = D \cap (X_1 - \{p\})$ is dense in $X_1 - \{p\}$ and $D \cap X_2 = D \cap (X_2 - \{p\})$ is dense in $X_2 - \{p\}$. Thus, $D \cap X_1$ is dense in $X_1$ and $D \cap X_2$ is dense in $X_2$. Since $X_1$ and $X_2$ are M I, $D \cap X_1$ and $D \cap X_2$ are open in $X_1$ and $X_2$ respectively. So, $D$ is open in $Z$.

Case 2. $p \in D$.

Since $D \cap X_1$ is dense in $X_1$ and $X_1$ is M I, $D \cap X_1$ is open in $X_1$. Similarly, $D \cap X_2$ is open in $X_2$. Then, $D$ is open in $Z$.

3. This is evident.

4. It is clear that $p \in X_1 - \{p\} \cap X_2 - \{p\}$. Since $X_1 - \{p\}$ and $X_2 - \{p\}$ are disjoint open subsets of $X_1 \cup_f X_2$, we obtain that $X_1 \cup_f X_2$ cannot be extremally disconnected.

5. The discrete union of $X_1 - \{p\}$ and $X_2$ gives a Tychonoff topology on $X_1 \cup_f X_2$ stronger than the original one. \qed
Example 1.10. [Folklore] There exists a Tychonoff MI space which is not maximal Tychonoff.

Proof: Let $X$ be a Tychonoff MI space. Let $X_1 = X \times \{0\}$ and $X_2 = X \times \{1\}$. Fix a point $p \in X$ and define $f : \{(p,0)\} \rightarrow X_2$ by $f((p,0)) = (p, 1)$. Then by Theorem 1.9, $X_1 \cup f X_2$ is a Tychonoff MI space which is not maximal Tychonoff. □

In brief, we have seen that the space obtained by identifying two points of a Tychonoff MI space is MI but not maximal Tychonoff. Examples of Tychonoff MI-spaces, in $ZFC$, are given in [El], [vD] and [LP].

Theorem 1.9 also implies that the union of two maximal Tychonoff subspaces is not necessary maximal Tychonoff. On the other hand, Example 2 shows that, in general, a crowded topology cannot be extended to an MI Tychonoff topology. This suggests the following question:

Question 1.11. What kind of spaces can be extended to maximal Tychonoff MI spaces?

We remark that if $(X, \tau)$ is MI, then every Tychonoff crowded extension of $X$ is MI. We also notice that, from Lemma 1.6 and [vD, Th. 2.2], a Tychonoff space is Hausdorff maximal iff it is a maximal Tychonoff MI-space.

References


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