Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY
PROCEEDINGS
Volume 24, Spring 1999

THE STRUCTURE AND TOPOLOGY OF THE BRJUNO NUMBERS

EILEEN F. LEE

ABSTRACT. We construct a naturally occurring *planar straight brush* associated with the set of highly irrational numbers known as the *Brjuno numbers*, which are important in dynamical systems. Furthermore, we partition the Brjuno numbers by "degree of irrationality" and we construct an infinite sequence of straight brushes corresponding to this partition. We use continued fractions throughout this work; listing pertinent classical theory and proving special properties, as necessary.

0. INTRODUCTION

Over the last century, researchers in dynamical systems considered the "small divisors problem"; they studied linearization conditions for complex analytic maps with neutral fixed points at the origin and irrational rotation numbers. They considered maps of the form $f(z) = e^{2\pi i\alpha}z + a_2z^2 + \dots$ and sought necessary and sufficient conditions on the irrational rotation number, α to ensure linearization in some neighborhood of the origin for every such map. From the works of C.L. Siegel in 1942 and A.D. Brjuno in 1965, it was known that when the

Key words and phrases. Straight Brush, Brjuno Numbers, Continued Fractions, Irrationality Degree.

rotation number was a Diophantine number [8] or more generally, an irrational that was "poorly approximated by rationals" [2], then linearization was always possible. In 1987 J.C. Yoccoz proved that the Brjuno numbers possess the necessary and sufficient "degree of rational approximation" for a rotation number to always yield linearization [9].

We study the structure of the Brjuno numbers and the topology of a planar graph associated with the Brjuno function. In Section 1, we provide some necessary background in number theory and topology. From number theory, the use of continued fractions in the rational approximation of irrational numbers is presented to the extent necessary to define the Brjuno function and the Brjuno numbers. We produce examples of Brjuno and non-Brjuno numbers and prove that the Brjuno numbers are dense in the irrationals. Next, we review the definition of planar straight brushes and give an example of one that arises naturally in dynamics.

In Section 2 we present our main result, which is a construction of a planar straight brush over the Brjuno numbers. We rely heavily on continued fraction theory to construct irrational numbers with prescribed absolute value and prescribed Brjuno sum.

In Section 3, we show that the Brjuno numbers properly contain the Diophantine numbers along with Liouville numbers of varying degrees of irrationality. Furthermore, we construct a nested sequence of subsets inside the Brjuno numbers and we prove that each of these subsets yields a planar straight brush.

1. Preliminaries

Continued fractions and Brjuno numbers

By the simple regular continued fraction representation of an irrational number we mean an infinite expansion of the form

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

where $a_0 \in \mathbf{Z}$ and $a_n \in \mathbf{Z}^+$ for every $n > 0$.

We shall use standard shorthand notation for the continued fraction representation, letting $\alpha = [a_0, a_1, a_2, \dots]$ or $\alpha = [a_n]$.

An important example in the work that follows is the continued fraction representation for the golden ratio:

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}} = [1, 1, 1, \dots]$$

Truncation of the infinite continued fraction expansion for α after a finite number of steps yields rational approximations, known as partial fraction *convergents.* The nth convergent of α is given by:

$$\frac{P_n}{Q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}} = [a_0, a_1, a_2, \dots, a_n]$$

The convergents are the "closest" nearby rational numbers in the sense that if $\mid \alpha - \frac{a}{b} \mid < \mid \alpha - \frac{P_n}{Q_n} \mid$ for some $n \geq 0$, then $b > Q_n$. We are interested in approximating irrationals by rationals with the least denominators possible, so the convergents provide the best rationals for the job.

Below are some properties of the convergents that are used in our work. All, except for 1.3.B are elementary and are found in [5]. Property 1.3.B is easily proved by induction. See [6] for proof.

1.1 Recursive relations

$$P_0 = a_0,\quad P_1 = a_1 a_0 + 1,\quad P_n = a_n P_{n-1} + P_{n-2} \quad \text{for every } n \geq 2$$
$$Q_0 = 1,\quad Q_1 = a_1,\quad Q_n = a_n Q_{n-1} + Q_{n-2} \quad \text{for every } n \geq 2$$

1.2 The convergents converge to α in an oscillating manner: $\frac{P_{2k}}{Q_{2k}} < \alpha < \frac{P_{2k+1}}{Q_{2k+1}}$ for every $k \geq 0$

That is, the subsequence of even terms yields monotonic convergence from below and the subsequence of odd terms yields monotonic convergence from above.

1.3 (A) If $x = [a_0, a_1, \ldots, a_n, a_{n+1} \ldots]$ and $y = [a_0, a_1, \ldots, a_n, b_{n+1} \ldots]$ then $\mid x - y \mid < \frac{1}{2^{n-1}}$.

(B) If $x = [a_0, a_1, \ldots]$ and $\mid x - y \mid < \mid x - \frac{P_{n+1}}{Q_{n+1}} \mid$ then $y = [a_0, a_1, \ldots, a_n, b_{n+1} \ldots]$.

1.4 The proximity of nearby rational numbers is governed by the convergent fraction denominators:

$$\frac{1}{Q_n(Q_n + Q_{n+1})} < |\alpha - \frac{P_n}{Q_n}| < \frac{1}{Q_n Q_{n+1}} \text{for all } n \geq 0.$$

A close look at the lower bound in Property 1.4 reveals that the growth rate of the convergent denominator sequence actually determines the type of "rational barrier function" exhibited by α. That is to say, an irrational number with a fast growing denominator sequence will permit better rational approximations than found with an irrational having a denominator sequence exhibiting slower growth rate. Intuitively, the latter are considered "more irrational" than the former. The absolute slowest denominator growth rate possible is found with the golden ratio (the sequence is the Fibonacci sequence) and the golden ratio is considered the "most irrational" number. On the other extreme, varying degrees of fast growth rates are found within the set of Liouville numbers.

We study functions of an irrational number pertaining to continued fraction data. For starters, if $\{Q_n\}$ is the sequence of convergent denominators, we remark that the infinite sum $B_0(\alpha) = \sum_{n=0}^{\infty} \frac{\log Q_n}{Q_n}$ is finite for *every* irrational number [6]. This function does not distinguish between irrational numbers of different irrationality degrees, but things change when we choose a function that captures the growth rate of $\{Q_n\}$, such as the Brjuno function:

$$B(\alpha) = \sum_{n=0}^{\infty} \frac{\log Q_{n+1}}{Q_n}.$$

The set of **Brjuno numbers** is defined to be the domain of convergence of this function. That is,

$$B = \{\alpha \in (\mathbf{R} \backslash \mathbf{Q}) \mid B(\alpha) < \infty\}.$$

Proposition 1.1. *Not all irrational numbers are Brjuno numbers.*

Proof: If we let $\alpha = [a_n]$ where

$$a_n = \begin{cases} 10 & \text{if } n = 0, 1 \\ Q_{n-1}^{Q_{n-1}} & \text{if } n \geq 2 \end{cases}$$

then one can check easily that $Q_{n+1} > Q_n^{\frac{2Q_n}{Q_{n-1}}}$ and the Ratio Test guarantees divergence of the Brjuno sum for α. □

Rest assured that Brjuno numbers exist. Below we describe some and prove that the Brjuno numbers are dense in the irrationals.

Proposition 1.2. *The golden ratio is a Brjuno number. Moreover, any irrational number whose continued fraction expansion ends with a string of 1's is a Brjuno number.*

Proof: If $\alpha = [1, 1, 1, \dots]$ then $Q_{n+1} < Q_n^2$ when $n \geq 2$ and if $\alpha = [a_0, \dots, a_k, 1, 1, 1, \dots]$ for some integer sequence $a_0, \dots a_k$, then $Q_{n+1} < Q_n^2$ when $n > k$. Thus, eventually, $\frac{\log Q_{n+1}}{Q_n} < \frac{2 \log Q_n}{Q_n}$ holds in both cases and since $\sum_{n=0}^{\infty} \frac{\log Q_n}{Q_n} < \infty$, we have finite Brjuno sums in both cases. □

Remark: If $\alpha = [a_0, \dots a_k, 1, 1, 1, \dots]$ for some integer sequence $a_0, \dots, a_k$, then we say that α has a "golden tail".

Earlier we remarked that two numbers are close if their continued fractions share the same front end and moreover, the longer that the front ends match, the closer the two numbers will be. This observation, coupled with the second part of Proposition 1.2, proves that the set of all numbers that end with a golden tail is dense in the irrationals. Hence we have

Proposition 1.3. *The Brjuno numbers are dense in the irrationals.*

Straight brushes We follow J.M. Aarts and L.G. Oversteegen [1] in defining a canonical planar straight brush. A **straight brush** in $\mathbf{R}^2$ is a closed subset Λ of the set $\mathbf{R} \backslash \mathbf{Q} \times [0, \infty)$ with the following properties:

1. Hairiness

For each $(\alpha, y) \in \Lambda$ there is a $y_\alpha \in [0, \infty)$ such that $\{y \mid (\alpha, y) \in \Lambda\} = [y_\alpha, \infty)$. The point (α, y_α) is called the endpoint of the hair at α and the non-endpoints on the hair are called interior points of the hair.

2. Density

(i) The set $\{\alpha \mid (\alpha, y) \in \Lambda \text{ for some } y\}$ is dense in the irrational numbers and

(ii) for each $(\alpha, y) \in \Lambda$, there exist sequences of irrational numbers, $\{\beta_n\}$ and $\{\zeta_n\}$ such that $\beta_n \uparrow \alpha$, $\zeta_n \downarrow \alpha$, $y_{\beta_n} \to y$, and $y_{\zeta_n} \to y$.

On the other hand, from dynamics Julia sets of many entire functions of finite type have been shown to contain *Cantor bouquets* of rays in the complex plane. See, for example [4] where Devaney and Tangerman exhibit Cantor bouquets in the Julia sets for each member of the family of complex exponential maps $E_\lambda(z) = \lambda e^z$ for a real parameter λ, $0 < \lambda < 1/e$. Aarts and Oversteegen [1] proved that a Cantor bouquet is homeomorphic to a straight brush and that all planar straight brushes are homeomorphic. The Julia sets mentioned above were celebrated as the first "naturally occurring" straight brushes and the Brjuno brushes presented here provide more such examples.

2. The Brjuno Brush

What does the set of Brjuno numbers look like? We begin by graphing the Brjuno function in the plane, and we find that $\{(\alpha, B(\alpha)) | \alpha \in B\}$ is a totally disconnected and perfect set. Next, we attach a vertical ray to each endpoint and define $\Lambda = \{(\alpha, y) \in \mathbf{R}^2 | \alpha \in B \text{ and } y \geqslant B(\alpha)\}$.

By construction, Λ satisfies the necessary hairiness property for a straight brush and with Propositions 2.1 and 2.2, we show that is a closed subset of the plane satisfying the endpoint density property for a straight brush.

Proposition 2.1. *$\Lambda = \{(\alpha, y) \in \mathbf{R}^2 \mid \alpha \in B$ and $y \geq B(\alpha)\}$ is a closed subset of the plane.*

Proof: We assume $\{(x_n, y_n)\} \in \Lambda$ is an infinite sequence converging to (α, y) where $\alpha = [a_i]$ and we must prove that $(\alpha, y) \in \Lambda$. We fix an $n \geq 0$ and we let $\frac{P_i}{Q_i}$ and $\frac{P_i^n}{Q_i^n}$ be the ith convergents for α and x_n, respectively.

By assumption, $x_n \in B$ and $y_n \geq B(x_n)$ we must show that $y \geq B(\alpha)$ and $\alpha \in B$. First, we observe that Property 1.3.B implies that we may assume, without loss of generality, that the continued fraction representation for x_n has a front end that matches the front end for α.

This implies that $Q_i^n = Q_i$ for $0 \leq i \leq n$, which implies that the nth partial Brjuno sum for x_n is identical to the nth partial Brjuno sum for α, yielding $B(x_n) > \sum_{i=0}^{i=n-1} \frac{\log Q_{i+1}}{Q_i}$. By assumption $y_n \geq B(x_n)$, hence $y_n > \sum_{i=0}^{i=n-1} \frac{\log Q_{i+1}}{Q_i}$ for every $n \geq 0$, and $\lim_{n\to\infty} y_n \geq \lim_{n\to\infty} \sum_{i=0}^{i=n-1} \frac{\log Q_{i+1}}{Q_i}$, that is $y \geq B(\alpha)$

This last inequality forces $B(\alpha)$ to be finite since y was assumed to be finite in the first place. Thus α is a Brjuno number and we have shown that $(\alpha, y) \in \Lambda$. □

The endpoint density requirement of a straight brush requires the construction of two sequences that converge on α ; one monotonically increasing and the other monotonically decreasing. Our approach involves constructing a single sequence that converges in an oscillating way, where the subsequence of even terms is eventually strictly increasing and the subsequence of odd terms eventually is strictly decreasing.

Proposition 2.2. *For each $(\alpha, y) \in \Lambda$, there exists a sequence $\{(x_n, B(x_n))\}$ in Λ with $x_n \to \alpha$ and $B(x_n) \to y$.*

Proof: Suppose $(\alpha, y) \in \Lambda$ where $\alpha = [a_n]$ and $z = y - B(\alpha)$. We form a sequence of irrational numbers, $\{x_n\}$ where the front end of the continued fraction for each x_n matches the continued fraction for α through the first n places and the continued fraction tail for each x_n is golden. We custom design one integer in the continued fraction for x_n so as to control its Brjuno sum. That is, we let
$x_n = [a_0, \ldots, a_n, [| \, b_n \, |], 1, 1, 1, \ldots]$ where $b_n = \frac{Q_{n+1} e^{zQ_n} - Q_{n-1}}{Q_n}$, and $[| \, b_n \, |]$ means the greatest integer in b_n.

We denote the kth convergent denominators for α and x_n by Q_k and Q_k^n, respectively and observe that $Q_k^n = Q_k$ for $0 \leq k \leq n$.

By the remarks made in Section 1, it is obvious that $x_n \in B$ for every $n \geq 0$ and $x_n \to \alpha$.

Next, we fix an $n \geq 0$ and we consider $B(x_n) - B(\alpha)$:

$$B(x_n) - B(\alpha) = \sum_{k=0}^{\infty} \frac{\log Q_{k+1}^n}{Q_k^n} - \sum_{k=0}^{\infty} \frac{\log Q_{k+1}}{Q_k}$$

$$\begin{aligned} B(x_n) - B(\alpha) = & \sum_{k=0}^{k=n-1} \left(\frac{\log Q_{k+1}^n}{Q_k^n} - \frac{\log Q_{k+1}}{Q_k} \right) \\ & + \left(\frac{\log Q_{n+1}^n}{Q_n^n} - \frac{\log Q_{n+1}}{Q_n} \right) \\ & + \left(\sum_{k>n} \frac{\log Q_{k+1}^n}{Q_k^n} - \sum_{k>n} \frac{\log Q_{k+1}}{Q_k} \right) \end{aligned}$$

$$\begin{aligned} B(x_n) - B(\alpha) = & \, 0 + \frac{\log \frac{Q_{n+1}^n}{Q_{n+1}}}{Q_n} \\ & + \sum_{k>n} \frac{\log Q_{k+1}^n}{Q_k^n} - \sum_{k>n} \frac{\log Q_{k+1}}{Q_k} \end{aligned}$$

In the limit we have:

$$\begin{aligned} \lim_{n\to\infty} (B(x_n) - B(\alpha)) = & \lim_{n\to\infty} \left(\frac{\log \frac{Q_{n+1}^n}{Q_{n+1}}}{Q_n} \right) \\ & + \lim_{n\to\infty} \left(\sum_{k>n} \frac{\log Q_{k+1}^n}{Q_k^n} \right) \\ & - \lim_{n\to\infty} \left(\sum_{k>n} \frac{\log Q_{k+1}}{Q_k} \right) \end{aligned}$$

The second and third terms on the right hand side drop out. In [6] a comparison test is used to show that $\lim_{n\to\infty} \left(\sum_{k>n} \frac{\log Q_{k+1}^n}{Q_k^n} \right) = 0$. On the other hand, $\lim_{n\to\infty} \left(\sum_{k>n} \frac{\log Q_{k+1}}{Q_k} \right) = 0$ because this is the limit of the tail of the converging series corresponding to the Brjuno sum for α.

Lastly, the number b_n was custom designed to give $\lim_{n\to\infty}(\frac{Q_{n+1}^n}{Q_{n+1}e^{zQ_n}}) = 1$. See [6] for details. Thus, $\lim_{n\to\infty}(B(x_n) - B(\alpha)) = z$ or equivalently, $B(x_n) \to y$. □

Using elementary properties of continued fractions, we find that the even terms of $\{x_n\}$ eventually are strictly increasing and the odd terms are eventually strictly decreasing. Thus we have proved the following corollary which gives the necessary endpoint density property for a planar straight brush.

Corollary 2.3. *For each* $(\alpha, y) \in \Lambda$, *there exist sequences of irrational numbers* $\{\beta_n\}$ *and* $\{\zeta_n\}$ *such that* $\beta_n \uparrow \alpha$, $\zeta_n \downarrow \alpha$, $y_{\beta_n} \to y$, *and* $y_{\zeta_n} \to y$.

3. Structure of the Brjuno numbers and a Sequence of Brushes

The Brjuno brush is just the beginning of the story! In this section we describe a sequence of nested subsets inside the Brjuno numbers. Each subset admits a straight brush construction similar to the one we constructed for the Brjuno numbers.

Suppose that the convergents for the irrational number α are given by $\{\frac{P_n}{Q_n}\}$ and suppose that the integer $j \geq 1$ is fixed.

Definitions. The **Brjuno function of order j** is given by $B_j(\alpha) = \sum_{n=0}^{\infty} \frac{\log Q_{n+j}}{Q_n}$ and
the **Brjuno numbers of order j** are given by
$B_j = \{\alpha \in \mathbf{R}\backslash\mathbf{Q} | B_j(\alpha) < \infty\}$.

Of course, $B_1(\alpha)$ is the Brjuno function and B_1 is the set of Brjuno numbers.

In [6] we prove that these sets are properly nested inside the Brjuno numbers:

$$\ldots B_{j+1} \subsetneq B_j \subsetneq B_{j-1} \ldots B_2 \subsetneq B_1 = B$$

As j increases and we go from B_j to B_{j+1}, we squeeze out irrationals which are easier to approximate by rationals, that is to say, the irrational numbers in B_{j+1} are "more irrational" than those in the difference set $B_j \setminus B_{j+1}$. This partition of the irrational numbers leads to the following:

Definition. If α is an irrational number and $\alpha \in B_j$ but $\alpha \notin B_{j+1}$ then the **degree of irrationality** of α is j.

We let $B_\infty = \bigcap_{j \geq 1} B_j$ and we say that a number has an infinite degree of irrationality if it is contained in B_∞. In [6] we show that the Diophantine numbers are contained in B_∞, which is consistent with their reputation as being most poorly approximated by rationals, or most irrational.

We construct a straight brush over each B_j similarly to the Brjuno brush construction for each positive j. We let $\Lambda_j = \{(\alpha, y) \in \mathbf{R}^2 | \alpha \in B_j \text{ and } y \geq B_j(\alpha)\}$. This set is easily shown to be closed in the plane using a nearly identical argument to the one used in Proposition 2.1 for the Brjuno brush case. The endpoint density proof requires a small change.

Proposition 3.1. *For each* $(\alpha, y) \in \Lambda_j$, *there exists a sequence* $\{(x_n, B_j(x_n))\}$ *in* Λ_j *with* $x_n \to \alpha$ *and* $B_j(x_n) \to y$.

Outline of Proof: As with the Brjuno brush case, we define a sequence of irrational numbers $\{x_n\}$ where the front end of the continued fraction for x_n matches that for α and x_n ends with the golden tail. However, whereas in the Brjuno brush proof we inserted one custom-designed integer in the middle, for the Λ_j case, we need to insert j custom-designed and distinct integers in the middle.

In [6] we prove that the following recipe does the job:

$$[a_0, a_1, \dots, a_n, [|b_{n+1}|], [|b_{n+2}|], \dots, [|b_{n+j}|], 1, 1, 1, \dots]$$

where

$$b_{n+1} = \frac{Q_{n+1} e^{\frac{z Q_{n+1-j}}{j}} - Q_{n-1}}{Q_n},$$

$$b_{n+2} = \frac{Q_{n+2}e^{\frac{zQ_{n+2-j}}{j}} - Q_n}{Q_{n+1}e^{\frac{zQ_{n+1-j}}{j}}}, \text{ and}$$

$$b_{n+i} = \frac{Q_{n+i}e^{\frac{zQ_{n+i-j}}{j}} - Q_{n+i-2}e^{\frac{zQ_{n+i-2-j}}{j}}}{Q_{n+i-1}e^{\frac{zQ_{n+i-1-j}}{j}}} \text{ for } 3 \le i \le j.$$

4. Final Comments and Question

We have not explored questions of Hausdorff dimension and Lebesque measure for the straight brushes constructed here. C. McMullen showed that not every straight brush has the same Lebesque measure. In [7] he showed that the Cantor bouquets found in the Julia sets for the exponential family mentioned earlier, have Lebesque measure 0, while on the other hand the Cantor bouquets present in the Julia sets for certain functions involving the sine function have infinite Lebesque measure. In both cases he found the straight brushes to have Hausdorff dimension 2. What about our straight brushes?

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TRINITY COLLEGE, HARTFORD, CT, 10616
E-mail address: eleeQmail.trincoll.edu