MUTUAL APOSYNDESIS OF SYMMETRIC PRODUCTS

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ABSTRACT. It is shown that the $n^{th}$ symmetric product of nondegenerate continua is mutually aposyndetic for all $n \geq 3$.

1. INTRODUCTION

Throughout this paper, $X$ denotes a continuum (nondegenerate, compact, connected, metric space), $2^X$ is the hyperspace of all nonempty closed subsets of $X$ with the Hausdorff metric $H_d$ ([IN], p. 11) and $C(X) = \{K \in 2^X : K$ is connected}; $F_n(X) = \{K \in 2^X : K$ has at most $n$ points$, n = 1, 2, ..., F_n(X)$ is called the $n^{th}$ symmetric product of $X$.

We say that a continuum $X$ is aposyndetic provided that for any two different points $x, y \in X$ there is a subcontinuum $K$ of $X$ such that $x \in \text{int}K$ and $y \notin K$.

We say that a continuum $X$ is mutually aposyndetic provided that for any two different points $x, y \in X$ there exist disjoint subcontinua $K$ and $L$ of $X$ such that $x \in \text{int}K$, $y \in \text{int}L$.

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Aposyndesis was first studied in connection with hyperspaces by Goodykoontz, who proved that $2^X$ and $C(X)$ are aposyndetic ([G], Theorem 1). Also, Illanes has some results about aposyndesis in hyperspaces ([I]). Macías has recently proved that if $X$ is a chainable continuum such that its second symmetric product is mutually aposyndetic, then $X$ is homeomorphic to $[0,1]$ ([M], Theorem 15); therefore, the second symmetric product of a continuum is not always mutually aposyndetic.

Illanes asked the author if $F_n(X)$ is mutually aposyndetic when $n \geq 3$. Our purpose here is to answer Illanes’ question affirmatively.

We note that our result is the analogue for symmetric products of the following fact about cartesian products: The cartesian product of three nondegenerate continua is mutually aposyndetic ([H], Theorem 2).

Our proof involves a number of technical details. After the proof, we comment about another, natural approach which, unfortunately, does not work.

2. MUTUAL APOSYNDESIS IN SYMMETRIC PRODUCTS

Let $X$ be a continuum and let $A_1, \ldots, A_m$ be subsets of $X$. We let $\langle A_1, \ldots, A_m \rangle$ denote $\{ K \in F_n(X) : K \subset \bigcup_{i=1}^{m} A_i \text{ and } K \cap A_i \neq \emptyset \text{ for each } i \in \{1, \ldots, m\} \}.$

It can be proved easily that if $A_1, \ldots, A_m$ are closed (open) subsets of $X$ then $\langle A_1, \ldots, A_m \rangle$ is closed (open) in $F_n(X)$.

**Lemma 1.** Let $X$ be a continuum and let $C_1, \ldots, C_n$ be connected subsets of $X$. If $m \geq n$, then the set $\langle C_1, \ldots, C_n \rangle$ is a connected subset of $F_m(X)$.

**Proof:** Take two different points $\{x_1, \ldots, x_r\}, \{y_1, \ldots, y_s\} \in \langle C_1, \ldots, C_n \rangle$. For each $i \in \{1, \ldots, n\}$, there is a point $x_{j_i} \in C_i$ for some $j_i \in \{1, \ldots, r\}$ and there is a point $y_{k_i} \in C_i$ for some $k_i \in \{1, \ldots, s\}$. We are going to prove the following fact:

(*) There is a connected subset of $\langle C_1, \ldots, C_n \rangle$ which contains both $\{x_1, \ldots, x_r\}$ and $\{x_{j_1}, \ldots, x_{j_n}\}$. 
To prove (*), assume \( \{x_1, \ldots, x_r \} \neq \{x_{j_1}, \ldots, x_{j_n} \} \). Then there is a point \( x_t \in \{x_1, \ldots, x_r \} - \{x_{j_1}, \ldots, x_{j_n} \} \). We know that \( x_t \in C_u \) for some \( u \in \{1, \ldots, n\} \). Define \( f : \{x_{j_1}, \ldots, x_{j_n}, c\} \times C_u \rightarrow \langle C_1, \ldots, C_n \rangle \) by \( f(x_{j_1}, \ldots, x_{j_n}, c) = \{x_{j_1}, \ldots, x_{j_n}, c\} \). It is easy to see that \( f \) is continuous. Consider \( A = f(\{x_{j_1}, \ldots, x_{j_n}\} \times C_u) \). We have that \( A \) is a connected subset of \( \langle C_1, \ldots, C_n \rangle \) and that \( \{x_{j_1}, \ldots, x_{j_n}\} \) and \( \{x_{j_1}, \ldots, x_{j_n}, x_t\} \) are points of \( A \).

If \( \{x_1, \ldots, x_r\} = \{x_{j_1}, \ldots, x_{j_n}, x_t\} \), then (*) is proved. So, assume that \( \{x_1, \ldots, x_r\} \neq \{x_{j_1}, \ldots, x_{j_n}, x_t\} \). Then there is a point \( x_v \in \{x_1, \ldots, x_r\} - \{x_{j_1}, \ldots, x_{j_n}, x_t\} \). Applying the previous argument to \( x_v \), we can find a connected subset of \( \langle C_1, \ldots, C_n \rangle \) that contains both \( \{x_{j_1}, \ldots, x_{j_n}, x_t\} \) and \( \{x_{j_1}, \ldots, x_{j_n}, x_t, x_v\} \).

In this fashion we can construct a connected subset of \( \langle C_1, \ldots, C_n \rangle \) that contains both \( \{x_{j_1}, \ldots, x_{j_n}\} \) and \( \{x_1, \ldots, x_r\} \). Therefore (*) is proved.

Now, define \( g_1 : C_1 \times \{x_{j_2}\} \times \cdots \times \{x_{j_n}\} \rightarrow \langle C_1, \ldots, C_n \rangle \) by \( g_1(c, x_{j_2}, \ldots, x_{j_n}) = \{c, x_{j_2}, \ldots, x_{j_n}\} \). We know that \( g_1 \) is continuous. Consider \( B_1 = g_1(C_1 \times \{x_{j_2}\} \times \cdots \times \{x_{j_n}\}) \). We have that \( B_1 \) is a connected subset of \( \langle C_1, \ldots, C_n \rangle \) that contains both \( \{x_{j_1}, \ldots, x_{j_n}\} \) and \( \{y_{k_1}, x_{j_2}, \ldots, x_{j_n}\} \).

Define \( g_2 : \{y_{k_1}\} \times C_2 \times \{x_{j_3}\} \times \cdots \times \{x_{j_n}\} \rightarrow \langle C_1, \ldots, C_n \rangle \) by \( g_2(y_{k_1}, c, x_{j_3}, \ldots, x_{j_n}) = \{y_{k_1}, c, x_{j_3}, \ldots, x_{j_n}\} \). We know that \( g_2 \) is continuous. Consider \( B_2 = g_2(\{y_{k_1}\} \times C_2 \times \{x_{j_3}\} \times \cdots \times \{x_{j_n}\}) \). We have that \( B_2 \) is a connected subset of \( \langle C_1, \ldots, C_n \rangle \) that contains both \( \{y_{k_1}, x_{j_2}, \ldots, x_{j_n}\} \) and \( \{y_{k_1}, y_{k_2}, x_{j_3}, \ldots, x_{j_n}\} \).

Similarly, we define \( B_3, \ldots, B_n \).

Using \( B_1, B_2, \ldots, B_n \), we can construct a connected subset of \( \langle C_1, \ldots, C_n \rangle \) that contains both \( \{x_{j_1}, \ldots, x_{j_n}\} \) and \( \{y_{k_1}, \ldots, y_{k_n}\} \).

Similar to the proof of (*), we can construct a connected subset of \( \langle C_1, \ldots, C_n \rangle \) that contains both \( \{y_{k_1}, \ldots, y_{k_n}\} \) and \( \{y_1, \ldots, y_s\} \).

Thus, we have constructed a connected subset of \( \langle C_1, \ldots, C_n \rangle \) that contains both \( \{x_1, \ldots, x_r\} \) and \( \{y_1, \ldots, y_s\} \).

Therefore \( \langle C_1, \ldots, C_n \rangle \) is connected. □

**Lemma 2.** Let \( m \geq 3 \). Let \( X \) be a continuum, let \( U, W \) be nonempty proper open subsets of \( X \) and let \( x, y \) be two different
points in $X$. Then the following set is a connected subset of $F_m(X)$:

$$\langle \overline{U}, \overline{W}, X \rangle \cup \langle BdU, BdW, X \rangle \cup \langle BdU, \{x\}, X \rangle \cup \langle \{x\}, \{y\}, X \rangle.$$  

**Proof:** Let $A = \langle BdU, \{x\}, X \rangle \cup \langle \{x\}, \{y\}, X \rangle$. We will show, first, that $A$ is connected. Given $q \in BdU$. Then $\{q, x, y\} \in \langle \{q\}, \{x\}, X \rangle \cap \langle \{x\}, \{y\}, X \rangle$ and, hence, $\langle \{q\}, \{x\}, X \rangle \cap \langle \{x\}, \{y\}, X \rangle \neq \emptyset$. Thus, since $\langle \{q\}, \{x\}, X \rangle$ and $\langle \{x\}, \{y\}, X \rangle$ are connected by Lemma 1, $\langle \{q\}, \{x\}, X \rangle \cup \langle \{x\}, \{y\}, X \rangle$ is connected. Therefore, since

$$A = \left( \bigcup \{\{q\}, \{x\}, X : q \in BdU\} \right) \cup \{\{x\}, \{y\}, X\},$$

we have that $A$ is connected.

Now, let $B = \langle BdU, BdW, X \rangle \cup A$. We will show that $B$ is connected. Given $q \in BdU$ and $r \in BdW$. Then, noticing that $\{q, r, x\} \in \langle BdU, \{x\}, X \rangle$, we see that $\langle \{q\}, \{r\}, X \rangle \cap A \neq \emptyset$. Thus, since $\langle \{q\}, \{r\}, X \rangle$ is connected by Lemma 1 and $A$ is connected (as already proved), $\langle \{q\}, \{r\}, X \rangle \cup A$ is connected. Therefore, since

$$B = \left( \bigcup \{\{q\}, \{r\}, X : q \in BdU, r \in BdW\} \right) \cup A,$$

we have that $B$ is connected.

Finally, let $C = \langle \overline{U}, \overline{W}, X \rangle \cup B$. We will show that $C$ is connected. Let $C_1$ be a component of $\overline{U}$ and let $C_2$ be a component of $\overline{W}$. Then, by the boundary bumping theorem ([N], p. 73), we know that there exist $c_1 \in C_1 \cap BdU$ and $c_2 \in C_2 \cap BdW$; hence, $\{c_1, c_2\} \in \langle C_1, C_2, X \rangle \cap B$. Thus, since $\langle C_1, C_2, X \rangle$ is connected by Lemma 1 and $B$ is connected (as already proved), $\langle C_1, C_2, X \rangle \cup B$ is connected. Therefore, since
\[ C = \left( \bigcup \{ \langle C_1, C_2, X \rangle : C_1 \text{ is a component of } \overline{U} \text{ and } C_2 \text{ is a component of } \overline{W} \} \right) \cup B, \]

we have that \( C \) is connected.

Therefore, \( \langle U, W, X \rangle \cup \langle BdU, BdW, X \rangle \cup \langle BdU, \{x\}, X \rangle \cup \langle \{x\}, \{y\}, X \rangle \) is connected. \( \square \)

**Lemma 3.** Let \( X \) be a continuum and let \( U_1 \subset U_2 \subset \ldots \subset U_n \) be nonempty proper open subsets of \( X \). Then the following set is a connected subset of \( F_m(X) \) for each \( m \geq n+1 \):

\[ D = \langle \overline{U}_1 \rangle \cup \langle BdU_1, \overline{U}_2 \rangle \cup \langle BdU_1, BdU_2, \overline{U}_3 \rangle \cup \ldots \cup \langle BdU_1, \ldots, BdU_{n-1}, \overline{U}_n \rangle \cup \langle BdU_1, \ldots, BdU_n, X \rangle. \]

**Proof:** Take two elements \( A, B \in D \). We will construct a connected subset of \( D \) which contains \( A \) and \( B \). We only consider the case when \( A, B \in \langle \overline{U}_1 \rangle \); the other cases can be reduced to this case by easy arguments.

Take \( A, B \in \langle \overline{U}_1 \rangle, A = \{a_{01}, \ldots, a_{0n}\}, B = \{b_{01}, \ldots, b_{0n}\} \); take \( C_{11}, \ldots, C_{1n} \) and \( K_{11}, \ldots, K_{1n} \) to be the components of \( \overline{U}_1 \) such that \( a_{0i} \in C_{1i} \) and \( b_{0i} \in K_{1i} \) for each \( i \in \{1, \ldots, n\} \). Consider

\[ \langle C_{11}, \ldots, C_{1n} \rangle \subset \langle \overline{U}_1 \rangle \text{ and } \langle K_{11}, \ldots, K_{1n} \rangle \subset \langle \overline{U}_1 \rangle. \]

By Lemma 1, \( \langle C_{11}, \ldots, C_{1n} \rangle \) and \( \langle K_{11}, \ldots, K_{1n} \rangle \) are connected. By the boundary bumping theorem ([N], p. 73), we can take \( \{a_{11}, \ldots, a_{1n}\} \) and \( \{b_{11}, \ldots, b_{1n}\} \) such that \( a_{1i} \in C_{1i} \cap BdU_1 \) and \( b_{1i} \in K_{1i} \cap BdU_1 \) for each \( i \in \{1, \ldots, n\} \); take \( C_{22}, \ldots, C_{2n} \) and \( K_{22}, \ldots, K_{2n} \) to be the components of \( \overline{U}_2 \) such that \( a_{1i} \in C_{2i} \) and \( b_{1i} \in K_{2i} \) for each \( i \in \{2, \ldots, n\} \). Consider

\[ \langle \{a_{11}\}, C_{22}, \ldots, C_{2n} \rangle \subset \langle BdU_1, \overline{U}_2 \rangle \text{ and } \langle \{b_{11}\}, K_{22}, \ldots, K_{2n} \rangle \subset \langle BdU_1, \overline{U}_2 \rangle. \]
By Lemma 1, \(\{a_{11}\}, C_{22}, ..., C_{2n}\) and \(\{b_{11}\}, K_{22}, ..., K_{2n}\)
are connected; thus, since
\[
\{a_{11}, ..., a_{1n}\} \in \langle C_{11}, ..., C_{1n}\rangle \cap \langle \{a_{11}\}, C_{22}, ..., C_{2n}\rangle
\]
and
\[
\{b_{11}, ..., b_{1n}\} \in \langle K_{11}, ..., K_{1n}\rangle \cap \langle \{b_{11}\}, K_{22}, ..., K_{2n}\rangle,
\]
we have that the following two sets are connected:
\[
\langle C_{11}, ..., C_{1n}\rangle \cup \langle \{a_{11}\}, C_{22}, ..., C_{2n}\rangle,
\]
\[
\langle K_{11}, ..., K_{1n}\rangle \cup \langle \{b_{11}\}, K_{22}, ..., K_{2n}\rangle.
\]
Now, by the boundary bumping theorem ([N], p. 73), we can take \(\{a_{22}, ..., a_{2n}\}\) and \(\{b_{22}, ..., b_{2n}\}\) such that \(a_{2i} \in C_{2i} \cap BdU_2\) and \(b_{2i} \in K_{2i} \cap BdU_2\) for each \(i \in \{2, ..., n\}\); take \(C_{33}, ..., C_{3n}\) and \(K_{33}, ..., K_{3n}\) the components of \(U_3\) such that \(a_{2i} \in C_{3i}\) and \(b_{2i} \in K_{3i}\) for each \(i \in \{3, ..., n\}\). Consider
\[
\langle \{a_{11}\}, \{a_{22}\}, C_{33}, ..., C_{3n}\rangle \subset \langle BdU_1, BdU_2, \overline{U_3}\rangle
\]
and
\[
\langle \{b_{11}\}, \{b_{22}\}, C_{33}, ..., C_{3n}\rangle \subset \langle BdU_1, BdU_2, \overline{U_3}\rangle.
\]
By lemma 1, \(\{a_{11}\}, \{a_{22}\}, C_{33}, ..., C_{3n}\) and
\(\{b_{11}\}, \{b_{22}\}, K_{33}, ..., K_{3n}\) are connected; thus, since
\[
\langle a_{11}, a_{22}, a_{23}, ..., a_{2n}\rangle \in \langle \{a_{11}\}, C_{22}, ..., C_{2n}\rangle
\]
\[
\cap \langle \{a_{11}\}, \{a_{22}\}, C_{33}, ..., C_{3n}\rangle
\]
and
\[
\{b_{11}, b_{22}, b_{23}, ..., b_{2n}\} \in \langle \{b_{11}\}, K_{22}, ..., K_{2n}\rangle
\]
\[
\cap \langle \{b_{11}\}, \{b_{22}\}, K_{33}, ..., K_{3n}\rangle
\]
we have that the following two sets are connected:
\[
\langle C_{11}, ..., C_{1n}\rangle \cup \langle \{a_{11}\}, C_{22}, ..., C_{2n}\rangle \cup \langle \{a_{11}\}, \{a_{22}\}, C_{33}, ..., C_{3n}\rangle,
\]
\[
\langle K_{11}, ..., K_{1n}\rangle \cup \langle \{b_{11}\}, K_{22}, ..., K_{2n}\rangle \cup \langle \{b_{11}\}, \{b_{22}\}, K_{33}, ..., K_{3n}\rangle.
\]
Repeating the procedure indicated above, we can find connected sets $C_1$ and $K_1$ which contain $A$ and $B$, respectively, of the form

$$C_1 = \{C_{11}, \ldots, C_{1n}\} \cup \{a_{11}\}, C_{22}, \ldots, C_{2n}\} \cup \ldots \cup \{a_{11}\}, \ldots, \{a_{n-1n-1}\}, C_{nn}\}$$

and

$$K_1 = \{K_{11}, \ldots, K_{1n}\} \cup \{b_{11}\}, K_{22}, \ldots, K_{2n}\} \cup \ldots \cup \{b_{11}\}, \ldots, \{b_{n-1n-1}\}, K_{nn}\}$$

where $a_{ii}, b_{ii} \in BdU_i$ for each $i \in \{1, \ldots, n-1\}$, $C_{11}, \ldots, C_{1n}$, $K_{11}, \ldots, K_{1n}$ are components of $\overline{U_1}$, $C_{22}, \ldots, C_{2n}$, $K_{22}, \ldots, K_{2n}$ are components of $\overline{U_2}$, ..., and $C_{nn}, K_{nn}$ are components of $\overline{U_n}$. Now, by the boundary bumping theorem ([N], p. 73), we can take points $a_{nn}$ and $b_{nn}$ such that $a_{nn} \in C_{nn} \cap BdU_n$ and $b_{nn} \in K_{nn} \cap BdU_n$. Consider

$$C_2 = \{a_{11}, \ldots, \{a_{nn}\}, X\} \subset \{BdU_1, \ldots, BdU_n, X\}$$

and

$$K_2 = \{b_{11}, \ldots, \{b_{nn}\}, X\} \subset \{BdU_1, \ldots, BdU_n, X\}.$$ 

By Lemma 1, $\{a_{11}, \ldots, \{a_{nn}\}, X\}$ and $\{b_{11}, \ldots, \{b_{nn}\}, X\}$ are connected; thus, since

$$\{a_{11}, \ldots, a_{nn}\} \in \{a_{11}, \ldots, \{a_{nn}\}, X\} \cap \{a_{11}, \ldots, \{a_{n-1n-1}\}, C_{nn}\}$$

and

$$\{b_{11}, \ldots, b_{nn}\} \in \{b_{11}, \ldots, \{b_{nn}\}, X\} \cap \{b_{11}, \ldots, \{b_{n-1n-1}\}, C_{nn}\}$$

we have that $C_1 \cup C_2$ and $K_1 \cup K_2$ are connected. Consider

$$J_1 = \{b_{11}, \{a_{22}\}, \ldots, \{a_{nn}\}, X\} \subset \{BdU_1, \ldots, BdU_n, X\},$$
\[ J_2 = \langle \{b_{11}\}, \{b_{22}\}, \{a_{33}\}, \ldots, \{a_{nn}\}, X \rangle \subseteq \langle BdU_1, \ldots, BdU_n, X \rangle, \]

\[
J_{n-1} = \langle \{b_{11}\}, \ldots, \{b_{n-1n-1}\}, \{a_{nn}\}, X \rangle \subseteq \langle BdU_1, \ldots, BdU_n, X \rangle;
\]

by Lemma 1, \( J_1, J_2, \ldots, J_{n-1} \) are connected; also, since
\[ \{a_{11}, \ldots, a_{nn}, b_{11}\} \in C_2 \cap J_1 \]
\[ \{a_{22}, \ldots, a_{nn}, b_{11}, b_{22}\} \in J_1 \cap J_2 \]

\[
\vdots
\]
we have that \( C_1 \cup C_2 \cup J_1 \cup \cdots \cup J_{n-1} \cup K_2 \cup K_1 \) is a connected set. Hence, we found a connected set that contains both \( A \) and \( B \) and is contained in \( D \).

Therefore \( D \) is connected. \( \Box \)

**Theorem 1.** Let \( X \) be a continuum. Then \( F_n(X) \) is mutually aposyndetic for each \( n \geq 3 \).

**Proof:** Take two different elements \( A, B \in F_n(X) \). We need to construct two disjoint subcontinua \( A \) and \( B \) of \( F_n(X) \) such that \( A \in \text{int} \mathcal{A} \) and \( B \in \text{int} \mathcal{B} \). We will consider two cases.

**Case 1.** Suppose that \( A \cap B = \emptyset \). Then we can find \( U_1 \subset U_2 \subset \cdots \subset U_{n-1} \) and \( V_1 \subset V_2 \subset \cdots \subset V_{n-1} \) open subsets of \( X \) such that \( A \subset U_1, B \subset V_1, \overline{U_{n-1}} \cap \overline{V_{n-1}} = \emptyset \) and for each \( i \in \{1, \ldots, n-2\} \), \( \overline{U_i} \subset U_{i+1} \) and \( \overline{V_i} \subset V_{i+1} \). Consider \( A = \langle U_1 \rangle \cup \langle BdU_1, U_2 \rangle \cup \langle BdU_1, BdU_2, U_3 \rangle \cup \cdots \cup \langle BdU_1, \ldots, BdU_{n-2}, U_{n-1} \rangle \) and \( B = \langle V_1 \rangle \cup \langle BdV_1, V_2 \rangle \cup \langle BdV_1, BdV_2, V_3 \rangle \cup \cdots \cup \langle BdV_1, \ldots, BdV_{n-2}, \overline{V_{n-1}} \rangle \cup \langle BdV_1, \ldots, BdV_{n-1}, X \rangle \), we know that
\(A\) and \(B\) are closed subsets of \(F_n(X)\) then by Lemma 3 we have that \(A\) and \(B\) are subcontinua of \(F_n(X)\). We will show that \(A\) and \(B\) are disjoint as follows:

First, since \(U_i \cap V_j = \emptyset\) for each \(i, j \in \{1, \ldots, n-1\}\), it follows that for each \(k, l \in \{1, \ldots, n-1\}\)

\[
\langle BdU_1, \ldots, BdU_k, \overline{U_{k+1}} \rangle \cap \langle BdV_1, \ldots, BdV_l, \overline{V_{l+1}} \rangle = \emptyset.
\]

Second, for each \(k \in \{1, \ldots, n-2\}\), since \(U_{k+1} \cap V_j = \emptyset\) for each \(j \in \{1, \ldots, n-1\}\), we have that

\[
\langle BdU_1, \ldots, BdU_k, \overline{U_{k+1}} \rangle \cap \langle BdV_1, \ldots, BdV_{n-1}, X \rangle = \emptyset.
\]

And third, since \(U_{n-1} \cap V_{n-1} = \emptyset\) and for each for each \(i \in \{1, \ldots, n-2\}\), \(U_i \cap BdU_{i+1} = \emptyset\) and \(V_i \cap BdV_{i+1} = \emptyset\), it follows that

\[
\langle BdU_1, \ldots, BdU_{n-1}, X \rangle \cap \langle BdV_1, \ldots, BdV_{n-1}, X \rangle = \emptyset.
\]

Therefore \(A \cap B = \emptyset\). Also, since \(A \subset U_1, B \subset V_1\) and \(U_1, V_1\) are open subsets of \(X\), we have that \(A \in \langle U_1 \rangle \subset A\) and \(B \in \langle V_1 \rangle \subset B\). Therefore, \(A\) and \(B\) are disjoint subcontinua of \(F_n(X)\) such that \(A \in intA\) and \(B \in intB\).

Case 2. Suppose that \(A \cap B \neq \emptyset\). Since \(A \neq B\), we can suppose, without loss of generality, that there exists a point \(p \in A - B\). Clearly, there exists a point \(q \in A\) different from \(p\). Since \(X\) is metric, we can find open subsets \(U, W\) and \(V_1 \subset V_2 \subset \cdots \subset V_{n-1}\) of \(X\) and two different points \(x, y \in X\) with the following properties: \(p \in U; q \in W; B \subset V_1; U \cap V_{n-1} = \emptyset; x, y \notin (U \cup W \cup V_{n-1}); \) for each \(i \in \{1, \ldots, n-1\}\), \(W \cap BdV_i = \emptyset\); and for each \(i \in \{1, \ldots, n-2\}\), \(V_i \subset V_{i+1}\).

Consider \(A = \langle U, W, X \rangle \cup \langle BdU, BdW, X \rangle \cup \langle BdU, \{x\}, X \rangle \cup \{\{x\}, \{y\}, X\}\) and \(B = \langle V_1 \rangle \cup \langle BdV_1, V_2 \rangle \cup \langle BdV_1, BdV_2, V_3 \rangle \cup \cdots \cup \langle BdV_1, \ldots, BdV_{n-2}, V_{n-1} \rangle \cup \langle BdV_1, \ldots, BdV_{n-1}, X \rangle\). We know that \(A\) and \(B\) are closed subsets of \(F_n(X)\); hence, by Lemma 2 and Lemma 3 (correspondingly), we have that \(A\) and \(B\) are subcontinua of \(F_n(X)\). We will show that \(A\) and \(B\) are disjoint as follows:
First, since $\overline{U} \cap \overline{V_{n-1}} = \emptyset$, it follows that for each $k \in \{1, \ldots, n-2\}$
\[
(\langle \overline{U}, \overline{W}, X \rangle \cup \langle BdU, BdW, X \rangle \cup \langle BdU, \{x\}, X \rangle)
\cap \langle BdV_1, \ldots, BdV_k, \overline{V_{k+1}} \rangle = \emptyset.
\]
Second, since $x, y \notin \overline{V_{n-1}}$, we have that
\[
\langle \{x\}, \{y\}, X \rangle \cap \langle BdV_1, \ldots, BdV_{n-1}, X \rangle = \emptyset.
\]
Third, since $\overline{U} \cap \overline{V_{n-1}} = \emptyset$, for each $i \in \{1, \ldots, n-1\}$, $\overline{W} \cap BdVi = \emptyset$ and, for each $i \in \{1, \ldots, n-2\}$, $\overline{V_i} \cap BdV_{i+1} = \emptyset$, it follows that
\[
\langle \overline{U}, \overline{W}, X \rangle \cap \langle BdV_1, \ldots, BdV_{n-1}, X \rangle = \emptyset
\]
and
\[
\langle BdU, BdW, X \rangle \cap \langle BdV_1, \ldots, BdV_{n-1}, X \rangle = \emptyset.
\]
And fourth, since $\overline{U} \cap \overline{V_{n-1}} = \emptyset$ and $x, y \notin \overline{V_{n-1}}$, we have that
\[
(\langle BdU, \{x\}, X \rangle \cup \langle \{x\}, \{y\}, X \rangle) \cap \langle BdV_1, \ldots, BdV_{n-1}, X \rangle = \emptyset.
\]
Therefore $A \subset B = \emptyset$.

Next, note that $p \in U$, $q \in W$, $B \subset V_1$ and $U, W, V_1$ are open subsets of $X$; hence, we have that $A \subset \langle U, W, X \rangle \subset A$ and $B \subset \langle V_1 \rangle \subset B$. Therefore, $A$ and $B$ are disjoint subcontinua of $F_n(X)$ such that $A \in intA$ and $B \in intB$.

Therefore $F_n(X)$ is mutually aposyndetic.

Symmetric products are related to cartesian products; in particular, letting $X^n$ denote cartesian product, there is a natural map $\pi_n : X^n \rightarrow F_n(X)$ given by $\pi_n(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\}$. If $\pi_n$ is open for $n \geq 3$, a simple proof of our theorem might be possible using the Theorem 2 of [H]. It is known that $\pi_2$ is open ([M], Lemma 9). However, we now show, $\pi_n$ never is open for $n \geq 3$:

**Proposition 1.** Let $X$ be a continuum. Then $\pi_n$ is not open for $n \geq 3$. 

Proof: Let $p, q$ be two different points of $X$, take sequences
\[ \{p_k\}_{k \in \mathbb{N}} \text{ and } \{q_k\}_{k \in \mathbb{N}} \]
such that $p_k \to q$, $q_k \to q$ and $p_k \neq q_l$ for every \( k, l \in \mathbb{N} \). Let \( \epsilon > 0 \) such that $B_\epsilon(p) \cap B_\epsilon(q) = \emptyset$. Let \( U = B_\epsilon(p) \times B_\epsilon(p) \times \cdots \times B_\epsilon(p) \times B_\epsilon(q) \subset X^n \). Then $U$ is an open subset of $X^n$ that contains $(p, p, \ldots, p, q)$.

Suppose that $\pi_n$ is open. Then $\pi_n(U)$ is an open set of $F_n(X)$ which contains $\{p, q\}$; since $p_n \to q$ and $q_n \to q$, there exits $N \in \mathbb{N}$ such that $\{p, p_N, q_N\} \in \pi_3(U)$ and $p_N, q_N \in B_\epsilon(q)$. Note that there are $n!$ points in $\pi_n^{-1}(\{p, p_N, q_N\})$; each of those points has two coordinates in $B_\epsilon(q)$. Hence, $\{p, p_N, q_N\} \notin \pi_n(U)$ which is a contradiction.

Therefore $\pi_n$ is not open. \( \square \)

References


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