Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
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E-mail: topolog@auburn.edu
ISSN: 0146-4124

Topology Proceedings
Volume 24, Summer 1999, 71–88

UNIFORM AND PROXIMAL HYPERSPACES FOR BITOPOLOGICAL SPACES

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Abstract

Hyperspace operators are defined for biquasiproximity spaces and biquasiuniform spaces analogous to the asymmetric hyperspace operator for bitopological spaces which we investigated in previous papers. We find characterizations of those bispaces for which the different structures on the hyperspace are compatible, and we look at various completeness properties in these hyperspaces.

We introduced a bitopological hyperspace operator and investigated some of its properties in [B2], [B3], and [B4]. Given a bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$ let 2^X be the set of non-empty subsets of X which are closed relative to $\mathcal{T}$. If $\mathcal{A}$ is a family of subsets of X, let $\langle \mathcal{A} \rangle = \{B \in 2^X | \ B \subseteq \cup \mathcal{A}$ and for each $A \in \mathcal{A}, A \cap B \neq \phi\}$. If $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ we may write $\langle \mathcal{A} \rangle$ as $\langle A_1, A_2, \ldots, A_n \rangle$. Let $L(\mathcal{T})$ be the topology on 2^X generated by the subbasis consisting of sets of the form $\langle O, X \rangle$ where $O \in \mathcal{T}$ and let $U(\mathcal{T}^*)$ be the topology on 2^X generated by the basis consisting of sets of the form $\langle O \rangle$ where $O \in \mathcal{T}^*$. Then $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$ is the *hyperspace* of $(X, \mathcal{T}, \mathcal{T}^*)$. Sometimes it will be convenient to denote this hyperspace by $(2^X, \mathcal{L}, \mathcal{U})$.

Our terminology is largely influenced by [Ko]. The *dual* of the bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$ is the bitopological space

Mathematics Subject Classification: 54B20, 54E05, 54E15, 54E35, 54E50, 54E55

Key words: Bitopological Space, Hyperspace, Proximity, Quasiproximity, Uniformity, Quasiuniformity, Metric, Quasimetric, Completeness

$(X, \mathcal{T}^*, \mathcal{T})$. When convenient we may designate a bitopological space by X and its dual by X^*. The $\mathcal{T}$-closure operator in X is designated by c and the $\mathcal{T}^*$-closure operator is c^*. We define interior operators Int and Int* analogously to c and c^*. For any $A \subseteq X$ define sat $A = \cap\{O \in \mathcal{T} | \ A \subseteq O\}$ and let cosat $A = X - \text{sat}\ (X - A)$.

A map $f : X \to Y$ is a *continuous* map $f : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ of bitopological spaces if it is continuous with respect to first topologies $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ and with respect to second topologies $f : (X, \mathcal{T}^*) \to (Y, \mathcal{U}^*)$. The standard bitopological structure for the unit interval I is (I, L, U) where L is the topology consisting of the empty set together with sets of the form $(a, 1]$ and U is the topology consisting of the empty set together with sets of the form $[0, a)$.

A space $(X, \mathcal{T}, \mathcal{T}^*)$ is an R_0 space if for any $x \in X$ and any $O \in \mathcal{T}$ if $x \in O$ then $c^*x \subseteq O$. We will designate by R_0^* the property that the dual space is R_0. A space $(X, \mathcal{T}, \mathcal{T}^*)$ is a *completely regular* space if for any $x \in X$ and any $O \in \mathcal{T}$ if $x \in O$ then there is a continuous function $f : (X, \mathcal{T}, \mathcal{T}^*) \to (I, L, U)$ such that $f(x) = 1$ and $f[X - O] = \{0\}$. A space $(X, \mathcal{T}, \mathcal{T}^*)$ is a *normal* space if for any $\mathcal{T}^*$-closed $A \subseteq X$ and any $O \in \mathcal{T}$ if $A \subseteq O$ then there is an $O' \in \mathcal{T}$ such that $A \subseteq O'$ and $c^*O' \subseteq O$. Note that normal is a self-dual property; X is normal iff X^* is. (See [Ke] for a treatment of normal and completely regular under slightly different names.)

A bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$ is *compact* if every $\mathcal{T}^*$-closed subset of X is $\mathcal{T}$-compact [B3]. A bispace $(X, \mathcal{T}, \mathcal{T}^*)$ is *sup-compact* if $(X, \mathcal{T} \vee \mathcal{T}^*)$ is compact. We note that by the Alexander subbase theorem a bispace X is sup-compact if and only if both X and X^* are compact. For a bispace $(X, \mathcal{T}, \mathcal{T}^*)$ define $K(X)$ to be the set of non-empty, $\mathcal{T}$-closed, $\mathcal{T}^*$-compact subsets of X.

When we use a result about spaces with only one topology we will refer to this as a result from *traditional* topology. For a traditional space $(X, \mathcal{T})$ the topologies $L(\mathcal{T})$ and $U(\mathcal{T})$ are respectively the *lower Vietoris* and *upper Vietoris* topologies on 2^X, and their supremum is the *Vietoris* topology [Vi], [Mi].

1. The Weak (Nachman) Biquasiproximal Hyperspace

Proximity structures for hyperspaces of proximity spaces were defined via hyperuniformities by Nachman [Na]. We give a hyperspace construction here which does not make use of (quasi) uniformities and we will show in the next section that this construction is the bispace version of Nachman's "weak hyperproximity".

Suppose $\ll$ and $\ll^*$ are relations on $\mathcal{P}(X)$, the power set of X, where the statement $A \ll B$ is to be interpreted as A is strongly in B. If $\ll$ and $\ll^*$ satisfy the axioms for strong containment relations generated by quasiproximities (see [De]) then $(X, \ll, \ll^*)$ is a biquasiproximity space. If $\ll$ and $\ll^*$ generate the topologies $\mathcal{T}$ and $\mathcal{T}^*$, respectively, then we will say $(X, \ll, \ll^*)$ is *compatible* with $(X, \mathcal{T}, \mathcal{T}^*)$. A map $f : X \to Y$ is a *proximal* map $f : (X, \ll, \ll^*) \to (Y, \ll^{**}, \ll^{***})$ of bitopological spaces if it is a proximal map (i.e., strong containment is preserved by f^{-1}) with respect to $f : (X, \ll) \to (Y, \ll^{**})$ and with respect to $f : (X, \ll^*) \to (Y, \ll^{***})$.

The *dual* of $(X, \ll, \ll^*)$ is $(X, \ll^*, \ll)$. The operation that corresponds to the inverse quasiproximity is the following: Given the relation $\ll$ define $\ll^\dagger$ by $A \ll^\dagger B$ iff $X - B \ll X - A$. Then the *conjugate* of $(X, \ll, \ll^*)$ is $(X, \ll^\dagger, \ll^{*\dagger})$ and the *adjoint* of $(X, \ll, \ll^*)$ is $(X, \ll^{*\dagger}, \ll^\dagger)$, the dual of the conjugate. A space $(X, \ll, \ll^*)$ is self-dual iff $\ll = \ll^*$ and is self-adjoint iff $\ll = \ll^{*\dagger}$. The advantage of using two structures instead of one, in this section and the next two sections, is that in the last section, on completeness, we will have a notational framework which will make it possible to contrast the self-adjoint and self-dual cases.

For any bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$, there is a pair of quasiproximities $\ll$ and $\ll^*$ on X which generate $\mathcal{T}$ and $\mathcal{T}^*$, respectively. However, for the bispace $(X, \ll, \ll^*)$ to be self-adjoint it is necessary and sufficient that X and X^* be completely regular [De, Theorem 9.3].

For a space $(X, \ll, \ll^*)$ let 2^X be the set of non-empty $\ll$-closed subsets of X. For a relation $\ll$ on $\mathcal{P}(X)$ define a relation $\ll^{\mathcal{H}}$ by saying that $\mathcal{A}, \mathcal{B} \in \mathcal{P}(2^X)$ satisfy $\mathcal{A} \ll^{\mathcal{H}} \mathcal{B}$ iff there exists a finite set of pairs $\{(A_1, B_1), \ldots, (A_n, B_n)\} \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ such that $A_i \ll B_i$ for each i, $\mathcal{A} \subseteq \langle X, A_1, \ldots, A_n\rangle$, and $\langle X, B_1, \ldots, B_n\rangle \subseteq \mathcal{B}$.

Proposition 1.1. *If $\ll$ is a quasiproximity then $\ll^{\mathcal{H}}$ is a quasiproximity.*

Proof. If $\langle X, B_1, \ldots, B_m\rangle \subseteq \mathcal{B}$ and $\langle X, C_1, \ldots, C_n\rangle \subseteq \mathcal{C}$ then $\langle X, B_1, \ldots, B_m, C_1, \ldots, C_n\rangle \subseteq \mathcal{B} \cap \mathcal{C}$. This shows that if $\mathcal{A} \ll^{\mathcal{H}} \mathcal{B}$ and $\mathcal{A} \ll^{\mathcal{H}} \mathcal{C}$ then $\mathcal{A} \ll^{\mathcal{H}} \mathcal{B} \cap \mathcal{C}$.

If $A \ll C$ and $B \ll D$ then $A \cup B \ll C \cup D$. If $\mathcal{A} \subseteq \langle X, A_1, \ldots, A_m\rangle$ and $\mathcal{B} \subseteq \langle X, B_1, \ldots, B_n\rangle$ then $\mathcal{A} \cup \mathcal{B} \subseteq \langle \{X\} \cup \{A_i \cup B_j |\ i = 1, \ldots, m; j = 1, \ldots, n\}\rangle$. If $\langle X, C_1, \ldots, C_m\rangle \subseteq \mathcal{C}$ and $\langle X, D_1, \ldots, D_n\rangle \subseteq \mathcal{C}$ then

$$\langle \{X\} \cup \{C_i \cup D_j |\ i = 1, \ldots, m; j = 1, \ldots, n\}\rangle \subseteq \mathcal{C}.$$

This shows that if $\mathcal{A} \ll^{\mathcal{H}} \mathcal{C}$ and $\mathcal{B} \ll^{\mathcal{H}} \mathcal{C}$ then $\mathcal{A} \cup \mathcal{B} \ll^{\mathcal{H}} \mathcal{C}$.

The other properties of a quasiproximity are easily verified. $\square$

We define $(2^X, \ll^{\mathcal{H}}, \ll^{*\dagger\mathcal{H}\dagger})$ to be the *hyperspace* of $(X, \ll, \ll^*)$. Note that the hyperspace of a self-adjoint space is self-adjoint.

We see $\mathcal{A} \ll^{*\dagger\mathcal{H}\dagger} \mathcal{B}$ iff there exists a finite set of pairs $\{(A_1, B_1), \ldots, (A_n, B_n)\} \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ such that $A_i \ll^* B_i$ for each i, $\mathcal{A} \subseteq \cup_{i=1}^n \langle A_i\rangle$, and $\cup_{i=1}^n \langle B_i\rangle \subseteq \mathcal{B}$. This implies that the topology generated by $\ll^{*\dagger\mathcal{H}\dagger}$ has for a basis sets of the form $\{A \in 2^X |\ A \ll^* O\}$ where O ranges over the $\ll^*$-open sets. Thus this topology is a quasiproximal form of the *proximal hypertopology.* (See [DN] for a comparison of this to related hypertopologies.)

Proposition 1.2. *Let $(X, \ll, \ll^*)$ be a biquasiproximity space compatible with the bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$. Then the following are equivalent:*

(1) $(2^X, \ll^{\mathcal{H}}, \ll^{*\dagger\mathcal{H}\dagger})$ is compatible with $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$.

(2) For any $\mathcal{T}$-closed set A and any $O \in \mathcal{T}^*$ with $A \subseteq O$ there is an $O' \in \mathcal{T}^*$ with $A \ll^* O'$ and $\operatorname{cosat} O' \subseteq O$.

Let $(I, \ll^L, \ll^U)$ be the unique biquasiproximal space compatible with (I, L, U). If we restrict to the case where $(X, \ll, \ll^*)$ is self-adjoint then we may add the following property to the list above:

(3) $(X, \mathcal{T}, \mathcal{T}^*)$ is normal and for any continuous function $f : (X, \mathcal{T}, \mathcal{T}^*) \to (I, L, U)$ we have $f : (X, \ll, \ll^*) \to (I, \ll^L, \ll^U)$ is a proximal map.

Proof. (1) implies (2): Given a $\mathcal{T}$-closed set A and an $O \in \mathcal{T}^*$ with $A \subseteq O$ we have $A \in \langle O \rangle$ and the basic open set $\langle O \rangle \in U(\mathcal{T}^*)$ is $\ll^{*\dagger\mathcal{H}\dagger}$-open. So there is an $O'' \in \mathcal{T}^*$ such that $A \ll^* O''$ and for any $\mathcal{T}$-closed $B \ll^* O''$ we have $B \in \langle O \rangle$. Take $O' \in \mathcal{T}^*$ with $A \ll^* O' \ll^* O''$. Then for any $\mathcal{T}$-closed $B \subseteq O'$ we have $B \in \langle O \rangle$. This last is equivalent to $\operatorname{cosat} O' \subseteq O$.

(2) implies (1): $\ll^{\mathcal{H}}$ generates $L(\mathcal{T})$ even without assuming (2). Given a subbasic open set $\langle X, O \rangle \in L(\mathcal{T})$ and given $A \in \langle X, O \rangle$, choose $x \in A \cap O$. Then $A \in \langle X, \{x\} \rangle \subseteq \langle X, O \rangle$. Thus $\langle X, O \rangle$ is $\ll^{\mathcal{H}}$-open.

Conversely, if $\mathcal{O}$ is $\ll^{\mathcal{H}}$-open, and $A \in \mathcal{O}$, then we have a finite set of pairs $\{(A_1, B_1), \ldots, (A_n, B_n)\} \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ such that $A_i \ll B_i$ for each i, $A \in \langle X, A_1, \ldots, A_n \rangle$, and $\langle X, B_1, \ldots, B_n \rangle \subseteq \mathcal{O}$. Then $A \in \langle X, \operatorname{Int} B_1, \ldots, \operatorname{Int} B_n \rangle \subseteq \mathcal{O}$.

With regard to second topologies, we note that the basis elements for the $\ll^{*\dagger\mathcal{H}\dagger}$-open sets are always $U(\mathcal{T}^*)$-open, and by property (2) the basis elements for $U(\mathcal{T}^*)$ are $\ll^{*\dagger\mathcal{H}\dagger}$-open.

Self-adjoint plus (2) implies (3): By self-adjointness X^* is completely regular and so X is R_0^*. By R_0^*, $\operatorname{cosat} O = O$ for any $O \in \mathcal{T}^*$. So in property (2) we may simply say that $A \ll^* O$.

Given a $\mathcal{T}$-closed set A with $A \subseteq O \in \mathcal{T}^*$, by (2) we have $A \ll^* O$. Then there is a B such that $A \ll^* B \ll^* O$. But this gives $A \subseteq \operatorname{Int}^* B \subseteq cB \subseteq O$, where the closure c would, a priori, be relative to $\ll^{*\dagger}$, but by self-adjointness this is the same as $\ll$. So $(X, \mathcal{T}, \mathcal{T}^*)$ is normal.

Given a continuous function $f : (X, \mathcal{T}, \mathcal{T}^*) \to (I, L, U)$, suppose that $A, B \subseteq I$ with $A \ll^U B$. Then $cA \subseteq \mathrm{Int}^* B$, assuming self-adjointness as above. So $f^{-1}[cA] \ll^* f^{-1}[\mathrm{Int}^* B]$ by property (2). Then $f : (X, \ll^*) \to (I, \ll^U)$ is proximal, and by self-adjointness so is $f : (X, \ll, \ll^*) \to (I, \ll^L, \ll^U)$.

(3) implies (2): Given a $\mathcal{T}$-closed set A with $A \subseteq O \in \mathcal{T}^*$, by normality there is a continuous function $f : (X, \mathcal{T}, \mathcal{T}^*) \to (I, L, U)$ such that $f[A] = \{0\}$ and $f[X - O] = \{1\}$ [Ke]. By (3) this f is proximal and so since $\{0\} \ll^U [0, 1)$ we have $A \ll^* O$. □

This proposition above is intended as the biquasiproximal version of the Theorem 3.4 in [Mi].

2. The Bourbaki Biquasiuniform Hyperspace

Our reference for quasiuniformities is [FL]. Quasiuniformities for hyperspaces were defined by Stager [Sta] and were studied by Levine and Stager [LS], Berthiaume [Br], and Francaviglia, Lechicki, and Levi [FLL].

Suppose $\mathcal{U}$ and $\mathcal{U}^*$ are quasiuniformities for X. Then $(X, \mathcal{U}, \mathcal{U}^*)$ is a biquasiuniform space. The *dual* of $(X, \mathcal{U}, \mathcal{U}^*)$ is $(X, \mathcal{U}^*, \mathcal{U})$ and the *inverse* of $(X, \mathcal{U}, \mathcal{U}^*)$ is $(X, \mathcal{U}^{-1}, \mathcal{U}^{*-1})$. The *adjoint* of a space is the dual of the inverse. The properties of self-dual and self-adjoint have the obvious definitions.

For any bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$, there is a biquasiuniform space $(X, \mathcal{U}, \mathcal{U}^*)$ compatible with $(X, \mathcal{T}, \mathcal{T}^*)$. However, if we require this $(X, \mathcal{U}, \mathcal{U}^*)$ to be self-adjoint it is necessary and sufficient that X and X^* be completely regular. [De, Theorem 9.3].

If $(X, \mathcal{U}, \mathcal{U}^*)$ is a space let 2^X be the set of non-empty $\mathcal{U}$-closed subsets of X. If R is a relation on X let $H(R)$ be the set of pairs $(A, B) \in 2^X \times 2^X$ such that $A \subseteq R[B]$. For a quasiuniformity $\mathcal{V}$ on X let $\mathcal{H}(\mathcal{V})$ be the quasiuniformity generated by the basis consisting of relations $H(V)$ for $V \in \mathcal{V}$.

The *hyperspace* of $(X,\mathcal{U},\mathcal{U}^*)$ is $(2^X,\mathcal{H}(\mathcal{U}^{-1}),(\mathcal{H}(\mathcal{U}^*))^{-1})$. Note that the hyperspace of a self-adjoint space is self-adjoint. For a relation V, note that $(A,B)\in H(V^{-1})$ iff $B\in\langle\{X\}\cup\{V[x]|\ x\in A\}\rangle$ and $(A,B)\in H(V)^{-1}$ iff $B\in\langle V[A]\rangle$.

If $(X,\mathcal{U},\mathcal{U}^*)$ is self-adjoint then $\sup\{\mathcal{H}(\mathcal{U}^{-1}),(\mathcal{H}(\mathcal{U}^*))^{-1}\}$ is a uniformity on 2^X. If $(X,\mathcal{U},\mathcal{U}^*)$ is self-dual then the supremum of $\mathcal{H}(\mathcal{U}^{-1})$ and $(\mathcal{H}(\mathcal{U}^*))^{-1}$ is the quasiuniformity $2^{\mathcal{U}}$ on 2^X defined by Levine and Stager [LS] (but in [KR] this is called the Bourbaki quasiuniformity). If $(X,\mathcal{U},\mathcal{U}^*)$ is self-dual and self-adjoint (i.e., $\mathcal{U}=\mathcal{U}^*$ is a uniformity) then the supremum of $\mathcal{H}(\mathcal{U}^{-1})$ and $(\mathcal{H}(\mathcal{U}^*))^{-1}$ is the Bourbaki uniformity $2^{\mathcal{U}}$ [Bo, Chap. II, §2., Ex. 7].

Proposition 2.1. [KR] *$\mathcal{H}(\mathcal{U})$ is a totally bounded quasiuniformity if and only if $\mathcal{U}$ is.*

Proposition 2.2. *Suppose $\mathcal{U}$ and $\mathcal{U}^*$ are totally bounded quasiuniformities and $(X,\mathcal{U},\mathcal{U}^*)$ is a biquasiuniform space compatible with the biquasiproximity space $(X,\ll,\ll^*)$. Then $(2^X,\mathcal{H}(\mathcal{U}^{-1}),(\mathcal{H}(\mathcal{U}^*))^{-1})$ is compatible with $(2^X,\ll^{\mathcal{H}},\ll^{*\dagger\mathcal{H}\dagger})$.*

Proof. Suppose $\mathcal{A}\ll^{\mathcal{H}}\mathcal{B}$. Then there is a set of pairs $\{(A_1,B_1),\ldots,(A_n,B_n)\}$ and a $U\in\mathcal{U}$ such that

$$\mathcal{A}\subseteq\langle X,A_1,\ldots,A_n\rangle,\langle X,B_1,\ldots,B_n\rangle\subseteq\mathcal{B},$$

and for each i, $U[A_i]\subseteq B_i$. If $C\in H(U^{-1})[\mathcal{A}]$ then for some $A\in\mathcal{A}$ we have $C\cap U[x]\neq\phi$ for each $x\in A$. Therefore $C\cap U[A\cap A_i]\neq\phi$ for each i, and so $C\in\langle X,B_1,\ldots,B_n\rangle$. Thus $H(U^{-1})[\mathcal{A}]\subseteq\mathcal{B}$.

On the other hand, suppose $H(U^{-1})[\mathcal{A}]\subseteq\mathcal{B}$. Choose a $V\in\mathcal{U}$ with $V\circ V\subseteq U$. Since $\mathcal{U}$ is totally bounded we may choose sets S_i for $i=1,\ldots,m$ such that $\cup_{i=1}^m S_i=X$ and for each i, $S_i\times S_i\subseteq V$. Let D_j for $j=1,\ldots,n$ be an enumeration of all the unions of the S_i's which intersect each $A\in\mathcal{A}$, noting that one of these D_j's, say D_1, is equal to X. Let $E_j=V[D_j]$ for each j. Then $\mathcal{A}\subseteq\langle D_j|\ j=1,\ldots,n\rangle\ll^{\mathcal{H}}\langle E_j|\ j=1,\ldots,n\rangle\subseteq\mathcal{B}$, with the last inclusion following from the observation that if $B\notin\mathcal{B}$ then for each $A\in\mathcal{A}$ there is an $x_A\in A$ with $B\cap V\circ V[x_A]=\phi$.

Each of these x_A's is in some S_i; so some D_j is the union of these S_i's as A ranges over $\mathcal{A}$. $B \cap V[D_j] = \phi$ for this j.

The same proof can be used to show that $(\mathcal{H}(\mathcal{U}^*))^{-1}$ is compatible with $\ll^{*\dagger\mathcal{H}\dagger}$. □

Corollary 2.1. *Suppose* $(X, \ll, \ll^*)$ *is a biquasiproximity space and* $(X,\mathcal{U},\mathcal{U}^*)$ *is the coarsest biquasiuniform space compatible with* $(X, \ll, \ll^*)$. *Then* $(2^X, \mathcal{H}(\mathcal{U}^{-1}), (\mathcal{H}(\mathcal{U}^*))^{-1})$ *is the coarsest biquasiuniform space compatible with* $(2^X, \ll^{\mathcal{H}}, \ll^{*\dagger\mathcal{H}\dagger})$.

Proof. By Proposition 2.2, $(2^X, \mathcal{H}(\mathcal{U}^{-1}), (\mathcal{H}(\mathcal{U}^*))^{-1})$ is compatible with $(2^X, \ll^{\mathcal{H}}, \ll^{*\dagger\mathcal{H}\dagger})$ and by Proposition 2.1 $(2^X, \mathcal{H}(\mathcal{U}^{-1}), (\mathcal{H}(\mathcal{U}^*))^{-1})$ is totally bounded, making it the coarsest biquasiuniform space compatible with $(2^X, \ll^{\mathcal{H}}, \ll^{*\dagger\mathcal{H}\dagger})$ [FL]. □

This last corollary shows that our biquasiproximal hyperspace is the bispace version of the "weak hyperproximity" in [Na]. The weak hyperproximity was defined as the proximity generated by the hyperuniformity of the coarsest uniform space compatible with the given proximity.

Combining Proposition 2.2 with Proposition 1.2 could give us conditions under which the hyperspace quasiuniformities are compatible with the hyperspace topologies, but it turns out that weaker conditions are sufficient. The following lemma is essentially Berthiaume's [Br] Lemma 2.4. It has been altered to apply to the non-empty closed sets 2^X rather than the power set of X. (Compare with Lemma 3.2 in [Mi].)

Lemma 2.1. *For a biquasiuniform space* $(X,\mathcal{U},\mathcal{U}^*)$ *compatible with* $(X, \mathcal{T}, \mathcal{T}^*)$ *and an* $A \in 2^X$ *we have the following:*

(1) The neighborhood system of A *in* $L(\mathcal{T})$ *is the same as the neighborhood system of* A *in* $\mathcal{H}(\mathcal{U}^{-1})$ *if and only if* A *is precompact in* $(X,\mathcal{U}^{-1})$.

(2) The neighborhood system of A *in* $U(\mathcal{T}^*)$ *is the same as the neighborhood system of* A *in* $(\mathcal{H}(\mathcal{U}^*))^{-1}$ *if and only if for each* $\mathcal{U}^*$*-open* O *with* $A \subseteq O$ *there is a* $U \in \mathcal{U}^*$ *with* $\operatorname{cosat}(U[A]) \subseteq O$.

Proposition 2.3. *Let $(X,\mathcal{U},\mathcal{U}^*)$ be an R_0^* biquasiuniform space generating the bitopological space $(X,\mathcal{T},\mathcal{T}^*)$. Then $(2^X,\mathcal{H}(\mathcal{U}^{-1}),(\mathcal{H}(\mathcal{U}^*))^{-1})$ is compatible with $(2^X,L(\mathcal{T}),U(\mathcal{T}^*))$ if and only if the following conditions are met:*

(1) $(X,\mathcal{U}^{-1})$ is hereditarily precompact.

(2) For any $\mathcal{T}$-closed set A and any $O\in\mathcal{T}^*$ with $A\subseteq O$ there is a $U\in\mathcal{U}^*$ such that $U[A]\subseteq O$.

Proposition 2.4. *For a self-adjoint bispace $(X,\mathcal{U},\mathcal{U}^{-1})$ which is compatible with $(X,\mathcal{T},\mathcal{T}^*)$ it is the case that*

$$(K(X),\mathcal{H}(\mathcal{U}^{-1})|_{K(X)},(\mathcal{H}(\mathcal{U}^{-1}))^{-1}|_{K(X)})$$

is compatible with

$$(K(X),L(\mathcal{T})|_{K(X)},U(\mathcal{T}^*)|_{K(X)}).$$

In particular, if X^* is compact then $2^X=K(X)$ and so $(2^X,\mathcal{H}(\mathcal{U}^{-1}),(\mathcal{H}(\mathcal{U}^{-1}))^{-1})$ is compatible with $(2^X,L(\mathcal{T}),U(\mathcal{T}^*))$.

Corollary 2.2. *For a self-adjoint sup-compact space $(X,\mathcal{U},\mathcal{U}^{-1})$, we have $(2^X,\mathcal{H}(\mathcal{U}^{-1}),(\mathcal{H}(\mathcal{U}^{-1}))^{-1})$ is sup-compact.*

Proof. If X is sup-compact then X and X^* are compact. In [B2] we showed that if X is compact then $(2^X,L(\mathcal{T}),U(\mathcal{T}^*))$ is sup-compact. By Proposition 2.4 if X^* is compact then $(2^X,\mathcal{H}(\mathcal{U}^{-1}),(\mathcal{H}(\mathcal{U}^{-1}))^{-1})$ is compatible with $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$. So $(2^X,\mathcal{H}(\mathcal{U}^{-1}),(\mathcal{H}(\mathcal{U}^{-1}))^{-1})$ is sup-compact. □

Corollary 2.2 stands in contrast to Example 1 of [KR] which shows that in the self-dual case $(X,\mathcal{U},\mathcal{U})$ may be sup-compact while $(2^X,\mathcal{H}(\mathcal{U}^{-1}),(\mathcal{H}(\mathcal{U}))^{-1})$ isn't.

Corollary 2.3. *For a bitopological space $(X,\mathcal{T},\mathcal{T}^*)$ let $\mathcal{U}$ and $\mathcal{U}^*$ be the Pervin quasiunifomities for $\mathcal{T}$ and $\mathcal{T}^*$, respectively. Then $(2^X,\mathcal{H}(\mathcal{U}^{-1}),(\mathcal{H}(\mathcal{U}^*))^{-1})$ is compatible with $(2^X,L(\mathcal{T}),U(\mathcal{T}^*))$.*

This last result was stated by Levine and Stager [LS, Theorem 2.1.5]. A stronger statement was erroneously made by Berthiaume [Br, Lemma 3.1] to the effect that if $\mathcal{U}$ and $\mathcal{U}^*$ are the Pervin quasiunifomities for $\mathcal{T}$ and $\mathcal{T}^*$, respectively, then $\mathcal{H}(\mathcal{U}^{-1})$ and $(\mathcal{H}(\mathcal{U}^*))^{-1}$ are the Pervin quasiunifomities for $L(\mathcal{T})$

and $U(\mathcal{T}^*)$, respectively. This is not true, as the following example shows.

Example 2.1. Let $(X,\mathcal{T})$ be an infinite discrete space and let $\mathcal{U}$ be the Pervin quasiuniformity for $\mathcal{T}$. ($\mathcal{U}$ is actually a uniformity when X is discrete.) Let $\mathbf{O} = 2^X - \{\{x\}|\ x \in X\} \in L(\mathcal{T})$. $\mathcal{H}(\mathcal{U}^{-1})$ is not the Pervin quasiuniformity for $L(\mathcal{T})$ because there is no $\mathbf{U} \in \mathcal{H}(\mathcal{U}^{-\mathbf{1}})$ with $\mathbf{U}[\mathbf{O}] = \mathbf{O}$. Similarly, let $\mathbf{O}' = \{\{x\}|\ x \in X\} \in U(\mathcal{T})$. $(\mathcal{H}(\mathcal{U}))^{-1}$ is not the Pervin quasiuniformity for $U(\mathcal{T})$ because there is no $\mathbf{U} \in (\mathcal{H}(\mathcal{U}))^{-1}$ with $\mathbf{U}[\mathbf{O}'] = \mathbf{O}'$. The point in both cases is that for basic $U \in \mathcal{U}$ there are only finitely many values that $U[A]$ can have as A ranges over 2^X, so there must be some $x, y \in X$ with $x \neq y$ such that $\{x,y\} \subseteq U[x]$.

The case where $\mathcal{T} = \mathcal{T}^*$ in the following result is Theorem 5.3 in [FLL].

Proposition 2.5. *Given a bitopological space $(X,\mathcal{T},\mathcal{T}^*)$, for there to be a self-adjoint biquasiuniform space $(X,\mathcal{U},\mathcal{U}^{-1})$ such that $(X,\mathcal{U},\mathcal{U}^{-1})$ is compatible with $(X,\mathcal{T},\mathcal{T}^*)$ and $(2^X, \mathcal{H}(\mathcal{U}^{-1}), (\mathcal{H}(\mathcal{U}^{-1}))^{-1})$ is compatible with $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$ it is necessary and sufficient that $(X,\mathcal{T},\mathcal{T}^*)$ be normal, R_0, and R_0^*.*

Proof. If there is such a $(X,\mathcal{U},\mathcal{U}^{-1})$ then, by Proposition 2.3, if we are given a $\mathcal{T}$-closed set A and any $O \in \mathcal{T}^*$ with $A \subseteq O$ there is a $U \in \mathcal{U}^{-1}$ such that $U \circ U[A] \subseteq O$. Then since $U[A] \cap U^{-1}[X-O] = \phi$ we have $A \subseteq \mathrm{Int}^* U[A] \subseteq c(U[A]) \subseteq O$. So X is normal. What's more, since $(X,\mathcal{U},\mathcal{U}^{-1})$ is compatible with $(X,\mathcal{T},\mathcal{T}^*)$, both X and X^* are completely regular, and so both X and X^* are R_0.

Conversely, if $(X,\mathcal{T},\mathcal{T}^*)$ is normal, R_0, and R_0^* define a relation $\ll$ on $\mathcal{P}(X)$ by $A \ll B$ iff $c^*A \subseteq \mathrm{Int}\, B$. By normality, this defines a quasiproximity. Note that $A \ll^\dagger B$ iff $cA \subseteq \mathrm{Int}^* B$. By R_0 and R_0^* we have $(X, \ll, \ll^\dagger)$ is compatible with $(X,\mathcal{T},\mathcal{T}^*)$. Let $\mathcal{U}$ be the coarsest quasiuniformity compatible with $\ll$. Then by Corollary 2.1 and Proposition 1.2 we have $(2^X, \mathcal{H}(\mathcal{U}^{-1}), (\mathcal{H}(\mathcal{U}^{-1}))^{-1})$ is compatible with $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$. $\square$

3. The Hausdorff Biquasimetric Hyperspace

A quasimetric d on a set X will be assumed to satisfy $d(a,a)=0$ and $d(a,c) \leq d(a,b)+d(b,c)$. This technically should be called a quasipseudometric but we would find that term cumbersome in the following discussion.

A biquasimetric space is a triple (X,d,d^*) where d and d^* are quasimetrics on X. The *dual* of (X,d,d^*) is (X,d^*,d) and the *inverse* of (X,d,d^*) is (X,d^{-1},d^{*-1}) where, for a quasimetric d, we define $d^{-1}(x,y)=d(y,x)$. The *adjoint* of a space is the dual of the inverse. The properties of self-dual and self-adjoint have the obvious definitions. Note that a space (X,d,d^*) is equal to its inverse iff both d and d^* are (pseudo)metrics.

We will assume for the purposes of defining the hyperspace that all quasimetrics are bounded. For any quasimetric d on X and sets $A,B \subseteq X$ let the equation

$$d^{\mathcal{H}_1}(A,B)=\sup_{a\in A}\inf_{b\in B} d(a,b)$$

define the first Hausdorff quasimetric and let the equation

$$d^{\mathcal{H}_2}(A,B)=\sup_{b\in B}\inf_{a\in A} d(a,b)$$

define the second Hausdorff quasimetric. (See [Ha] for the traditional treatment, or, more recently [Be].) Given (X,d,d^*) let 2^X be the collection of non-empty d-closed subsets of X and regard $d^{\mathcal{H}_1}$ and $d^{*\mathcal{H}_2}$ as defined on 2^X. Then the *hyperspace* of (X,d,d^*) is $\left(2^X,d^{\mathcal{H}_1},d^{*\mathcal{H}_2}\right)$. Note that the hyperspace of a self-adjoint space is self-adjoint.

Proposition 3.1. *If (X,d,d^*) is compatible with $(X,\mathcal{U},\mathcal{U}^*)$ then $(2^X,d^{\mathcal{H}_1},d^{*\mathcal{H}_2})$ is compatible with $(2^X,\mathcal{H}(\mathcal{U}^{-1}),(\mathcal{H}(\mathcal{U}^*))^{-1})$.*

See [KR] and before that [Br] for a similar result for the Stager [Sta] quasiuniform hyperspace. The proof is practically the same.

4. Completeness

Completeness results for the traditional hyperspace can be found in [I1], [I2], and [B1][1].

Definition 4.1. If $(X, \mathcal{T})$ is a traditional space and $S : D \to \mathcal{P}(X)$ is a set-valued net, then define $\overline{\lim}\ S$ to be the set of points $x \in X$ such that for any $O \in \mathcal{T}$ with $x \in O$ the set of $d \in D$ such that $S_d \cap O \neq \phi$ is cofinal in D.

The limit in this last definition was (for sequences) called $\overline{\mathrm{Fl}}$ by Hausdorff [Ha]. A few years later it was (still for sequences) called Ls by Kuratowski [Ku]. One sees this same Ls used by Beer [Be] for nets, although lim sup was used for the same definition by Klein and Thompson [KT]. When this is paired with $\underline{\lim}$ (or Li, etc.) it is used to define the *Kuratowski-Painlevé* convergence. (See [Be, p.146] for a list of further references.)

Definition 4.2. For a traditional quasiuniform space $(X,\mathcal{U})$ we will say a net $S : D \to X$ is *right K-Cauchy* [Sto] [KR] if for any $U \in \mathcal{U}$ there is a $d \in D$ such that $d \leq d_1 \leq d_2$ implies $S_{d_1} \in U[S_{d_2}]$ for all d_1, d_2. We say S is *almost Cauchy* [B1] [KR] if for any $U \in \mathcal{U}$ there is a $d \in D$ such that for every $d' \in D$ if $d' \geq d$ then $S^{-1}[U^{-1}[S(d')]]$ is cofinal in D. A filter $\mathcal{F}$ on X is *stable* [I1] [I2] [KR] if for any $U \in \mathcal{U}$ there is some $F \in \mathcal{F}$ with $F \subseteq U[F']$ for each $F' \in \mathcal{F}$.

Observation 4.1. For a traditional quasiuniform space $(X,\mathcal{U})$ and any topology $\mathcal{T}$ on X, the following are equivalent:

(1) Every $\mathcal{U}$-almost Cauchy net has a $\mathcal{T}$-cluster point.

(2) Every $\mathcal{U}$-stable filter has a $\mathcal{T}$-cluster point.

In the results below we will regard our hyperspace structures to be extended to the set $\mathcal{P}_0(X)$ of non-empty subsets of X where appropriate.

[1] At the time of writing [B1] we had not seen the paper [I1]. We note that both Corollary 1 and Corollary 2 of [B1] are essentially stated in the last paragraph of [I1].

Lemma 4.1. *For a traditional quasiuniform space* $(X,\mathcal{U})$ *suppose a net* $S : D \to \mathcal{P}_0(X)$ *is right K-Cauchy with respect to* $\mathcal{H}(\mathcal{U}^{-1})$. *Then* $A \in \mathcal{P}_0(X)$ *is an* $\mathcal{H}(\mathcal{U}^{-1})$*-limit of* S *if and only if* $A \subseteq \overline{\lim}\ S$.

Lemma 4.2. *For a traditional quasiuniform space* $(X,\mathcal{U})$, *with* c^* *denoting closure with respect to* $\mathcal{U}^{-1}$, *and any topology* $\mathcal{T}$ *on* X*:*

(1) [Künzi and Ryser] *Suppose that every* $\mathcal{U}$*-stable filter has a* $\mathcal{U}$*-cluster point. Suppose* $S : D \to \mathcal{P}_0(X)$ *is right K-Cauchy with respect to* $(\mathcal{H}(\mathcal{U}))^{-1}$. *Then* $A \in \mathcal{P}_0(X)$ *is an* $(\mathcal{H}(\mathcal{U}))^{-1}$*-limit of* S *if* $\overline{\lim}_{\mathcal{U}}\ S \subseteq c^*A$.

(2) Suppose that for every $\mathcal{U}$*-stable filter there is a point which is both a* $\mathcal{U}$*-cluster point and a* $\mathcal{T}$*-cluster point. Suppose* $S : D \to \mathcal{P}_0(X)$ *is right K-Cauchy with respect to* $(\mathcal{H}(\mathcal{U}))^{-1}$. *Then* $A \in \mathcal{P}_0(X)$ *is an* $(\mathcal{H}(\mathcal{U}))^{-1}$*-limit of* S *if* $\overline{\lim}_{\mathcal{U}}\ S \cap \overline{\lim}_{\mathcal{T}}\ S \subseteq c^*A$.

In either case a necessary condition for $A \in \mathcal{P}_0(X)$ *to be an* $(\mathcal{H}(\mathcal{U}))^{-1}$*-limit of* S *is that* $\overline{\lim}_{\mathcal{U}^{-1}}\ S \subseteq c^*A$.

Proof. (1) is essentially Lemma 6 in [KR]. In this context we note that $\overline{\lim}_{\mathcal{U}}\ S$ is the set of $\mathcal{U}$-cluster points of the filter generated by the sets $\cup_{d' \geq d} S_{d'}$ as d ranges over D.

(2) requires only a minor modification of the proof of (1). □

Proposition 4.1. [Künzi and Ryser] *For a self-dual biquasiuniform space* $(X,\mathcal{U},\mathcal{U})$ *for every right K-Cauchy net* $S : D \to \mathcal{P}_0(X)$ *to converge to* $\overline{\lim}\ S$ *with respect to the supremum quasiuniformity on the hyperspace* $(\mathcal{P}_0(X), \mathcal{H}(\mathcal{U}^{-1}), (\mathcal{H}(\mathcal{U}))^{-1})$ *it is necessary and sufficient that every* $\mathcal{U}$*-stable filter has a* $\mathcal{U}$*-cluster point.*

Proof. This is Proposition 6 in [KR]. Sufficiency follows from our statements here by combining Lemma 4.1 with part (1) of Lemma 4.2. □

Lemma 4.3. *For a self-adjoint biquasiuniform space* $(X,\mathcal{U},\mathcal{U}^{-1})$ *let* $\hat{\mathcal{U}}$ *be the supremum of* $\mathcal{U}$ *and* $\mathcal{U}^{-1}$*, and suppose that every* $\mathcal{U}^{-1}$*-stable filter has a* $\hat{\mathcal{U}}$*-cluster point. Suppose* $S : D \to 2^X$ *is Cauchy with respect to the supremum uniformity on the hyperspace* $(2^X, \mathcal{H}(\mathcal{U}^{-1}), (\mathcal{H}(\mathcal{U}^{-1}))^{-1})$*. Then* S *converges (with respect to the supremum uniformity) to* $A \in 2^X$ *if and only if* $A = \overline{\lim}_{\mathcal{U}}\ S$*.*

Proof. Suppose A is a limit of S as above. Applying the last line of Lemma 4.2 to the quasiuniformity $\mathcal{U}^{-1}$ gives $\overline{\lim}_{\mathcal{U}}\ S \subseteq cA$. Applying Lemma 4.1 gives $A \subseteq \overline{\lim}_{\mathcal{U}}\ S$. So $A = \overline{\lim}_{\mathcal{U}}\ S$.

Conversely, suppose that $A = \overline{\lim}_{\mathcal{U}}\ S$. We have, therefore, $\overline{\lim}_{\mathcal{U}^{-1}}\ S \cap \overline{\lim}_{\mathcal{U}}\ S \subseteq A = cA$. Applying property (2) of Lemma 4.2 to the quasiuniformity $\mathcal{U}^{-1}$, with $\mathcal{T}$ being the topology generated by $\mathcal{U}$, we find that A is an $(\mathcal{H}(\mathcal{U}^{-1}))^{-1}$-limit of S. By Lemma 4.1, A is an $\mathcal{H}(\mathcal{U}^{-1})$-limit of S. So A is a limit of S with respect to the supremum uniformity. □

Proposition 4.2. *For a self-adjoint biquasiuniform space* $(X,\mathcal{U},\mathcal{U}^{-1})$ *let* $\hat{\mathcal{U}}$ *be the supremum of* $\mathcal{U}$ *and* $\mathcal{U}^{-1}$*, and suppose that every* $\mathcal{U}^{-1}$*-stable filter has a* $\hat{\mathcal{U}}$*-cluster point. It follows that the supremum uniformity on the hyperspace* $(2^X, \mathcal{H}(\mathcal{U}^{-1}),$ $(\mathcal{H}(\mathcal{U}^{-1}))^{-1})$ *is complete.*

Proof. Suppose $S : D \to 2^X$ is Cauchy with respect to the supremum uniformity on the hyperspace $(2^X, \mathcal{H}(\mathcal{U}^{-1}), (\mathcal{H}(\mathcal{U}^{-1}))^{-1})$. Then $\overline{\lim}_{\mathcal{U}}\ S \neq \phi$ and so $\overline{\lim}_{\mathcal{U}}\ S \in 2^X$. □

The following is [KR, Proposition 5].

Proposition 4.3. [Künzi and Ryser] *For a self-dual biquasimetric space* (X, d, d) *we have* $\left(2^X, d^{\mathcal{H}_1}, d^{\mathcal{H}_2}\right)$ *with the supremum quasimetric is right K-sequentially complete if and only if* (X, d, d) *is.*

Our notation has allowed us to contrast the self-dual and self-adjoint cases in this section. The self-dual biquasiuniform spaces reflect the main stream of the literature from [Sta] and [LS] on. We have offered the class of self-adjoint spaces as an alternative and we have several results (Propostion 1.2, Proposition 2.4, Corollary 2.2, Proposition 2.5) whose theme would be that traditonal results in uniform mathematics transfer to the self-adjoint bitopological setting without becoming unrecognizable. The difference for hyperspaces is this: If one prefers the self-adjoint spaces then one would like to see a quasiuniformity $\mathcal{U}$ which generates $\mathcal{T}$, and whose inverse generates $\mathcal{T}^*$, to lift to a quasiuniformity $\mathcal{H}(\mathcal{U}^{-1})$ which, for nice enough spaces, generates $L(\mathcal{T})$ and whose inverse generates $U(\mathcal{T}^*)$. If, on the other hand, one prefers the self-dual spaces then one would like to see a quasiuniformity $\mathcal{U}$ which generates $\mathcal{T}$, and whose inverse generates $\mathcal{T}^*$, to lift to a quasiuniformity $\mathcal{H}(\mathcal{U}^{-1}) \vee \mathcal{H}(\mathcal{U})^{-1}$ which, for nice enough spaces, generates $2^{\mathcal{T}}$, the Vietoris topology, and whose inverse generates $2^{\mathcal{T}^*}$. There are advantages to both points of view, but we have not favored the second one since it negates any role for the asymmetric bitopological hyperspace $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$, to which we have devoted three papers.

We must note, however, that the paper by Künzi and Ryser has shown us that completeness results for the self-dual case are much nicer. We end with the following example, which shows that Proposition 4.3 is not true in the self-adjoint case. It also shows that in proposition 4.2 we cannot weaken the hypothesis by replacing $\mathcal{U}^{-1}$-stable filter with $\hat{\mathcal{U}}$-stable filter because that would imply, by Hahn's Theorem (see [B1]), that the hyperspace $\left(2^X, d^{\mathcal{H}_1}\right)$ of a bicomplete quasimetric space (X, d) is bicomplete.

Example 4.1. With $N = \{1, 2, 3 \dots\}$ let $X = N \times N$ and define a quasimetric d on X by

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} 0, & \text{if } (x_1, y_1) = (x_2, y_2) \\ \max\left\{\frac{1}{x_1}, \frac{1}{x_2}\right\}, & \text{if } y_1 < y_2 \\ 1, & \text{otherwise.} \end{cases}$$

Then the symmetrization $\hat{d}(P_1, P_2) = \max\{d(P_1, P_2), d^{-1}(P_1, P_2)\}$ is the discrete metric, and therefore the metric space $\left(X, \hat{d}\right)$ is complete. But in the hyperspace

$$\left(2^X, d^{\mathcal{H}_1}, (d^{-1})^{\mathcal{H}_2}\right) = \left(2^X, d^{\mathcal{H}_1}, (d^{\mathcal{H}_1})^{-1}\right)$$

the symmetrization metric (i.e., the maximum of the two Hausdorff quasimetrics) makes the sequence $\{S_n\}$, where $S_n = \{n\} \times N \subseteq X$, Cauchy. But $\overline{\lim}_d\, S = \phi$ since the d-topology is discrete. Therefore the hyperspace is not complete with respect to the supremum metric.

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