COVERING PROPERTIES AND METRISATION OF MANIFOLDS 2

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Abstract

There are many conditions equivalent to metrisability for a topological manifold which are not equivalent to metrisability for topological spaces in general. What are the weakest such? We show that a number of weak covering properties which are equivalent to metrisability for a manifold, for example metaLindelöf, may be further weakened by considering only covers of cardinality the first uncountable ordinal. Extensions to higher cardinals are discussed.

1. Introduction and Definitions

By a topological manifold we mean a connected Hausdorff space each point of which has a neighbourhood homeomorphic to euclidean space. In [4] there is a list of over 50 conditions which are equivalent to metrisability for a manifold but not for a topological space in general. As one might expect, some of these conditions are strictly stronger than metrisability and some are strictly weaker than metrisability in a general space. In this paper we investigate just how weak covering properties can be made while still being equivalent to metrisability for a manifold.

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All cardinals are assumed infinite. We denote the cardinality of a set $X$ by $|X|$. If $x \in X$ and $\mathcal{F}$ is a family of subsets of $X$ then $\text{ord}(x, \mathcal{F})$ is the order of $\mathcal{F}$ at $x$, i.e. $|\{F \in \mathcal{F} \mid x \in F\}|$. When $X$ is a topological space, we denote by $\chi(x, X)$ the character of $x$ in $X$, i.e. the least infinite cardinality of a local basis at $x$. A good reference for the set theory used in this paper is [10].

The following properties are studied in [1] where Theorem 4.1 states that every locally metrisable, linearly Lindelöf space is hereditarily Lindelöf. They observe that their proof may be modified to show that every locally metrisable $\omega_1$-Lindelöf space is hereditarily Lindelöf. (As noted in [1] and in Proposition 15 below, every linearly Lindelöf space is $\omega_1$-Lindelöf.) Setting $\kappa = \omega_1$ in Proposition 12 shows that local metrisability can be replaced by local hereditary Lindelöfness.

**Definition 1.** A space $X$ is linearly Lindelöf provided that every open cover of $X$ which is a chain has a countable subcover. A family $\mathcal{F}$ of subsets of a set $X$ is a chain provided that $\forall F, G \in \mathcal{F}$ either $F \subset G$ or $G \subset F$.

A space $X$ is $\omega_1$-Lindelöf provided that every open cover of $X$ of cardinality $\omega_1$ has a countable subcover.

Recall also the following definition.

**Definition 2.** Let $\kappa$ and $\lambda$ be two cardinal numbers. A topological space $X$ is $[\kappa, \lambda]$-compact, [12], if and only if every open cover of $X$ of cardinality at most $\lambda$ has a subcover of cardinality less than $\kappa$.

If $\kappa = \omega$ then $[\kappa, \lambda]$-compact is also called initially $\lambda$-compact. If $\lambda \geq |X|$ then $[\kappa, \lambda]$-compact is also called finally $\kappa$-compact.

Motivated by these definitions we formulate the following definitions, where $\kappa$ and $\lambda$ are two cardinal numbers:

**Definition 3.** A space $X$ is linearly $[\kappa, \lambda]$-compact provided that every open cover $\mathcal{U}$ of $X$ which is a chain and satisfies $|\mathcal{U}| \leq \lambda$ has a subcover $\mathcal{V}$ with $|\mathcal{V}| < \kappa$. 
A space \( X \) is (linearly) \([\kappa, \lambda]\)-metacompact provided that every open cover \( U \) of \( X \) which (is a chain and) satisfies \(|U| \leq \lambda\) has an open refinement \( V \) such that \( \text{ord}(x, V) < \kappa \) for each \( x \in X \). If \( \lambda \geq |X| \) then \([\kappa, \lambda]\)-metacompact is also called finally \(\kappa\)-metacompact.

A space is nearly (linearly) \([\kappa, \lambda]\)-metacompact if we merely demand that \( \text{ord}(x, V) < \kappa \) for each point \( x \) in some dense subset of \( X \).

An \([\omega_1, \omega_1]\)-metacompact space may also be called an \(\omega_1\)-metaLindelöf space, and is a weak form of metaLindelöfness as it requires point-countability of a refinement only for open covers of cardinality \( \omega_1 \). Theorem 13 tells us that under appropriate conditions, which all manifolds satisfy, an \(\omega_1\)-metaLindelöf space is in fact metaLindelöf. (Nearly) linearly metaLindelöf and nearly \(\omega_1\)-metaLindelöf are defined analogously. The ultimate must be the following: a space is (nearly) linearly \(\omega_1\)-metaLindelöf provided that for every open cover \( U \) which is a chain and which satisfies \(|U| \leq \omega_1\) there is an open refinement \( V \) which is point-countable (on a dense subset).

Given a set \( X \) and a collection \( S \) of subsets of \( X \), a choice function is a function \( f : S \to X \) such that \( f(S) \in S \) for each \( S \in S \).

**Definition 4.** A space \( X \) has property \((\omega_1)\)pp, \([7]\), provided that each open cover \( U \) of \( X \) (with \(|U| = \omega_1\)) has an open refinement \( V \) such that for each choice function \( f : V \to X \) with \( f(V) \in V \) for each \( V \in V \) the set \( f(V) \) is closed and discrete in \( X \).

The main result in this paper is the following.

**Theorem 5.** Let \( M \) be a manifold. Then the following are equivalent:

(a) \( M \) is metrisable;

(b) \( M \) is nearly linearly \(\omega_1\)-metaLindelöf;
(c) for every open cover \( U \) of \( M \) with \( |U| = \omega_1 \) there is an open refinement \( V \) such that for every choice function \( f : V \rightarrow M \) the set \( f(V) \) is closed and discrete;

(d) for every open cover \( U \) of \( M \) with \( |U| = \omega_1 \) there is an open refinement \( V \) such that for every choice function \( f : V \rightarrow M \) the set \( f(V) \) is closed;

(e) for every open cover \( U \) of \( M \) with \( |U| = \omega_1 \) there is an open refinement \( V \) such that for every choice function \( f : V \rightarrow M \) the set \( f(V) \) is discrete.

Of course with the Continuum Hypothesis this tells us no more than what we already know from [4], that every (nearly) meta-Lindelöf manifold (equivalently, manifold with property pp) is metrisable, as every manifold has the cardinality of the continuum, by [9, Theorem 2.9].

2. Finally \( \kappa \)-metacompact Spaces

Recall that the character of a space \( X \) is the least cardinal \( \kappa \) for which every point of \( X \) has a local base of cardinality at most \( \kappa \).

We say that a sequence \( \langle V_\alpha \rangle \) of subsets of a space is strongly increasing provided that \( V_\alpha \subset V_{\alpha+1} \) for each \( \alpha \).

**Lemma 6.** Let \( \kappa \) be a regular cardinal. Suppose that \( X \) is a space such that \( \chi(x, X) < \kappa \) for each \( x \in X \) and \( \langle V_\alpha \rangle \) is a strongly increasing \( \kappa \)-sequence of subsets of \( X \). Then \( \bigcup_{\alpha<\kappa} V_\alpha \) is closed in \( X \).

**Proof.** Suppose that \( x \in \overline{\bigcup_{\alpha<\kappa} V_\alpha} \). Let \( \{U_\beta \mid \beta \leq \theta \} \) be a neighbourhood base at \( x \), where \( \theta < \kappa \). For each \( \beta \) we have \( U_\beta \cap (\bigcup_{\alpha<\kappa} V_\alpha) \neq \emptyset \) so \( U_\beta \cap V_{\alpha_\beta} \neq \emptyset \) for some \( \alpha_\beta < \kappa \). Let \( \alpha = \sup \{\alpha_\beta \mid \beta \leq \theta \} \). Then \( \alpha < \kappa \) and \( U_\beta \cap V_\alpha \neq \emptyset \) for all \( \beta \), and hence \( x \in \overline{V_\alpha} \subset \overline{V_{\alpha+1}} \). Thus \( \overline{\bigcup_{\alpha<\kappa} V_\alpha} \subset \bigcup_{\alpha<\kappa} V_\alpha \). \( \square \)
Lemma 7. Let $\kappa$ be a regular cardinal. Suppose that $X$ is a connected space and that $\mathcal{V}$ is an open cover of $X$ such that $\text{ord}(x, \mathcal{V}) < \kappa$ for each $x \in X$ and each member of $\mathcal{V}$ has density $< \kappa$. Then $|\mathcal{V}| < \kappa$.

Proof. We may assume that $\emptyset \not\in \mathcal{V}$.

Pick any $V_0 \in \mathcal{V}$ and set $\mathcal{V}_0 = \{V_0\}$. Assuming that $\mathcal{V}_i \subset \mathcal{V}$ has been defined, let $V_i = \cup \mathcal{V}_i$ and set $\mathcal{V}_{i+1} = \{V \in \mathcal{V} | \forall V \cap V_i \neq \emptyset\}$. It suffices to show that $|\mathcal{V}_i| < \kappa$ and that $\mathcal{V} = \cup_{i=0}^{\infty} \mathcal{V}_i$.

(i) We show that $|\mathcal{V}_i| < \kappa$ by induction on $i$, the result being trivial when $i = 0$. Suppose that $|\mathcal{V}_i| < \kappa$. Then because $\kappa$ is regular, $V_i$ has a dense subset, say $D_i$, with $|D_i| < \kappa$. For each $V \in \mathcal{V}_{i+1}$ we have $V \cap V_i \neq \emptyset$ so $V \cap D_i \neq \emptyset$. Again because $\kappa$ is regular, $\mathcal{V}_{i+1} = \cup_{d \in D_i} \{V \in \mathcal{V} | d \in V\}$ has cardinality less than $\kappa$ since $\text{ord}(x, \mathcal{V}) < \kappa$ for each $x \in X$.

(ii) $\mathcal{V} = \cup_{i=0}^{\infty} \mathcal{V}_i$ follows from connectedness via the fact that any two points of $X$ are chained to each other by members of $\mathcal{V}$: thus for any $x \in V_0 \in \mathcal{V}$ and any $y \in V \in \mathcal{V}$ there is a finite sequence $\langle W_i \rangle$ of members of $\mathcal{V}$ such that $x \in W_0$, $y \in W_n$ and $W_{i-1} \cap W_i \neq \emptyset$ for each $i = 0, \ldots, n$. We may assume that $W_0 = V_0$ and $W_n = V$. Then for each $i$, $W_i \in \mathcal{V}_i$. In particular $V \in \mathcal{V}_n$. \qed

Corollary 8. Let $\kappa$ be a regular cardinal. Then any connected and finally $\kappa$-metacompact space which is locally of density $< \kappa$ is finally $\kappa$-compact.

In particular every connected, locally separable, metaLindelöf space is Lindelöf. We also obtain:

Corollary 9. Let $\kappa$ be a regular cardinal and $\lambda$ any cardinal. Every connected, $[\kappa, \lambda]$-metacompact space of density $< \kappa$ is $[\kappa, \lambda]$-compact.
Proof. Suppose that $X$ is a connected, $[\kappa, \lambda]$-metacompact space of density $< \kappa$ and let $\mathcal{U}$ be an open cover of $X$ with $|\mathcal{U}| = \lambda$. Let $\mathcal{V}$ be an open refinement of $\mathcal{U}$ such that $\text{ord}(x, \mathcal{V}) < \kappa$ for each $x \in X$. As an open subset of a space of density $< \kappa$, each member of $\mathcal{V}$ has density $< \kappa$. By Lemma 7, $|\mathcal{V}| < \kappa$ and hence $\mathcal{U}$ has a subcover of cardinality less than $\kappa$. \hfill \Box

Let $X$ be a topological space and $A$ a non-empty subset of $X$. A point $x \in X$ is a point of complete accumulation of $A$ if and only if for every neighbourhood $N$ of $x$ we have $|A \cap N| = |A|$.

**Proposition 10.** [2, page 17] and [13, Theorem 1] Let $\kappa$ be a regular cardinal. A space $X$ is $[\kappa, \kappa]$-compact if and only if every $A \subset X$ such that $|A| = \kappa$ has a point of complete accumulation.

**Proposition 11.** Let $\kappa$ be a regular cardinal. Let $X$ be a space which is not hereditarily finally $\kappa$-compact. Then there is a subspace $Y \subset X$ such that $|Y| = \kappa$ and that no subset $Z \subset Y$ of cardinality $\kappa$ is finally $\kappa$-compact.

**Proof.** (cf [11, Theorem 3.1]). Because $X$ is not hereditarily finally $\kappa$-compact there is a strictly increasing sequence $\langle U_\alpha \rangle_{\alpha < \kappa}$ of open sets. For each $\alpha < \kappa$ choose $y_\alpha \in U_{\alpha+1} - U_\alpha$ and set $Y = \{y_\alpha \mid \alpha < \kappa\}$. \hfill \Box

The following result generalises [1, theorem 4.1]. The proof may be obtained by appropriate generalisation of the proof of that result using Propositions 10 and 11.

**Proposition 12.** Let $\kappa$ be a regular cardinal. Every locally hereditarily finally $\kappa$-compact, $[\kappa, \kappa]$-compact space is hereditarily finally $\kappa$-compact.

**Theorem 13.** Let $\kappa$ be a regular cardinal. Suppose that $X$ is a space which is of character $< \kappa$, is locally connected, locally hereditarily finally $\kappa$-compact and locally hereditarily of density $< \kappa$. If $X$ is $[\kappa, \kappa]$-metacompact then $X$ is the topological direct sum of finally $\kappa$-compact spaces.
Proof. As $X$ is locally connected, every component is open so by looking at each component separately if necessary we may assume that $X$ is connected also. We construct a strongly increasing $\kappa$-sequence $\langle V_\alpha \rangle$ of non-empty, connected, open and finally $\kappa$-compact subsets of $X$.

Because $X$ is locally connected and locally hereditarily finally $\kappa$-compact we may begin by choosing any non-empty, connected, open, finally $\kappa$-compact subset $V_0 \subset X$. For any other limit ordinal $\alpha$, if $V_\beta$ has already been constructed for all $\beta < \alpha$, let $V_\alpha = \bigcup_{\beta < \alpha} V_\beta$.

Suppose that $V_\alpha$ has been constructed. Because $V_\alpha$ is finally $\kappa$-compact it also has a dense subset of cardinality $< \kappa$. Thus $\bar{V}_\alpha$ has a dense subset of cardinality $< \kappa$. $\bar{V}_\alpha$ is also connected as $V_\alpha$ is. Furthermore, as a closed subset of a $[\kappa, \kappa]$-metacompact space $\bar{V}_\alpha$ is also $[\kappa, \kappa]$-metacompact. Thus by Corollary 9 $\bar{V}_\alpha$ is $[\kappa, \kappa]$-compact. It now follows from Proposition 12 that $\bar{V}_\alpha$ is finally $\kappa$-compact. For each $x \in \bar{V}_\alpha - V_\alpha$ choose $U_x \subset X$ open and finally $\kappa$-compact such that $x \in U_x$. Then $\{U_x \mid x \in \bar{V}_\alpha - V_\alpha\}$ is an open cover of the finally $\kappa$-compact subset $\bar{V}_\alpha - V_\alpha$ so has a subcover of cardinality $< \kappa$. The collection consisting of this subcover together with $V_\alpha$ is a collection of fewer than $\kappa$ many open finally $\kappa$-compact subsets of $X$ so their union is also open and finally $\kappa$-compact and contains $\bar{V}_\alpha$. Let $V_{\alpha+1}$ be the component of this union containing $V_\alpha$.

Suppose that $\mathcal{U}$ is an open cover of $X$. Then for each $\alpha < \kappa$, $\mathcal{U}$ is also an open cover of the finally $\kappa$-compact set $V_\alpha$: let $\mathcal{U}_\alpha$ be a subcover of cardinality $< \kappa$. Then $\bigcup_{\alpha < \kappa} \mathcal{U}_\alpha$ is a subfamily of $\mathcal{U}$ of cardinality at most $\kappa$ which covers $\bigcup_{\alpha < \kappa} V_\alpha$, hence the connected space $X$, by Lemma 6 because this union is non-empty, open and closed. As $X$ is $[\kappa, \kappa]$-metacompact it follows that this subfamily has an open refinement whose order at each point is less than $\kappa$ and hence so does $\mathcal{U}$. Now it follows from Corollary 8 that $X$ is finally $\kappa$-compact. \qed
Remark. The three local properties ‘of character $< \kappa$, locally
hereditarily finally $\kappa$-compact and locally hereditarily of density
$< \kappa$’ of Theorem 13 are all implied by the single local prop-
erty: locally of weight $< \kappa$. In the case where $\kappa = \omega_1$ these
four properties are, respectively, first countable, locally heredi-
tarily Lindelöf, locally hereditarily separable and locally second
countable and in this case, Theorem 13 gives:

**Corollary 14.** Every connected, locally connected, locally sec-
ond countable, $\omega_1$-metaLindelöf space is Lindelöf.

This corollary has an obvious generalisation to higher regular
cardinal $\kappa$ in place of $\omega_1$.

**Proposition 15.** (cf [1]) Every linearly $\omega_1$-(meta)Lindelöf space
is $\omega_1$-(meta)Lindelöf.

*Proof.* We will just consider the metaLindelöf case. Let $U$ be an
open cover of the linearly $\omega_1$-metaLindelöf space $X$ such that
$|U| = \omega_1$. Then we can write $U = \{U_\alpha \mid \alpha < \omega_1\}$. For each
$\alpha < \omega_1$ let $V_\alpha = \bigcup \{U_\beta \mid \beta < \alpha\}$. Then $V = \{V_\alpha \mid \alpha < \omega_1\}$
is an open cover of $X$ which is a chain. Thus as $X$ is linearly
$\omega_1$-metaLindelöf it follows that there is a point-countable open
refinement, say $\mathcal{W}$.

For each $W \in \mathcal{W}$ there is $\alpha(W) < \omega_1$ such that $W \subset V_{\alpha(W)}$.
Let $S = \{W \cap U_\beta \mid W \in \mathcal{W} \text{ and } \beta \leq \alpha(W)\}$. Then $S$ is a
point-countable open refinement of $U$. □

**Proof of the equivalence of (a) and (b) of Theorem 5**

As every metrisable space is paracompact, it is also nearly
linearly $\omega_1$-metaLindelöf so (a)$\Rightarrow$(b) in Theorem 5. For the
converse, suppose that $M$ is a nearly linearly $\omega_1$-metaLindelöf
manifold. Clearly one can modify the proof of [5, Lemma 3.2] to
conclude that $M$ is linearly $\omega_1$-metaLindelöf. As every manifold
is $T_3$, connected, locally connected and locally second count-
able, it follows from Corollary 14 and Proposition 15 that $M$
is Lindelöf, hence second countable and therefore metrisable by
Urysohn’s Metrisation Theorem. □
3. Spaces with Property pp

**Lemma 16.** A point \( x \in X \) is a limit point of \( X \) if and only if for each collection \( \mathcal{V} \) of open sets containing \( x \), with \(|\mathcal{V}| \geq \chi(x,X)\), there exists a choice function \( f : \mathcal{V} \to X \), such that \( x \in f(\mathcal{V}) - f(\mathcal{V}) \).

*Proof.* \( \Rightarrow \): Suppose that \( \mathcal{V} \) is a collection of open sets containing \( x \) with \(|\mathcal{V}| \geq \chi(x,X)\), say \( \{V_\alpha \mid \alpha < \chi(x,X)\} \subset \mathcal{V} \) satisfies \( V_\alpha \neq V_\beta \) whenever \( \alpha \neq \beta \). Let \( \{W_\alpha \mid \alpha < \chi(x,X)\} \) be a neighbourhood basis at \( x \). Then we may define \( f : \mathcal{V} \to X \) so that \( f(V_\alpha) \in V_\alpha \cap W_\alpha - \{x\} \). Then \( x \in f(\mathcal{V}) - f(\mathcal{V}) \).

\( \Leftarrow \): Let \( U \) be any neighbourhood of \( x \) and take \( \mathcal{V} \) to be a collection of open neighbourhoods of \( x \) forming a neighbourhood basis at \( x \). Then \(|\mathcal{V}| \geq \chi(x,X)\). Let \( f : \mathcal{V} \to X \) be a choice function such that \( x \in f(\mathcal{V}) - f(\mathcal{V}) \). Then \( f(U) \in U - \{x\} \), so \( x \) is a limit point of \( X \). \( \square \)

**Lemma 17.** Let \( \mathcal{V} \) be an open cover of a \( T_1 \) space \( X \). Then the following are equivalent:

(a) For every choice function \( f : \mathcal{V} \to X \), the set \( f(\mathcal{V}) \) is closed and discrete;

(b) For every choice function \( f : \mathcal{V} \to X \), the set \( f(\mathcal{V}) \) is closed;

(c) For every choice function \( f : \mathcal{V} \to X \), the set \( f(\mathcal{V}) \) is discrete.

*Proof.* It suffices to show that (b) and (c) are equivalent.

(b) \( \Rightarrow \) (c). Suppose that \( f : \mathcal{V} \to X \) is a choice function but \( f(\mathcal{V}) \) is not discrete. Then there is \( x \in f(\mathcal{V}) \) every neighbourhood of which meets \( f(\mathcal{V}) \) in some point other than \( x \). Define \( g : \mathcal{V} \to X \) by \( g(V) = f(V) \) if \( f(V) \neq x \) and \( g(V) \in V - \{x\} \) if \( f(V) = x \). Then \( x \in g(\mathcal{V}) - g(\mathcal{V}) \) so \( g(\mathcal{V}) \) is not closed.
(c)⇒(b). Suppose that $f : \mathcal{V} \to X$ is a choice function but $f(\mathcal{V})$ is not closed, say $x \in f(\mathcal{V}) - f(\mathcal{V})$. Pick $V_x \in \mathcal{V}$ such that $x \in V_x$. Define $g : \mathcal{V} \to X$ by $g(V) = f(V)$ unless $V = V_x$ and let $g(V_x) = x$. Because $X$ is $T_1$ it follows that every neighbourhood of $x$ meets $g(\mathcal{V})$ in some point other than $x$ so $g(\mathcal{V})$ is not discrete. □

**Proposition 18.** Let $\kappa$ be a cardinal. Suppose that $X$ has character at most $\kappa$ and has no isolated points, and that every open cover $\mathcal{U}$ of $X$ with $|\mathcal{U}| = \kappa^+$ has an open refinement $\mathcal{V}$ such that for every choice function $f : \mathcal{V} \to X$ the set $f(\mathcal{V})$ is closed. Then $X$ is $[\kappa^+, \kappa^+]$-metacompact.

**Proof.** Let $\mathcal{U}$ be an open cover of $X$ with $|\mathcal{U}| = \kappa^+$. Apply Lemma 16 to the open refinement $\mathcal{V}$ given by hypothesis: then $\text{ord}(x, \mathcal{V}) < \kappa < \kappa^+$ for each $x \in X$. □

We can now complete the proof of Theorem 5.

By Lemma 17 (c), (d) and (e) are equivalent. By Proposition 18 with $\kappa = \omega$, (d) implies (b). Finally every metrisable manifold is pp and hence satisfies (c).

### 4. Some Questions

Are there even weaker covering conditions which are equivalent to metrisability for a manifold?

Using [6, Theorems 1 and 2] (or see [3, Theorem 8.11]) and [9, Theorem 2.5] we find that the following conditions are each equivalent to metrisability for a manifold:

- $M$ is normal and $\theta$-refinable;
- $M$ is normal and subparacompact.

Let $X$ be a space.

$X$ is $\theta$-refinable ([14]) (also called submetacompact) if every open cover can be refined to an open $\theta$-cover, i.e. a cover $\mathcal{U}$ which can be expressed as $\bigcup_{n \in \omega} \mathcal{U}_n$ where each $\mathcal{U}_n$ covers $X$ and for each $x \in X$ there is $n$ such that $\text{ord}(x, \mathcal{U}_n) < \omega$. 
X is subparacompact, [8] (where it is called $F_\sigma$-screenable), if every open cover has a $\sigma$-discrete closed refinement.

Our theme suggests the following definition.

**Definition 19.** Say that $X$ is $\omega_1$-$\theta$-refinable if every open cover $\mathcal{U}$ of $X$ with $|\mathcal{U}| = \omega_1$ has a $\theta$-refinement.

**Question 20.** Is every $\omega_1$-$\theta$-refinable manifold $\theta$-refinable?

**Question 21.** Must a manifold be metrisable if it is normal and every open cover of cardinality at most $\omega_1$ has an open $\theta$-refinement?

**Question 22.** Must a manifold be metrisable if it is normal and every open cover of cardinality at most $\omega_1$ has a $\sigma$-discrete closed refinement?

Comparing Corollary 8 with Corollary 9 leads to the following question.

**Question 23.** Let $\kappa$ be a regular cardinal. Must every connected and $[\kappa, \kappa]$-metacompact space which is locally of density $< \kappa$ be $[\kappa, \kappa]$-compact?

Note that in Proposition 18 we have only concluded that $X$ is $[\kappa^+, \kappa^+]$-metacompact rather than $[\kappa, \kappa^+]$-metacompact even though the open cover of size $\kappa^+$ has been refined to an open cover of order less than $\kappa$: we did not carry out a similar reduction of an open cover of cardinality $\kappa$ because we did not need to. This raises the following question.

**Question 24.** Is there a space $X$ with character at most $\kappa$ and having no isolated points such that every open cover of size $\kappa^+$ has an open refinement $\mathcal{V}$ whose order at each point is less than $\kappa$ but $X$ is not $[\kappa, \kappa^+]$-metacompact?

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