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ON COMPACTIFICATIONS OF THE REAL LINE AND UNIQUE HYPERSPACE

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ABSTRACT. In this paper we prove that the members of the class of metric compactifications of the space $(-\infty, \infty)$, with connected remainder, are C-determined.

1. INTRODUCTION

A **continuum** is a nonempty compact connected metric space. The **hyperspace of subcontinua** of a continuum X is denoted by $C(X)$. If two continua X and Y are homeomorphic, we write $X \approx Y$.

The members of a class Λ of continua are said to be C**-determined** ([14, Definition 0.61]) provided that if $X, Y \in \Lambda$ and $C(X) \approx C(Y)$, then $X \approx Y$. The members of the following classes of continua are known to be C-determined:

(1) Finite graphs different from an arc and a circle ([3, Theorem 9.1]).
(2) Hereditarily indecomposable continua ([14, Theorem 0.60]).
(3) Smooth fans ([4, Corollary 3.3]).
(4) Indecomposable continua such that all their proper and nondegenerate subcontinua are arcs ([12, Theorem 3]).
(5) Metric compactifications of the real line, different from an arc and with disconnected remainder ([2, Corollary 8]).

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(6) Metric compactifications $X = V \cup R$ of the real line $V = (-\infty, \infty)$ with connected remainder R such that $\mathrm{Cl}_X([0, \infty)) - [0, \infty) = \mathrm{Cl}_X((-\infty, 0]) - (-\infty, 0]$ ([1, Theorem 3.5]).
(7) Continua X such that $X = S_1 \cup R \cup S_2$, where $S_1 \cup R$ and $S_2 \cup R$ are metric compactifications of the disjoint rays S_1 and S_2, both with remainder R ([2, Theorem 10]).
(8) Continua X such that $X = V \cup R \cup S$, where $V \cup R$ is a metric compactification of the ray V, with an arc R as the remainder, and S is an arc that shares an end-point with R such that $S \cap (V \cup R)$ is the set consisting of the common end-point of R and S ([1, Theorem 4.6]).
(9) Arcwise connected circle-like continua ([1, Theorem 4.12]).

A continuum X has **unique hyperspace** ([2, Definition 1]) provided that if Y is a continuum and $C(X) \approx C(Y)$, then $X \approx Y$. The following continua have unique hyperspace:

- Finite graphs different from an arc and a circle ([2, Theorem 1]).
- Hereditarily indecomposable continua ([2, Theorem 2]).
- Indecomposable continua such that all their proper and nondegenerate subcontinua are arcs ([1, Theorem 2.3]).
- Metric compactifications of the real line, different from an arc and with disconnected remainder ([2, Theorem 7]).

In sections 3 and 4 of [1] it is proved that continua of the form (6)-(9) described above, does not have unique hyperspace (see [1, Theorems 3.3, 3.4, 4.4 and 4.11]).

The main purpose of this paper is to show that the members of the class of metric compactifications of the real line, with connected remainder, are C-determined (compare with continua of the form (6) described above).

The paper consists of four sections. After the introduction, in Section 2 we present some terminology and basic facts. In Section 3 we develop the techniques that we are going to use in Section 4 to proof the main result of the paper.

2. Terminology and Facts

All spaces considered in this paper are assumed to be metric. Given a space X, we denote by $B_X(x,\varepsilon)$ the (open) ball in X centered at a point $x \in X$ and having the radius ε. For a subset A of X, we define $N_X(A,\varepsilon) = \bigcup_{a\in A} B_X(a,\varepsilon)$, and we use the symbols $\mathrm{Cl}_X(A)$, $\mathrm{Int}_X(A)$ and $\mathrm{Bd}_X(A)$ to denote the closure, the interior and the boundary of A in X, respectively.

The letter I represents the closed unit interval $[0,1]$ in the real line $\mathbb{R}$ and the letter $\mathbb{N}$ the set of positive integers. By a **ray** we mean a one-to-one image of the real half-line $[0,\infty)$. The image of 0 is called the **end-point** of the ray. To simplify notation we often omit the mapping and refer to a ray as to the half-line itself.

The hyperspace of all nonempty closed subsets of a continuum X is denoted by 2^X and $F_1(X)$ denotes the hyperspace of singletons of X. Note that $X \approx F_1(X)$. The hyperspaces 2^X, $C(X)$ and $F_1(X)$ are metrized by the **Hausdorff metric** H. If $P \in C(X)$, we define $C(P,X) = \{A \in C(X) : P \subset A\}$.

A **map** is a continuous function. Let X be a continuum. If $A, B \in C(X)$ and $A \subsetneq B$, then an **order arc from A to B in $C(X)$** is a map $\lambda : I \to C(X)$ such that $\lambda(0) = A$, $\lambda(1) = B$ and $\lambda(s) \subsetneq \lambda(t)$ if $s < t$ (see [14, Definitions 1.2 and 1.7]). To simplify notation we often identify an order arc with its image. A **Whitney map in $C(X)$** is a map $\mu : C(X) \to [0,\infty)$ such that $\mu(\{x\}) = 0$ for each $x \in X$ and $\mu(A) < \mu(B)$ whenever $A, B \in C(X)$ and $A \subsetneq B$ (see [14, 0.50, p. 24]).

A continuum X is **decomposable** if there exist $A, B \in C(X) - \{X\}$ such that $X = A \cup B$. Otherwise, X is said to be **indecomposable**. A continuum X is **connected im kleinen at** $p \in X$ provided that, for each open subset U of X such that $p \in U$, the component C of U that contains p satisfies that $p \in \mathrm{Int}_X(C)$.

For $n \in \mathbb{N}$ an **n-od** in X is an element $B \in C(X)$ for which there exists $A \in C(B)$, called the **core** of B, such that $B - A$ has at least n components. A 3-od is also called a **triod**. An **n-cell** is a space homeomorphic to I^n. If D is a 2-cell in a space X, then ∂D denotes the manifold boundary of D, and the set $\mathsf{int}_X(D) = D - \partial D$ is called the **relative interior of D in X**.

The following results will be used in the next sections of the paper.

Lemma 2.1. [7, Theorem 1.9] *The hyperspace $C(X)$ of a continuum X contains an n-cell if and only if X contains an n-od.*

Lemma 2.2. [2, Lemma 8] *Let K be a subcontinuum of a continuum X, and for some $n \in \mathbb{N}$ and $\varepsilon > 0$ let T be an n-od in X such that $T \in B_{C(X)}\left(K, \frac{\varepsilon}{2}\right)$. Then, there exists an n-cell $\mathcal{T}$ in $C(X)$ such that $T \in \mathcal{T} \subset B_{C(X)}(K, \varepsilon)$.*

Lemma 2.3. [2, Theorem 3] *Let X be a continuum such that the hyperspace $C(X)$ contains a 2-cell $\mathcal{D}$. If $p \in X$ satisfies that $\{p\} \in \mathfrak{int}_{C(X)}(\mathcal{D})$, then for each $\varepsilon > 0$ there is $T \in B_{C(X)}(\{p\}, \varepsilon)$ such that T is a triod in X.*

Lemma 2.4. [1, Theorem 4.2] *Let $A, B \in C(X)$ be such that $A \subsetneq B$. If*

(1) *$A = A_1 \cup A_2$ for some $A_1, A_2 \in C(A) - \{A\}$,*
(2) *$B = B_1 \cup B_2$ for some $B_1, B_2 \in C(B) - \{B\}$,*
(3) *$A_1 \subsetneq B_1$, $A_2 \subsetneq B_2$ and $A_1 \cap A_2 = B_1 \cap B_2 \in C(X)$,*

then there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $A \in \mathfrak{int}_{C(X)}(\mathcal{D})$.

A more general version of Lemma 2.4 is given in the following result.

Lemma 2.5. *Let $A, B \in C(X)$ be such that $A \subsetneq B$. If*

(1) *$A = A_1 \cup A_2$ for some $A_1, A_2 \in C(A) - \{A\}$,*
(2) *$B = B_1 \cup B_2$ for some $B_1, B_2 \in C(B) - \{B\}$,*
(3) *$A_1 \subsetneq B_1$, $A_2 \subsetneq B_2$ and $B_0 = B_1 \cap B_2 \in C(A)$,*
(4) *$A_1 - B_0 \neq \varnothing$, $A_2 - B_0 \neq \varnothing$, $B_1 - (A_1 \cup B_2) \neq \varnothing$ and $B_2 - (A_2 \cup B_1) \neq \varnothing$,*

then there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $A \in \mathfrak{int}_{C(X)}(\mathcal{D})$.

Proof: Put $A_1' = A_1 \cup B_0$ and $A_2' = A_2 \cup B_0$. Since $B_0 \subset A$, we have $A_1' \cup A_2' = A$. Since $B_1 - (A_1 \cup B_2) \neq \varnothing$, it follows that $A_1' \subsetneq B_1$. Similarly, $A_2' \subsetneq B_2$. Moreover, $A_1' \cap A_2' = (A_1 \cap A_2) \cup B_0 = B_0 = B_1 \cap B_2 \in C(X)$. Since $A_2 - B_0 \neq \varnothing$, we have $A_1' \subsetneq A$. Similarly, $A_2' \subsetneq A$. Then, by Lemma 2.4, there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $A \in \mathfrak{int}_{C(X)}(\mathcal{D})$. □

Lemma 2.6. *Let a continuum X contain a subset S which is either a ray or a one-to-one image of the real line. If K is an arc contained* *in S (and if S is a ray with the end-point p, then $p \notin K$) then there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathrm{int}_{C(X)}(\mathcal{D})$.*

Proof: Let $K = ab \subset S$. Order S by $<$ in such a way that $a < b$. Fix points $a', c, b' \in S$ such that $a' < a < c < b < b'$ and define $A_1 = ac$, $A_2 = cb$, $B_1 = a'c$, $B_2 = cb'$ and $B = B_1 \cup B_2$. Then $A_1, A_2 \in C(K) - \{K\}$, $B_1, B_2 \in C(B) - \{B\}$, $A_1 \subsetneq B_1$, $A_2 \subsetneq B_2$ and $A_1 \cap A_2 = B_1 \cap B_2 = \{c\} \in C(X)$. Thus, by Lemma 2.4, there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathrm{int}_{C(X)}(\mathcal{D})$. □

Let us assume that $X = V_1 \cup S_1 \cup S_2 \cup V_2$ is a continuum such that $V_1 \cup S_1$ and $V_2 \cup S_2$ are compactifications of the disjoint rays $V_1 = [a, \infty)$ and $V_2 = (-\infty, b]$, respectively, such that S_1 is the remainder of $V_1 \cup S_1$, S_2 is the remainder of $V_2 \cup S_2$, $V_1 \cap S_2 = \varnothing$ and $V_2 \cap S_1 = \varnothing$. Then $S_1 \cap S_2 \neq \varnothing$ and both $V_1 \cup S_1$ and $V_2 \cup S_2$ are subcontinua of X. Moreover, it is easy to see that $\mathrm{Cl}_X([x, \infty)) = [x, \infty) \cup S_1$ for each $x \in V_1$ and $\mathrm{Cl}_X((-\infty, y]) = (-\infty, y] \cup S_2$ for each $y \in V_2$. If $i \in \{1, 2\}$, we denote by $C(V_i)$ the set of all bounded and closed subintervals of V_i. It is not difficult to see that if $A \in C(X)$, then A satisfies one of the following conditions:

(a) $A \in C(S_1 \cup S_2)$,
(b) $A \in C(V_1)$,
(c) $A \in C(V_2)$,
(d) $A \in C(S_1, X)$ and $A \cap V_1 \neq \varnothing$,
(e) $A \in C(S_2, X)$ and $A \cap V_2 \neq \varnothing$.

Let $A \in C(X)$. In the following lemma we give conditions under which it is possible to find a 2-cell $\mathcal{D}$ in $C(X)$ such that $A \in \mathrm{int}_{C(X)}(\mathcal{D})$.

Lemma 2.7. *If A is a nondegenerate subcontinuum of X such that $a, b \notin A$ and $A \cap (V_1 \cup V_2) \neq \varnothing$, then there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $A \in \mathrm{int}_{C(X)}(\mathcal{D})$.*

Proof: Let A be as assumed. Then $a, b \notin A$ and A satisfies one of the conditions (b)-(e) given above. If A satisfies either (b) or (c), the conclusion follows from Lemma 2.6. Let us assume that A satisfies (d). Note that each point $x \in (a, \infty)$ separates X into the sets $A_x = [a, x)$ and $B_x = (x, \infty) \cup S_1 \cup S_2 \cup V_2$. Using this, it can be seen that $A \cap [a, \infty) = [v, \infty)$ for some $v \in (a, \infty)$. Fix $x \in (v, \infty)$. Since $X - \{x\} = A_x \cup B_x$ is a separation, the sets $B_1 = A_x \cup \{x\}$ and $B_2 = B_x \cup \{x\}$ are proper subcontinua of X

such that $B_1 \cup B_2 = X$. Since $A - \{x\} = (A_x \cap A) \cup (B_x \cap A)$ is also a separation, the sets $A_1 = (A_x \cap A) \cup \{x\}$ and $A_2 = (B_x \cap A) \cup \{x\}$ are proper subcontinua of A such that $A = A_1 \cup A_2$. Clearly, $A_1 \subsetneq B_1$, $A_2 \subsetneq B_2$ and $A_1 \cap A_2 = \{x\} = B_1 \cap B_2 \in C(X)$. Then, by Lemma 2.4, there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $A \in \mathrm{int}_{C(X)}(\mathcal{D})$. If A satisfies (e) the proof is similar. □

3. Locating 2-cells or triods

Let X be a continuum and K be a proper subcontinuum of X. In this section we give conditions under which it is possible to find either a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathrm{int}_{C(X)}(\mathcal{D})$ or triods arbitrarily close to K, with respect to the Hausdorff metric H (see Theorems 3.15 and 3.17). For this purpose, the notion of semi-boundary given by A. Illanes is important.

If $K \in C(X) - \{X\}$, then the **semi-boundary of** K ([10, p. 63]) is the set $SB(K) = \{B \in C(K)$: there is a map $\alpha : I \to C(X)$ such that $\alpha(0) = B$ and $\alpha(t) - K \neq \varnothing$ for each $t > 0\}$. In [10, Theorem 1.2] it is shown that $K \in SB(K)$ and that $SB(K) = \{B \in C(K)$: there is an order arc $\alpha : I \to C(X)$ such that $\alpha(0) = B$ and $\alpha(t) - K \neq \varnothing$ for each $t > 0\}$. The next result is easy to prove.

Lemma 3.1. *Let X be a continuum and let $K \in C(X) - \{X\}$. Suppose that $A \in C(X)$ is such that $A \cap K \neq \varnothing$ and $A - K \neq \varnothing$. Then*

(1) *if C is a component of $A \cap K$, then $C \in SB(K)$,*
(2) *if D is a component of $A - K$, then $K \cap \mathrm{Cl}_X(D) \neq \varnothing$ and $Bd_X(D) - D \subset K$.*

Theorem 3.2. [10, Theorem 1.3] *Let K be a proper subcontinuum of a continuum X and let $(B_n)_n$ be a sequence in $C(K)$ converging to some $B \in C(K)$. If for each $n \in \mathbb{N}$, $B_n \in SB(K)$ and $B_n \cap B \neq \varnothing$, then $B \in SB(K)$.*

Using Theorem 3.2 and part (1) of Lemma 3.1, the next result is easy to prove.

Theorem 3.3. *Let K be a proper subcontinuum of a continuum X and let $B \in C(K)$ be such that, for each $\varepsilon > 0$, there is $L \in C(B, X)$ such that $L - K \neq \varnothing$ and $H(L, B) < \varepsilon$. Then $B \in SB(K)$.*

Lemma 3.4. *Let K be a proper subcontinuum of a continuum X such that $SB(K)$ has at least three mutually disjoint elements E_1, E_2 and E_3. Then, for each $\varepsilon > 0$, there exist three mutually disjoint subcontinua E_1', E_2' and E_3' of X such that*

(1) *$E_i \subsetneq E_i'$ and $E_i' - K \neq \varnothing$ for each $i = 1, 2, 3$,*
(2) *the set $T = K \cup E_1' \cup E_2' \cup E_3'$ is a triod in X such that $T \in B_{C(X)}(K, \varepsilon)$.*

Moreover, if $L_1, L_2, L_3 \in 2^X$ are such that $E_i \cap L_i = \varnothing$ for each $i = 1, 2, 3$, then E_i' can be constructed in such a way that $E_i' \cap L_i = \varnothing$.

Proof: Let $\varepsilon > 0$ and let $L_1, L_2, L_3 \in 2^X$ be such that $E_i \cap L_i = \varnothing$ for each $i = 1, 2, 3$. Given $i \in \{1, 2, 3\}$ let $\alpha_i : I \to C(X)$ be an order arc such that $\alpha_i(0) = E_i$ and $\alpha_i(t) - K \neq \varnothing$ for each $t > 0$. Since the set

$$\mathcal{U} = \left\{ A \in C(X) : A \subset (X - (E_2 \cup E_3 \cup L_1)) \cap N_X\left(K, \frac{\varepsilon}{3}\right) \right\}$$

is open in $C(X)$, $E_1 \in \mathcal{U}$ and α_1 is continuous, there exists $t > 0$ such that $E_1' = \alpha_1(t) \in \mathcal{U}$. Then, $E_1 \subsetneq E_1'$, $E_1' - K \neq \varnothing$, $E_1' \cap (E_2 \cup E_3 \cup L_1) = \varnothing$ and $E_1' \subset N_X\left(K, \frac{\varepsilon}{3}\right)$. This implies that $H(K, K \cup E_1') < \frac{\varepsilon}{3}$. Similarly, there is $E_2' \in C(X)$ such that $E_2 \subsetneq E_2'$, $E_2' - K \neq \varnothing$, $E_2' \cap (E_1' \cup E_3 \cup L_2) = \varnothing$, $H(K \cup E_1', K \cup E_1' \cup E_2') < \frac{\varepsilon}{3}$ and there is $E_3' \in C(X)$ such that $E_3 \subsetneq E_3'$, $E_3' - K \neq \varnothing$, $E_3' \cap (E_1' \cup E_2' \cup L_3) = \varnothing$ and

$$H\left(K \cup E_1' \cup E_2', K \cup E_1' \cup E_2' \cup E_3'\right) < \frac{\varepsilon}{3}.$$

Then $T = K \cup E_1' \cup E_2' \cup E_3'$ is a triod in X, with core K, such that $T \in B_{C(X)}(K, \varepsilon)$. Clearly, the sets E_1', E_2' and E_3' satisfy the required properties. □

As a direct consequence of Lemma 3.4 and part (1) of Lemma 3.1, we have the following result.

Lemma 3.5. *Let K be a proper subcontinuum of a continuum X. Suppose that there exists $A \in C(X)$ such that $A \cap K \neq \varnothing$ and $A - K \neq \varnothing$. If $A \cap K$ has at least three components C_1, C_2 and C_3 then, for each $\varepsilon > 0$, there exist three mutually disjoint subcontinua C_1', C_2' and C_3' of A, such that*

(1) *$C_i \subsetneq C_i'$ and $C_i' - K \neq \varnothing$ for each $i = 1, 2, 3$,*

(2) *the set* $T = K \cup C_1' \cup C_2' \cup C_3'$ *is a triod in* X *such that* $T \in B_{C(X)}(K, \varepsilon)$.

Moreover, if $L_1, L_2, L_3 \in 2^X$ *are such that* $C_i \cap L_i = \varnothing$ *for each* $i = 1, 2, 3$*, then* C_i' *can be constructed in such a way that* $C_i' \cap L_i = \varnothing$.

Let X be a continuum and let $K \in C(X) - \{X\}$. In [10, Theorem 1.4] it is shown that if $B \in SB(K)$, then there exists a minimal element (with respect to the inclusion) $C \in SB(K)$ such that $C \subset B$. We denote by $m(K)$ the set of minimal elements in $SB(K)$.

From now, in this section, the letter K denotes a proper and nondegenerate subcontinuum of a given continuum X. As a direct consequence of Lemma 3.4, we have the following result.

Theorem 3.6. *If* $m(K)$ *has at least three mutually disjoint elements then, for each* $\varepsilon > 0$*, there is a triod* T *in* X *such that* $T \in B_{C(X)}(K, \varepsilon)$.

Theorem 3.7. *Let* $\varepsilon > 0$ *be such that no* $T \in B_{C(X)}(K, \varepsilon)$ *is a triod in* X*. If* $E \in m(K)$*, then there is a decreasing sequence* $(E_n)_n$ *in* $C(X)$ *such that* $E_n \to E$ *and, for each* $n \in \mathbb{N}$,

(1) $E \subsetneq E_n$,
(2) $E_n - K$ *is nonempty and connected,*
(3) $E_n \cap K$ *is connected and belongs to* $SB(K)$,
(4) $E \subset E_n \cap K \subset Bd_X(E_n - K)$,
(5) $H(K, K \cup E_n) < \varepsilon$.

Moreover, if $L \in 2^X$ *is such that* $E \cap L = \varnothing$*, then each* E_n *can be constructed in such a way that* $E_n \cap L = \varnothing$.

Proof: Let $L \in 2^X$ be such that $E \cap L = \varnothing$. Since $E \in SB(K)$, there is an order arc $\alpha : I \to C(X)$ such that $\alpha(0) = E$ and $\alpha(t) - K \neq \varnothing$ for each $t > 0$. Since the set $\mathcal{U} = \{A \in C(X) : A \subset (X - L) \cap N_X(K, \varepsilon)\}$ is open in $C(X)$, $E \in \mathcal{U}$ and α is continuous, there exists $t_1 > 0$ such that $G_1 = \alpha(t_1) \in \mathcal{U}$. Let $(t_n)_{n \geq 2}$ be a strictly decreasing sequence such that, for each $n \in \mathbb{N} - \{1\}$, $0 < t_n < \frac{1}{n}$ and $t_2 < t_1$. Put $G_n = \alpha(t_n)$. Then $G_n \to E$ and, for each $n \in \mathbb{N} - \{1\}$, $G_n \subsetneq G_{n-1}$. Note that each G_n satisfies the following conditions:

a) $E \subsetneq G_n$,
b) $G_n - K \neq \varnothing$,

c) $G_n \cap L = \varnothing$ and
d) $H(K, K \cup G_n) < \varepsilon$.

Suppose that some set $G_n - K$ has at least three components C_1, C_2 and C_3. For each $i \in \{1,2,3\}$, define $D_i = \mathrm{Cl}_X(C_i)$ and let $T = K \cup D_1 \cup D_2 \cup D_3$. It it easy to see that T is a triod in X, with core K, such that $T \in B_{C(X)}(K, \varepsilon)$. Since this is a contradiction, we infer that

1) for each $n \in \mathbb{N}$, $G_n - K$ has at most two components.

We claim that

2) if for some $n \in \mathbb{N}$, $G_n - K$ has two components A_n and B_n, and A_m is a component of $G_m - K$ for some $m \geq n$, then the set $C_m = K \cap \mathrm{Cl}_X(A_m)$ is nonempty and connected.

By part (2) of Lemma 3.1, C_m is nonempty. Let us assume, without loss of generality, that $A_m \subset A_n$. If C_m is disconnected, then by Lemma 3.5 we infer that C_n has exactly two components K_1 and K_2. Using order arcs from K_1 to $\mathrm{Cl}_X(A_m)$ and from K_2 to $\mathrm{Cl}_X(A_m)$, respectively, we can find two subcontinua K_1' and K_2' of $\mathrm{Cl}_X(A_m)$ such that $K_1 \subsetneq K_1'$, $K_2 \subsetneq K_2'$ and $K_1' \cap K_2' = \varnothing$. Then, it is easy to see that $T = K \cup K_1' \cup K_2' \cup \mathrm{Cl}_X(B_n)$ is a triod in X, with core K, such that $T \in B_{C(X)}(K, \varepsilon)$. This contradiction shows that C_m is connected. Then 2) is established.

Now, we claim that

3) there is a subsequence $(n_i)_i$ of $(n)_n$ and a decreasing sequence $(A_{n_i})_i$ in which, for each $i \in \mathbb{N}$, A_{n_i} is a component of $G_{n_i} - K$ such that if $C_{n_i} = K \cap \mathrm{Cl}_X(A_{n_i})$, then $(C_{n_i})_i$ is a decreasing sequence of subcontinua of K such that, for each $i \in \mathbb{N}$,
   i) $C_{n_i} \in SB(K)$,
   ii) $E \subset C_{n_i}$ and
   iii) $C_{n_i} \to E$.

To show 3), let us assume first that $G_1 - K$ has two components A_1 and B_1. Suppose that, for each $n \in \mathbb{N} - \{1\}$, $G_n - K$ has two components A_n and B_n. If for some $n \in \mathbb{N} - \{1\}$, we have $A_n, B_n \subset A_{n-1}$, then it is easy to see that the set

$$T = K \cup \mathrm{Cl}_X(A_n) \cup \mathrm{Cl}_X(B_n) \cup \mathrm{Cl}_X(B_{n-1})$$

is a triod in X, with core K, such that $T \in B_{C(X)}(K, \varepsilon)$. This is a contradiction. If $A_n, B_n \subset B_{n-1}$, in a similar way we obtain a contradiction. Thus, we can assume that $A_n \subset A_{n-1}$ and $B_n \subset B_{n-1}$ for each $n \in \mathbb{N} - \{1\}$. Put $C_n = K \cap \mathrm{Cl}_X(A_n)$ and $D_n = K \cap \mathrm{Cl}_X(B_n)$ for any $n \in \mathbb{N}$. By 2), each C_n and each D_n is nonempty and connected. By part (1) of Lemma 3.1, $C_n, D_n \in SB(K)$ for each $n \in \mathbb{N}$. Clearly, $C_n \subset C_{n-1}$ and $D_n \subset D_{n-1}$ for $n > 1$, so $C_n \to C = \bigcap_{n\in\mathbb{N}} C_n$ and $D_n \to D = \bigcap_{n\in\mathbb{N}} D_n$. Then, by Theorem 3.2, $C, D \in SB(K)$. Since $C, D \subset G_n$ for each $n \in \mathbb{N}$ and $G_n \to E$, we have $C, D \subset E$. Then $C = D = E$ and we have proved that

3.1) if for each $n \in \mathbb{N}$, $G_n - K$ has two components A_n and B_n, then the sets $C_n = K \cap \mathrm{Cl}_X(A_n)$ and $D_n = K \cap \mathrm{Cl}_X(B_n)$ are elements in $SB(K)$ that contains E. Moreover, $C_n \to E$ and $D_n \to E$.

Therefore, the sequences $(A_n)_n$ and $(C_n)_n$ satisfy the required properties. Assume now that $G_t - K$ is connected for some $t \in \mathbb{N} - \{1\}$. Let m be the first natural number such that $G_m - K$ is connected. Since $G_1 - K$ is disconnected, we have $m > 1$. Put $A_m = G_m - K$. Note that G_{m-1} has two components A_{m-1} and B_{m-1}. Then we can assume, without loss of generality, that $A_m \subset A_{m-1}$. We will show that $G_n - K$ is connected for each $n > m$ so let us assume, to the contrary, that there is $n > m$ such that $G_n - K$ has two components A_n and B_n. Then $A_n, B_n \subset A_m$ and the set $T = K \cup \mathrm{Cl}_X(A_n) \cup \mathrm{Cl}_X(B_n) \cup \mathrm{Cl}_X(B_{m-1})$ is a triod in X, with core K, such that $T \in B_{C(X)}(K, \varepsilon)$, which is a contradiction. This shows that $A_n = G_n - K$ is connected for each $n > m$. Put $C_n = K \cap \mathrm{Cl}_X(A_n)$ for $n \geq m$. By 2) each C_n is nonempty and connected. By part (1) of Lemma 3.1, $C_n \in SB(K)$ for each $n \geq m$. Clearly $C_n \subset C_{n-1}$ for $n > m$, so $C_n \to C = \bigcap_{n\geq m} C_n$. By Theorem 3.2, $C \in SB(K)$. Since $C \subset G_n$ for any $n \in \mathbb{N}$ and $G_n \to E$, we have $C \subset E$. Then $C = E$, and we have proved that

3.2) if there exists a first natural number m such that $G_m - K$ is connected, then for each $n \geq m$, $A_n = G_n - K$ is connected and the set $C_n = K \cap \mathrm{Cl}_X(A_n)$ is an element in $SB(K)$ that contains E. Moreover, $C_n \to E$.

Therefore, the sequences $(A_n)_{n\geq m}$ and $(C_n)_{n\geq m}$ satisfy the required properties. Let us assume now that $G_1 - K$ is connected. If for some $n \in \mathbb{N} - \{1\}$, $G_n - K$ has two components, then following the same ideas that gave us 3.1) and 3.2), we obtain sequences $(A_{n_i})_i$ and $(C_{n_i})_i$ with the required properties. Then we can assume that $G_n - K$ is connected for each $n \in \mathbb{N}$. Put $A_n = G_n - K$ and $C_n = K \cap \mathrm{Cl}_X(A_n)$ for any $n \in \mathbb{N}$. By Lemma 3.5, each C_n has at most two components. We will show that there is $N \in \mathbb{N}$ such that C_n is connected for any $n \geq N$. To this aim let us assume, to the contrary, that there is a sequence of natural numbers $n_1 < n_2 < \cdots$ such that, for each $j \in \mathbb{N}$, C_{n_j} has two components K_j and L_j. By part (1) of Lemma 3.1, $K_j, L_j \in SB(K)$ for any $j \in \mathbb{N}$. If for some $j \in \mathbb{N}$ we have $K_{j+1}, L_{j+1} \subset K_j$ then K_{j+1}, L_{j+1} and L_j are three mutually disjoint elements in $SB(K)$. Thus, by Lemma 3.4, there is a triod T in X such that $T \in B_{C(X)}(K, \varepsilon)$. This is a contradiction. If $K_{j+1}, L_{j+1} \subset L_j$, in a similar way we obtain a contradiction. Thus we can assume that $K_{j+1} \subset K_j$ and $L_{j+1} \subset L_j$ for each $j \in \mathbb{N}$. Hence $K_j \to K_0 = \bigcap_{j\in\mathbb{N}} K_j$ and $L_j \to L_0 = \bigcap_{j\in\mathbb{N}} L_j$. Note that $K_0 \cap L_0 = \varnothing$. By Theorem 3.2, $K_0, L_0 \in SB(K)$. Since $K_0, L_0 \subset G_{n_j}$ for each $j \in \mathbb{N}$ and $G_{n_j} \to E$ we have $K_0, L_0 \subset E$. Then $K_0 = L_0 = E$. This implies that $K_0 \cap L_0 \neq \varnothing$, which is a contradiction. Then, there exists $N \in \mathbb{N}$ such that C_n is connected for each $n \geq N$. Hence $(C_n)_{n\geq N}$ is a decreasing sequence of elements in $SB(K)$, so $C_n \to C = \bigcap_{n\geq N} C_n$. Proceeding as before, it can be seen that $C = E$. Then, the sequences $(A_n)_{n\geq N}$ and $(C_n)_{n\geq N}$ satisfy the required properties, and 3) holds.

Let $(n_i)_i$, $(A_{n_i})_i$ and $(C_{n_i})_i$ be as assumed in 3). For each $i \in \mathbb{N}$ put $E_i = \mathrm{Cl}_X(A_{n_i})$. Given $i \in \mathbb{N}$, we show that E_i satisfies properties (1)-(5). To show (1) note that $K \cap E_i = K \cap \mathrm{Cl}_X(A_{n_i}) = C_{n_i}$ and that $E \subset C_{n_i}$. Thus $E \subset K \cap E_i \subset E_i$. Since $E \subset K$, $A_{n_i} \subset E_i$ and $A_{n_i} \cap K = \varnothing$ it follows that $E \subsetneq E_i$. To show (2) note that, by part (2) of Lemma 3.1, $\mathrm{Bd}_X(A_{n_i}) - A_{n_i} \subset K$. Thus $E_i - K = A_{n_i}$, so $E_i - K$ is nonempty and connected. To show (3) note that $K \cap E_i = C_{n_i}$ so, by property i) of C_{n_i}, we infer that $K \cap E_i$ is connected and belongs to $SB(K)$. Now we show (4). Since $K \cap A_{n_i} = \varnothing$ and $E_i - K = A_{n_i}$ we have

$$K \cap E_i = K \cap \mathrm{Cl}_X(A_{n_i}) = K \cap \mathrm{Bd}_X(A_{n_i}) \subset \mathrm{Bd}_X(A_{n_i}) = \mathrm{Bd}_X(E_i - K).$$

Thus $E \subset K \cap E_i \subset \mathrm{Bd}_X(E_i - K)$. To show (5) note that, since $E_i \subset G_{n_i}$, $G_{n_i} \cap L = \varnothing$ and $H(K, K \cup G_{n_i}) < \varepsilon$ we have $E_i \cap L = \varnothing$ and $H(K, K \cup E_i) < \varepsilon$.

It is clear that $(E_i)_i$ is a decreasing sequence. Since $E \subset E_i \subset G_{n_i}$ for each $i \in \mathbb{N}$ and $G_{n_i} \to E$, we have $E_i \to E$. This finishes the proof of the theorem. $\square$

As a simple application of Theorem 3.7, we have the following lemma.

Lemma 3.8. *Let $\varepsilon > 0$ be such that no $T \in B_{C(X)}(K, \varepsilon)$ is a triod in X. Let $A, B \in C(K) - \{K\}$ be such that $K = A \cup B$ and $H(A, K) < \varepsilon$. If $E \in m(K)$ satisfies $E \subset A - B$, then $A \cap B$ is connected.*

Proof: Assume, to the contrary, that $A \cap B$ has at least two components C and D. Let $C', D' \in C(B)$ be such that $C \subsetneq C'$, $D \subsetneq D'$ and $C' \cap D' = \varnothing$. By Theorem 3.7, there exists $E' \in C(X)$ such that $E \subsetneq E'$, $E' - K \neq \varnothing$, $E' \cap B = \varnothing$ and $H(K, K \cup E') < \varepsilon$. Then, it is easy to see that $T = A \cup C' \cup D' \cup E'$ is a triod in X, with core A, such that $T \in B_{C(X)}(K, \varepsilon)$, which is a contradiction. Then $A \cap B$ is connected. $\square$

Theorem 3.9. *Let $\varepsilon > 0$ be such that no $T \in B_{C(X)}(K, \varepsilon)$ is a triod in X. If $m(K)$ has at least two elements E and F, then there exist decreasing sequences $(E_n)_n$ and $(F_n)_n$ in $C(X)$ such that $E_n \to E$, $F_n \to F$ and, for each $n \in \mathbb{N}$,*

1. *$E \subsetneq E_n$ and $F \subsetneq F_n$,*
2. *$E_n - K$ and $F_n - K$ are nonempty and connected,*
3. *$E_n \cap K$ and $F_n \cap K$ are connected and belong to $SB(K)$,*
4. *$E \subset E_n \cap K \subset Bd_X(E_n - K)$,*
5. *$F \subset F_n \cap K \subset Bd_X(F_n - K)$,*
6. *$E_n \cap F_n = \varnothing$ if $E \cap F = \varnothing$; otherwise $E_n \cap F_n \subset K$,*
7. *$H(K, K \cup E_n) < \varepsilon$ and $H(K, K \cup F_n) < \varepsilon$,*
8. *if $T_n = K \cup E_n \cup F_n$, then $H(T_n, K) < \varepsilon$.*

Moreover, if $L_1, L_2 \in 2^X$ are such that $E \cap L_1 = \varnothing$ and $F \cap L_2 = \varnothing$, then each E_n and each F_n can be constructed in such a way that $E_n \cap L_1 = \varnothing$ and $F_n \cap L_2 = \varnothing$.

Proof: Let $L_1, L_2 \in 2^X$ be such that $E \cap L_1 = \varnothing$ and $F \cap L_2 = \varnothing$. Let us assume first that $E \cap F = \varnothing$. By Theorem 3.7 there is a decreasing sequence $(E_n)_n$ in $C(X)$ converging to E and such that, for each $n \in \mathbb{N}$, E_n satisfies properties (1)-(5) of the same theorem and $E_n \cap (F \cup L_1) = \varnothing$. Applying again Theorem 3.7 we infer that there is a decreasing sequence $(F_n)_n$ in $C(X)$ converging to F and such that, for each $n \in \mathbb{N}$, E_n satisfies properties (1)-(5) of the same theorem and $F_n \cap (E_1 \cup L_2) = \varnothing$. Thus, sequences $(E_n)_n$ and $(F_n)_n$ satisfy the required properties.

Let us assume now that $E \cap F \neq \varnothing$. Fix points $e \in E - F$ and $f \in F - E$. By Theorem 3.7, there exist decreasing sequences $(E_i)_i$ and $(F_i)_i$ such that $E_i \to E$, $F_i \to F$ and, for each $i \in \mathbb{N}$,

a) $E \subsetneq E_i$ and $F \subsetneq F_i$,
b) $E_i - K$ and $F_i - K$ are nonempty and connected,
c) $E_i \cap K$ and $F_i \cap K$ are connected and belong to $SB(K)$,
d) $E \subset E_i \cap K \subset \mathrm{Bd}_X(E_i - K)$,
e) $F \subset F_i \cap K \subset \mathrm{Bd}_X(F_i - K)$,
f) $E_i \cap (L_1 \cup \{f\}) = \varnothing$ and $F_i \cap (L_2 \cup \{e\}) = \varnothing$,
g) $H(K, K \cup E_i) < \varepsilon$ and $H(K, K \cup F_i) < \varepsilon$.

Define, for each $i \in \mathbb{N}$, $T_i = K \cup E_i \cup F_i$. By g), $H(T_i, K) < \varepsilon$. If for some $i \in \mathbb{N}$ we have $F_i \subset E_i \cup K$, then $F_i - K \subset E_i$. This implies, by e), that $F \subset \mathrm{Bd}_X(F_i - K) \subset E_i$. In particular, $f \in E_i$. Since this contradicts f), we have shown that

1) for each $i \in \mathbb{N}$, F_i is not contained in $E_i \cup K$.

Similarly

2) for each $i \in \mathbb{N}$, E_i is not contained in $F_i \cup K$.

We claim that

3) for each $i \in \mathbb{N}$ there exists $j(i) \geq i$ such that $E_i \cap F_{j(i)} \subset K$.

To show 3) let $i \in \mathbb{N}$ and assume, to the contrary, that $E_i \cap F_j \nsubseteq K$ for each $j \geq i$. For $j \geq i$ define

$$\mathcal{H}_j = \{\ C \subset E_i \cap F_j\ : C \text{ is a component of } E_i \cap F_j \text{ such that } C \cap K = \varnothing\}.$$

Let us assume that, for some $j \geq i$, $\mathcal{H}_j$ has at least three elements C_1, C_2 and C_3. Since $F_j \subset F_i$ and, by 2), $E_i \nsubseteq F_i \cup K$ it follows that $E_i \nsubseteq F_j \cup K$. By Lemma 3.5, there exist three mutually disjoint subcontinua C_1', C_2' and C_3' of E_i such that, for each $s \in \{1, 2, 3\}$, $C_s \subsetneq C_s'$, $C_s' - F_j \neq \varnothing$, $C_s' \cap K = \varnothing$ and $T = F_j \cup C_1' \cup C_2' \cup C_3'$ is a triod in X. Put $T' = K \cup T$. It is easy to see that T' is a triod in X such that $T' \in B_{C(X)}(K, \varepsilon)$. Since this is a contradiction, we have shown that

3.1) for each $j \geq i$, $\mathcal{H}_j$ has at most two elements.

We claim that

3.2) there exists $j_0 \geq i$ such that $\mathcal{H}_j = \varnothing$ for each $j \geq j_0$.

Assume, to the contrary, that for each $j \geq i$ there is $j' \geq j$ such that $\mathcal{H}_{j'} \neq \varnothing$. Given $j \geq i$ such that $\mathcal{H}_j \neq \varnothing$ we denote by C_1^j and C_2^j the elements of $\mathcal{H}_j$ and we agree that $C_1^j \neq \varnothing$. Let $j_1 \geq i$ be such that $\mathcal{H}_{j_1} \neq \varnothing$. Since the set $\mathcal{U}_1 = \left\{A \in C(X) : A \subset X - \left(C_1^{j_1} \cup C_2^{j_1}\right)\right\}$ is open in $C(X)$, $F \in \mathcal{U}_1$ and $F_j \to F$, there exists $J_1 \in \mathbb{N}$ such that $J_1 \geq j_1$ and $F_j \in \mathcal{U}_1$ for each $j \geq J_1$. Let $j_2 \geq J_1$ be such that $\mathcal{H}_{j_2} \neq \varnothing$. Note that $F_{j_2} \cap \left(C_1^{j_1} \cup C_2^{j_1}\right) = \varnothing$, so $j_2 > j_1$. Proceeding as before, we infer that there exists $J_2 \in \mathbb{N}$ such that $J_2 \geq j_2$ and $F_j \cap \left(C_1^{j_2} \cup C_2^{j_2}\right) = \varnothing$ for each $j \geq J_2$. Let $j_3 \geq J_2$ be such that $\mathcal{H}_{j_3} \neq \varnothing$. Note that $F_{j_3} \cap \left(C_1^{j_2} \cup C_2^{j_2}\right) = \varnothing$, so $j_3 > j_2$. Since $j_3 \geq J_1$ we also have $F_{j_3} \cap \left(C_1^{j_1} \cup C_2^{j_1}\right) = \varnothing$. Note that $C_1^{j_1}, C_1^{j_2}$ and $C_1^{j_3}$ are nonempty and mutually disjoint and that, for each $s \in \{1,2,3\}$, $C_1^{j_s} \cap K = \varnothing$ and $C_1^{j_s} \subsetneq F_{j_s}$. Let G_1, G_2 and G_3 be three mutually disjoint continua such that, for each $s \in \{1,2,3\}$, $C_1^{j_s} \subsetneq G_s \subset F_{j_s}$ and $G_s \cap K = \varnothing$. Note that, for each $s \in \{1,2,3\}$, we have $G_s - E_i \neq \varnothing$. Put $T = K \cup E_i \cup G_1 \cup G_2 \cup G_3$. It is easy to see that T is a triod in X, with core $K \cup E_i$, such that $T \in B_{C(X)}(K, \varepsilon)$. This contradiction shows 3.2).

Now, for each $j \geq j_0$ define

$$\mathcal{L}_j = \{\ C \subset E_i \cap F_j\ : C \text{ is a component of } E_i \cap F_j \text{ such that } C - K \neq \varnothing\}.$$

Let us assume that, for some $j \geq j_0$, $\mathcal{L}_j$ has at least three elements C_1, C_2 and C_3. For each $s \in \{1,2,3\}$ let D_s be a component of $C_s \cap K$. By part (1) of Lemma 3.1, $D_1, D_2, D_3 \in SB(K)$. Clearly, D_1, D_2 and D_3 are mutually disjoint. Then, by Lemma 3.4, there is a triod T in X such that $T \in B_{C(X)}(K, \varepsilon)$. Since this is a contradiction, we have shown that

3.3) for each $j \geq j_0$, $\mathcal{L}_j$ has at most two elements.

We claim that

3.4) there exists $m_0 \geq j_0$ such that $\mathcal{L}_j = \varnothing$ for each $j \geq m_0$.

Suppose, to the contrary, that for each $j \geq j_0$ there exists $j' \geq j$ such that $\mathcal{L}_{j'} \neq \varnothing$. Then, there is a subsequence $(j_s)_s$ of $(j)_{j \geq j_0}$ such that $\mathcal{L}_{j_s} \neq \varnothing$ for each $s \in \mathbb{N}$. It is easy to see that if $l < s$, then each element of $\mathcal{L}_{j_s}$ is contained in some element of $\mathcal{L}_{j_l}$. By 3.3) $\mathcal{L}_{j_1}$ has at most two elements D_1^1 and D_2^1. Define $U_1 = \{s > 1 : \mathcal{L}_{j_s}$ has an element contained in $D_1^1\}$ and $V_1 = \{s > 1 : \mathcal{L}_{j_s}$ has an element contained in $D_2^1\}$. Since each element of $\mathcal{L}_{j_s}$ $(s > 1)$ is either contained in D_1^1 or in D_2^1 we may assume, without loss of generality, that the set U_1 in infinite. Put $C_1 = D_1^1, s_1 = 1$ and $s_2 = \min U_1$. By 3.3) $\mathcal{L}_{j_{s_2}}$ has at most two elements $D_1^{s_2}$ and $D_2^{s_2}$ and, without loss of generality, $D_1^{s_2} \subset D_1^{s_1}$. We define a subcontinuum C_2 of X and a positive number $s_3 > s_2$ as follows. If $D_2^{s_2} \subset D_2^{s_1}$ we define $C_2 = D_1^{s_2}$ and $s_3 = \min\{s > s_2 : \mathcal{L}_{j_s}$ has an element contained in $D_1^{s_2}\}$. If $\varnothing \neq D_2^{s_2} \subset D_1^{s_1}$, we first consider the sets $U_2 = \{s > s_2 : \mathcal{L}_{j_s}$ has an element contained in $D_1^{s_2}\}$ and $V_2 = \{s > s_2 : \mathcal{L}_{j_s}$ has an element contained in $D_2^{s_2}\}$. Since each element of $\mathcal{L}_{j_s}$ $(s > s_2)$ is either contained in $D_1^{s_2}$ or in $D_2^{s_2}$ we may assume, without loss of generality, that the set U_2 in infinite. Define $C_2 = D_1^{s_2}$ and $s_3 = \min U_2$. Proceeding as before, it is possible to find $D_1^{s_3} \in \mathcal{L}_{j_{s_3}}$ such that $D_1^{s_3} \subset D_1^{s_2}$ and the set $U_3 = \{s > s_3 : \mathcal{L}_{j_s}$ has an element contained in $D_1^{s_3}\}$ is infinite. Define $C_3 = D_1^{s_3}$ and $s_4 = \min U_3$. Following this argument, we can obtain a subsequence $(j_{s_m})_m$ of $(j_s)_s$ and a decreasing sequence $(C_m)_m$ such that $s_1 = 1$ and, for each $m \in \mathbb{N}$, $C_m \in \mathcal{L}_{j_{s_m}}$. Then, $C_m \to C_0 = \bigcap_{m \in \mathbb{N}} C_m$.

Since $C_m \subset E_i \cap F_{j_{s_m}}$ for each $m \in \mathbb{N}$ and $F_{j_{s_m}} \to F$, it follows that $C_0 \subset E_i \cap F$. Moreover, by Theorem 3.3, $C_0 \in SB(K)$. Then, since $C_0 \subset F$ and $F \in m(K)$ we have $C_0 = F$. This implies that $F = C_0 \subset E_i \cap F \subset E_i$. In particular, $f \in E_i$, which contradicts f). Therefore, 3.4) is satisfied.

To conclude the proof of 3), just define $j(i) = m_0$.

Note that, by 3), $E_{j(i)} \cap F_{j(i)} \subset E_i \cap F_{j(i)} \subset K$ for each $i \in \mathbb{N}$. Define $n_1 = j(1)$ and, for each $s \geq 2$, $n_s = j(n_{s-1} + 1)$. It is clear that the decreasing sequences $(E_{n_s})_s$ and $(F_{n_s})_s$ satisfy properties (1)-(8). Moreover $E_{n_s} \to E$, $F_{n_s} \to F$ and, for each $s \in \mathbb{N}$, $E_{n_s} \cap L_1 = \varnothing$ and $F_{n_s} \cap L_2 = \varnothing$. □

Theorem 3.10. *Suppose that $m(K)$ has two elements E and F such that $E \cap F \neq \varnothing$ and $E \cup F \neq K$. Then, for each $\varepsilon > 0$, there is a triod T in X such that $T \in B_{C(X)}(K, \varepsilon)$.*

Proof: Assume, to the contrary, that there exists $\varepsilon > 0$ such that no $T \in B_{C(X)}(K, \varepsilon)$ is a triod in X. Fix a point $k \in K - (E \cup F)$. By Theorem 3.9, there exist $E', F' \in C(X)$ such that $E \subsetneq E'$, $F \subsetneq F'$, $E' - K$ and $F' - K$ are nonempty, $E' \cap K$ and $F' \cap K$ are connected, $E' \cap F' \subset K$, $k \notin E' \cup F'$ and $T = K \cup E' \cup F' \in B_{C(X)}(K, \varepsilon)$. Put $B = (E' \cup F') \cap K$. Since $E \cap F \neq \varnothing$ we have $B \in C(T)$. It is easy to see that T is a triod in X with core B. This contradiction finishes the proof of the theorem. □

Theorem 3.11. *Suppose that $m(K)$ has two elements E and F such that $E \cup F = K$. Then, either there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathrm{int}_{C(X)}(\mathcal{D})$ or, for each $\varepsilon > 0$, there is a triod T in X such that $T \in B_{C(X)}(K, \varepsilon)$.*

Proof: Let us assume that there is $\varepsilon > 0$ such that no $T \in B_{C(X)}(K, \varepsilon)$ is a triod in X. Fix points $e \in E - F$ and $f \in F - E$. By Theorem 3.9, there exist $E', F' \in C(X)$ such that $E \subsetneq E'$, $F \subsetneq F'$, $E' - K$ and $F' - K$ are nonempty, $E' \cap K$ and $F' \cap K$ are connected, $E' \cap F' \subset K$, $f \notin E'$, $e \notin F'$, $H(K, K \cup E') < \varepsilon$ and $H(K, K \cup F') < \varepsilon$. Put $E_0 = E' \cap K$ and $F_0 = F' \cap K$. Note that $K = E_0 \cup F_0$, $E_0 \cap F_0 = E' \cap F'$, $e \in E_0 - F_0$ and $f \in F_0 - E_0$. We claim that

1) there exists $L \in C(E_0) - \{E_0\}$ such that $E_0 \cap F_0 \subset L$.

To show 1) let us assume, to the contrary, that there is no $L \in C(E_0) - \{E_0\}$ such that $E_0 \cap F_0 \subset L$. Then $E_0 \cap F_0$ is disconnected. Fix a component C of $E_0 \cap F_0$ and let $\alpha_1 : I \to C(X)$ be an order arc from C to E_0. Define $\alpha : I \to C(X)$ by $\alpha(t) = F_0 \cup \alpha_1(t)$. Clearly, α is a map such that $\alpha(0) = F_0, \alpha(1) = K$ and $\alpha(s) \subset \alpha(t)$ whenever $s \leq t$. Let $t_0 = \min\{t \in I : \alpha(t) = K\}$ and let $t \in (0, t_0)$ be such that $H(\alpha(t), K) < \varepsilon$. Put $L = \alpha(t)$ and note that $E_0 - L \neq \varnothing$. If $\alpha_1(t)$ intersects all the components of $E_0 \cap F_0$, then $\alpha_1(t) \cup (E_0 \cap F_0)$ is a proper subcontinuum of E_0 that contains $E_0 \cap F_0$. Since this is a contradiction we infer that there is a component D of $E_0 \cap F_0$ such that $\alpha_1(t) \cap D = \varnothing$. Let $D', G \in C(E_0)$ be such that $D \subsetneq D'$,

$\alpha_1(t) \subsetneq G$ and $D' \cap G = \varnothing$. Put $L' = L \cup G$ and note that $L' \in C(K)$, $L \subsetneq L'$, $D' - F_0 \neq \varnothing$ and $D' \cap L' = D' \cap F_0 \subset F_0$. Hence, it is easy to see that the set $T = L' \cup D' \cup F'$ is a triod in X, with core L, such that $T \in B_{C(X)}(K, \varepsilon)$. Since this is a contradiction 1) is satisfied.

By 1) there is $L \in C(E_0) - \{E_0\}$ such that $E_0 \cap F_0 \subset L$. Define $A_1 = E_0$, $A_2 = F_0 \cup L$, $B_1 = A_1 \cup E'$, $B_2 = A_2 \cup F'$ and $B = B_1 \cup B_2$. Clearly $A_1, A_2 \in C(K) - \{K\}$, $B_1, B_2 \in C(B) - \{B\}$, $A_1 \cup A_2 = K$, $A_1 \subsetneq B_1$ and $A_2 \subsetneq B_2$. Moreover, $A_1 \cap A_2 = L = B_1 \cap B_2 \in C(X)$. Then, by Lemma 2.4, there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathrm{int}_{C(X)}(\mathcal{D})$. This finishes the proof of the theorem. □

As a direct consequence of Theorems 3.10 and 3.11, we have the following result.

Theorem 3.12. *If $m(K)$ has at least two elements E and F such that $E \cap F \neq \varnothing$, then either there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathrm{int}_{C(X)}(\mathcal{D})$ or, for each $\varepsilon > 0$, there is a triod T in X such that $T \in B_{C(X)}(K, \varepsilon)$.*

Now we consider the case in which $|m(K)| = 2$ and the two elements of $m(K)$ are disjoint.

Lemma 3.13. *Suppose that $m(K)$ has exactly two elements E and F. If $E \cap F = \varnothing$ and if there is $L \in C(E \cup F, K) - \{K\}$ then, for each $\varepsilon > 0$, there is a triod T in X such that $T \in B_{C(X)}(K, \varepsilon)$.*

Proof: Assume, to the contrary, that there exists $\varepsilon > 0$ such that no $T \in B_{C(X)}(K, \varepsilon)$ is a triod in X. Fix a point $k \in K - L$. By Theorem 3.9, there exist $E', F' \in C(X)$ such that $E \subsetneq E'$, $F \subsetneq F'$, $E' - K$ and $F' - K$ are nonempty, $E' \cap K$ and $F' \cap K$ are connected, $E' \cap F' = \varnothing$, $H(K, K \cup E') < \varepsilon$, $H(K, K \cup F') < \varepsilon$ and $k \notin E' \cup F'$. Define $C = L \cup (E' \cap K) \cup (F' \cap K)$ and $T = K \cup E' \cup F'$. It is easy to see that T is a triod in X, with core C, such that $T \in B_{C(X)}(K, \varepsilon)$. This contradiction finishes the proof of the lemma. □

Theorem 3.14. *Suppose that $m(K)$ has exactly two elements E and F. If K is decomposable and $E \cap F = \varnothing$, then either there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathrm{int}_{C(X)}(\mathcal{D})$ or, for each $\varepsilon > 0$, there is a triod T in X such that $T \in B_{C(X)}(K, \varepsilon)$.*

Proof: Assume that there is $\varepsilon > 0$ such that no $T \in B_{C(X)}(K, \varepsilon)$ is a triod in X. By Lemma 3.13

1) there is no any proper subcontinuum of K containing $E \cup F$.

We claim that

2) if $A, B \in C(K) - \{K\}$ satisfy $K = A \cup B$ and $E \subset (A - B) \cup (B - A)$, then there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathfrak{int}_{C(X)}(\mathcal{D})$.

To show this, let $A, B \in C(K) - \{K\}$ be as considered in 2). We may assume, without loss of generality, that $E \subset A - B$. Let $L \in C(K)$ be such that $A \subset L \subsetneq K$ and $H(K, L) < \varepsilon$. Clearly, $K = L \cup B$. By 1), $F \not\subset L$. We consider two cases. First assume that $F \cap L = \varnothing$. Let C be the component of $B - L$ that contains B and put $M = \mathrm{Cl}_X(C)$. By [13, Theorem 5.6], $M \cap L \neq \varnothing$. Then, by 1), $K = L \cup M$. Moreover, $E \subset L - M$ and $H(K, L) < \varepsilon$. Then, by Lemma 3.8, $L \cap M$ is connected. By Theorem 3.9, there exist $E', F' \in C(X)$ such that $E \subsetneq E'$, $F \subsetneq F'$, $E' - K$ and $F' - K$ are nonempty, $E' \cap F' = \varnothing$, $E' \cap M = \varnothing$ and $F' \cap L = \varnothing$. Define $A_1 = L$, $A_2 = M$, $B_1 = A_1 \cup E'$, $B_2 = A_2 \cup F'$ and $B' = B_1 \cup B_2$. Clearly $A_1, A_2 \in C(K) - \{K\}$, $B_1, B_2 \in C(B') - \{B'\}$, $A_1 \cup A_2 = K$, $A_1 \subsetneq B_1$ and $A_2 \subsetneq B_2$. Moreover, $B_1 \cap B_2 = L \cap M = A_1 \cap A_2 \in C(X)$. Then, by Lemma 2.4, there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathfrak{int}_{C(X)}(\mathcal{D})$.

Now assume that $F \cap L \neq \varnothing$. By 1) $K = L \cup F$ and, by Lemma 3.8, $L \cap F$ is connected. By Theorem 3.9, there exist $E', F' \in C(X)$ such that $E \subsetneq E'$, $F \subsetneq F'$, $E' \cap F' = \varnothing$, $E' - K$ and $F' - K$ are nonempty and $E' \cap K$ and $F' \cap K$ are nonempty and connected. By 1), $K = L \cup (F' \cap K)$. Since $H(L, K) < \varepsilon$ and $E \subset L - (F' \cap K)$, by Lemma 3.8, $L \cap (F' \cap K) = F' \cap L$ is connected. Define $A_1 = L$, $A_2 = F$, $B_1 = A_1 \cup E', B_2 = F'$ and $B' = B_1 \cup B_2$. Clearly $A_1, A_2 \in C(K) - \{K\}$, $B_1, B_2 \in C(B') - \{B'\}$, $A_1 \cup A_2 = K$, $A_1 \subsetneq B_1$ and $A_2 \subsetneq B_2$. Moreover, $B_0 = B_1 \cap B_2 = L \cap F' \in C(K)$. Note that $E \subset E' \cap L \subset A_1$ and $E' \cap F' = \varnothing$. Then $A_1 - B_0 \neq \varnothing$. Since $F - L \neq \varnothing$ we infer that $A_2 - B_0 \neq \varnothing$ and since $E' - K$ and $F' - K$ are nonempty, we have $B_1 - (A_1 \cup B_2) \neq \varnothing$ and $B_2 - (A_2 \cup B_1) \neq \varnothing$. Then, by Lemma 2.5, there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathfrak{int}_{C(X)}(\mathcal{D})$, so 2) is established.

Similarly we have

3) if $A, B \in C(K) - \{K\}$ satisfy $K = A \cup B$ and $F \subset (A - B) \cup (B - A)$, then there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathtt{int}_{C(X)}(\mathcal{D})$.

Since K is decomposable there exist $A, B \in C(K) - \{K\}$ such that $K = A \cup B$. By 2) and 3) we may assume that $E \cap (A \cap B) \neq \varnothing$ and $F \cap (A \cap B) \neq \varnothing$. By 1) $K = A \cup E \cup F$. If $E \subset A$, then A and F are proper subcontinua of K such that $K = A \cup F$ and $E \subset A - F$. Then, by 2), there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathtt{int}_{C(X)}(\mathcal{D})$. If $F \subset A$ we obtain the same conclusion. Hence let us assume that $E - A \neq \varnothing$ and $F - A \neq \varnothing$. By Theorem 3.9, there exist $E', F' \in C(X)$ such that $E \subsetneq E'$, $F \subsetneq F'$, $E' \cap F' = \varnothing$ and $E' - K$ and $F' - K$ are nonempty. Define $A_1 = A \cup E$, $A_2 = A \cup F$, $B_1 = A_1 \cup E', B_2 = A_2 \cup F'$ and $B' = B_1 \cup B_2$. Clearly $A_1, A_2 \in C(K) - \{K\}$, $B_1, B_2 \in C(B') - \{B'\}$, $A_1 \cup A_2 = K$, $A_1 \subsetneq B_1$ and $A_2 \subsetneq B_2$. Moreover, $A_1 \cap A_2 = A = B_1 \cap B_2 \in C(X)$. Then, by Lemma 2.4, there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathtt{int}_{C(X)}(\mathcal{D})$. This finishes the proof of the theorem □

As a direct consequence of Theorems 3.6, 3.12 and 3.14, we have the following result.

Theorem 3.15. *If K is decomposable and $m(K)$ has at least two elements, then either there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathtt{int}_{C(X)}(\mathcal{D})$ or, for each $\varepsilon > 0$, there is a triod T in X such that $T \in B_{C(X)}(K, \varepsilon)$.*

Corollary 3.16. *If X is atriodic, K is decomposable and $m(K)$ has at least two elements, then there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathtt{int}_{C(X)}(\mathcal{D})$.*

Theorem 3.17. *If K is indecomposable and $C(X) - \{K\}$ is arcwise connected then, for each $\varepsilon > 0$, there is a triod T in X such that $T \in B_{C(X)}(K, \varepsilon)$.*

Proof: We claim that

1) for each $a \in K$ there exists $E_a \in SB(K) - \{K\}$ such that $a \in E_a$.

Let $a \in K$. Since $C(X) - \{K\}$ is arcwise connected there is a map $\alpha : I \to C(X) - \{K\}$ such that $\alpha(0) = \{a\}$ and $\alpha(1) = X$. Let $\beta : I \to C(X)$ be the map $\beta(t) = \bigcup \alpha([0, t])$. Clearly, $\beta(0) = \{a\}$, $\beta(1) = X$ and $\beta(s) \subset \beta(t)$ whenever $s \leq t$. Define

$t_0 = \max\{t \in I : \beta(t) \subset K\}$ and $E_a = \beta(t_0)$. Note that, for each $t \in [0, t_0]$, $\alpha(t) \subset \beta(t_0) = E_a \subset K$, so $a \in E_a$. If $E_a = K$, then by [14, Theorem 1.50], $K \in \alpha([0,t])$. Since this is a contradiction we infer that $E_a \subsetneq K$. Since $X \nsubseteq K$ we have $t_0 < 1$, so the map $\gamma = \beta_{|[t_0,1]}$ satisfies that $\gamma(t_0) = E_a$ and $\gamma(t) - K \neq \varnothing$ for each $t > t_0$. Then, $E_a \in SB(K)$. This shows 1). Clearly, E_a is contained in the composant of a in K.

Take three points a, b and c in different composants of K. By 1), there exist $E_a, E_b, E_c \in SB(K) - \{K\}$ such that $a \in E_a$, $b \in E_b$ and $c \in E_c$. It is easy to see that the sets E_a, E_b and E_c are mutually disjoint. Hence, the conclusion follows from Lemma 3.4. □

4. Compactifications of $(-\infty, \infty)$

For a compactification of the space $V = (-\infty, \infty)$ and remainder R, we write $X = V \cup R$ and define

$$R_1 = \bigcap_{n \in \mathbb{N}} \mathrm{Cl}_X((n, \infty)) \text{ and } R_2 = \bigcap_{n \in \mathbb{N}} \mathrm{Cl}_X((-\infty, -n)).$$

Then $\mathrm{Cl}_X([0,\infty)) = [0,\infty) \cup R_1$ and $\mathrm{Cl}_X((-\infty, 0]) = (-\infty, 0] \cup R_2$. It is easy to see that R_1 and R_2 are subcontinua of R such that $R_1 \cup R_2 = R$. Moreover, for each $x \in V$ we have $\mathrm{Cl}_X([x,\infty)) = [x,\infty) \cup R_1$ and $\mathrm{Cl}_X((-\infty, x]) = (-\infty, x] \cup R_2$.

From now, the letter X denotes a compactification of V with nondegenerate and connected remainder R. Let $\mathcal{C}$ be the class of such continua. In this section we show that the members of $\mathcal{C}$ are C-determined. We denote by $C(V)$ the set of all closed and bounded subintervals of V.

4.1. **The set $\mathcal{M}_K$.** If K is a proper subcontinuum of R, we define the set

$$\mathcal{M}_K = \{\, A \in C(K) : \text{ there is a sequence } (A_n)_n \text{ in } C(V) \text{ such that } A_n \to A\}.$$

In the following lemma we write some properties of the set $\mathcal{M}_K$.

Lemma 4.1. *If K is a proper subcontinuum of R, then*

(1) *$\mathcal{M}_K$ is closed in $C(K)$ and $F_1(K) \subset \mathcal{M}_K$;*
(2) *$\mathcal{M}_K$ has maximal elements (with respect to the inclusion). Moreover, if $a \in K$, then there is a maximal element E in $\mathcal{M}_K$ such that $a \in E$;*

(3) *if* K *is nondegenerate and* E *is a maximal element in* $\mathcal{M}_K$, *then* $E \in SB(K)$.

Proof: The part (1) is easy to proof. Let $\mu : C(X) \to [0, \infty)$ be a Whitney map. To show (2) let $a \in K$ and define $\mathcal{L}_a = \{A \in \mathcal{M}_K : a \in A\}$. It is easy to see that $\mathcal{L}_a$ is a nonempty closed subset of $C(K)$, so $\mu(\mathcal{L}_a)$ is a nonempty closed subset of $[0, \mu(K)]$. Let $t_0 = \max \mu(\mathcal{L}_a)$ and let $E \in \mathcal{L}_a$ be such that $\mu(K) = t_0$. Then $a \in E$ and E is a maximal element in $\mathcal{M}_K$.

To show (3) let us assume that K is nondegenerate and that E is a maximal element in $\mathcal{M}_K$. Let $(E_n)_n$ be a sequence in $C(V)$ such that $E_n \to E$. Since $K \in SB(K)$ we may assume that $E \neq K$. Put $t_1 = \mu(E)$. Clearly, $0 \leq t_1 < \mu(K)$. Take $\varepsilon > 0$. By [14, Lemma 1.28] there is $\eta > 0$ such that if $A, B \in C(X)$, $A \subset B$ and $\mu(B) - \mu(A) < \eta$, then $H(A, B) < \varepsilon$. Let $t_2 \in (t_1, \mu(K))$ be such that $t_2 - t_1 < \eta$. Since $\mu(E_n) \to t_1$ we may assume that $\mu(E_n) < t_2$ for each $n \in \mathbb{N}$. Given $n \in \mathbb{N}$, taking an order arc from E_n to X in $C(X)$ we infer that there is $L_n \in C(V)$ such that $E_n \subsetneq L_n$ and $\mu(L_n) = t_2$. Let us assume, without loss of generality, that $L_n \to L \in C(X)$. Then $\mu(L) = t_2$ and $E \subsetneq L$, so $H(E, L) < \varepsilon$. Since E is maximal in $\mathcal{M}_K$, we have $L - K \neq \varnothing$. Therefore, by Theorem 3.3, $E \in SB(K)$. □

Theorem 4.2. *Let* K *be a proper and nondegenerate subcontinuum of* R *such that* $K \notin \mathcal{M}_K$. *If* $m(K) = \{E\}$ *and* $E \subsetneq K$ *then, for each* $\varepsilon > 0$, *there is a triod* T *in* X *such that* $T \in B_{C(X)}(K, \varepsilon)$.

Proof: Let us assume, to the contrary, that there is $\varepsilon > 0$ such that no element $T \in B_{C(X)}(K, \varepsilon)$ is a triod in X. Fix a point $e \in E$. By Lemma 4.1, there is a maximal element F in $\mathcal{M}_K$ such that $e \in F$ and $F \in SB(K)$, so $E \subset F$. Since $K \notin \mathcal{M}_K$ we have $F \subsetneq K$. Let $F' \in C(K)$ be such that $F \subsetneq F' \subsetneq K$ and $H(F', K) < \frac{\varepsilon}{2}$. Fix a point $x \in K - F'$. By Lemma 4.1, there is a maximal element G in $\mathcal{M}_K$ such that $x \in G$ and $G \in SB(K)$, so $E \subset G$. Fix a point $m \in F - G$.

Let us assume that $G \cap F'$ is disconnected. If $G \cap F'$ has at least three components, then it is easy to construct a triod T in X, with core F', such that $H(T, F') < \frac{\varepsilon}{2}$. Then $H(T, K) < \varepsilon$. Since this is a contradiction, $G \cap F'$ has only two components C and D. We may assume that $E \subset C$. Let $G' \in C(G)$ be such that $C \subsetneq G'$

and $G' \cap D = \varnothing$. It is easy to see that $G' - F' \neq \varnothing$ and that $G' \cap F' = C$. Fix a point $g \in G' - F'$. By Theorem 3.7, there is $E' \in C(X)$ such that $E \subsetneq E'$, $E' - K \neq \varnothing$, $E' \cap K \in SB(K)$, $H(K, K \cup E') < \varepsilon$ and $g, m \notin E'$. Put $T = E' \cup F' \cup G'$ and $A = (E' \cap K) \cup C$. It is easy to see that T is a triod in X, with core A, such that $T \in B_{C(X)}(K, \varepsilon)$. Since this is a contradiction, $G \cap F'$ is connected. Now, applying Theorem 3.7 again we infer that there is $E' \in C(X)$ such that $E \subsetneq E'$, $E' - K \neq \varnothing$, $E' \cap K \in SB(K)$, $H(K, K \cup E') < \varepsilon$ and $x, m \notin E'$. Put $T = E' \cup F' \cup G$ and $A = (E' \cap K) \cup (F' \cap G)$. It is easy to see that T is a triod in X, with core A, such that $T \in B_{C(X)}(K, \varepsilon)$. This contradiction finishes the proof of the theorem. □

Theorem 4.3. *Let $Y = W \cup S$ be a compactification of the space $W = (-\infty, \infty)$, with nondegenerate and connected remainder S such that $C(X) \approx C(Y)$. Let $h : C(Y) \to C(X)$ be a homeomorphism. If $w \in W$ is such that*

$$K = h(\{w\}) \in C(R) - (F_1(R) \cup \{R, R_1, R_2\}),$$

then $K \notin \mathcal{M}_K$ and every $E \in m(K)$ is a proper subset of K.

Proof: Let us assume that $K \in \mathcal{M}_K$ and take a sequence $(K_n)_n$ in $C(V)$ such that $K_n \to K$. It is not difficult to see that $K \in C(R_1) \cup C(R_2)$ so, without loss of generality, assume that $K \in C(R_1)$. Then, $K \in C(R_1) - \{R_1\}$, so there is $r \in R_1 - K$. Let U be an open subset of X such that $K \subset U$ and $r \notin U$. Take $\varepsilon > 0$ such that $N_X(K, \varepsilon) \subset U$ and let $N \in \mathbb{N}$ be such that $K_n \in B_{C(X)}(K, \varepsilon)$ for each $n \geq N$. Note that $K_n \subset N_X(K, \varepsilon) \subset U$ for each $n \geq N$. Let C be the component of U containing K. Since Y is connected im kleinen at w, by [5, Corollary 4], $C(Y)$ is connected im kleinen at $\{w\}$. Then $C(X)$ is connected im kleinen at K and by [6, Theorem 2] there is $M \geq N$ such that $K_n \subset C$ for each $n \geq M$. This implies that $R_1 \subset C$, so $r \in C \subset U$. This contradiction shows that $K \notin \mathcal{M}_K$

To show the other part of the theorem let us assume that $m(K) = \{K\}$. Fix a point $a \in K$. By part (2) of Lemma 4.1, there is a maximal element M in $\mathcal{M}_K$ such that $a \in M$ and $M \in SB(K)$, so $M = K$. This implies that $K \in \mathcal{M}_K$, which is a contradiction to the first part of the theorem. Then $m(K) \neq \{K\}$, so every element of $m(K)$ is a proper subset of K. □

4.2. **The Theorem.** We are now ready to prove that main result of this paper. Recall that the letter $\mathcal{C}$ denotes the class of compactifications of the real line, with nondegenerate and connected remainder.

Theorem 4.4. *The members of* $\mathcal{C}$ *are* C*-determined.*

Proof: Let $X = V \cup R$ and $Y = S \cup W$ be two compactifications of the space $V = W = (-\infty, \infty)$, with nondegenerate and connected remainder R and S, respectively. Let us assume that $C(X) \approx C(Y)$ and that $h : C(Y) \to C(X)$ is a homeomorphism. We claim that

1) if $w \in W$, then there is no 2-cell $\mathcal{D}$ in $C(X)$ such that $h(\{w\}) \in \mathsf{int}_{C(X)}(\mathcal{D})$.

To show 1) let $w \in W$ and assume, to the contrary, that there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $h(\{w\}) \in \mathsf{int}_{C(X)}(\mathcal{D})$. Then $\mathcal{D}' = h^{-1}(\mathcal{D})$ is a 2-cell in $C(Y)$ such that $\{w\} \in \mathsf{int}_{C(Y)}(\mathcal{D}')$. Take $\varepsilon > 0$ such that $N_Y(\{w\}, \varepsilon) \subset W$. By Lemma 2.3, there is a triod T in Y such that $T \in B_{C(Y)}(\{w\}, \varepsilon)$. Clearly, $T \subset W$ so W contains a triod. Since this is a contradiction, 1) holds.

Put $\Lambda = F_1(V) \cup (C(R) - F_1(R)) \cup \{X\}$. We claim that

2) $h(F_1(W)) \subset \Lambda$.

Assume, to the contrary, that there is a point $w \in W$ such that $K = h(\{w\}) \notin \Lambda$. Then either $K \in F_1(R)$ or K is a proper and nondegenerate subcontinuum of X such that $K \cap V \neq \varnothing$. Note that $C(Y)$ is locally connected at $\{w\}$, so $C(X)$ is locally connected at K. Since R is the remainder of the compactification X, we infer that $C(X)$ is not locally connected at any point of $F_1(R)$. This implies that $K \notin F_1(R)$. Since $K \neq X$ we have $V \not\subseteq K$. Then there exist $a, b \in V$ such that $b < a$ and $[b, a] \cap K = \varnothing$. Put $V_1 = [a, \infty)$, $V_2 = (-\infty, b]$ and $X_0 = V_1 \cup R \cup V_2$. Clearly, K is a nondegenerate subcontinuum of X_0 such that $K \cap (V_1 \cup V_2) \neq \varnothing$ and $a, b \notin K$. Then, by Lemma 2.7, there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathsf{int}_{C(X)}(\mathcal{D})$. Since this contradicts 1), 2) is established.

By 2) and the fact that the set $\{X\}$ is isolated with respect to Λ, it follows that

3) $h(F_1(W)) \subset F_1(V) \cup (C(R) - F_1(R))$.

Let us assume that there is a point $w \in W$ such that

$$K = h(\{w\}) \in C(R) - (F_1(R) \cup \{R, R_1, R_2\}).$$

By [14, Lemma 11.2], the set $C(Y)-\{\{w\}\}$ is arcwise connected. Then $C(X)-\{K\}$ is arcwise connected too. Let us consider the set $\mathcal{M}_K$ defined in Subsection 4.1. By Theorem 4.3, $K \notin \mathcal{M}_K$ and every element of $m(K)$ is a proper subset of K. Moreover, by 1) and Theorems 3.15, 3.17 and 4.2 it follows that, for each $\varepsilon > 0$, there is a triod T in X such that $T \in B_{C(X)}(K, \varepsilon)$.

Take $\delta, \varepsilon > 0$ such that $N_Y(\{w\}, \delta) \subset W$ and $h^{-1}(B_{C(X)}(K, \varepsilon)) \subset B_{C(Y)}(\{w\}, \delta)$. Let T be a triod in X such that $T \in B_{C(X)}\left(K, \frac{\varepsilon}{2}\right)$. By Lemma 2.2, there is a 3-cell $\mathcal{T}$ in $C(X)$ such that $T \in \mathcal{T} \subset B_{C(X)}(K, \varepsilon)$. Then, $\mathcal{T}_0 = h^{-1}(\mathcal{T})$ is a 3-cell in $C(Y)$ such that $\mathcal{T}_0 \subset B_{C(Y)}(\{w\}, \delta)$. Put $T_0 = \bigcup \mathcal{T}_0$. It is easy to see that $\mathcal{T}_0 \subset C(T_0)$ and that $T_0 \subset W$. Hence, by Lemma 2.1, T_0 contains a triod. Then W contains a triod too. Since this is a contradiction, we have shown that

4) $h(F_1(W)) \subset F_1(V) \cup \{R, R_1, R_2\}$.

By 4) and the fact that the set $\{R, R_1, R_2\}$ is isolated with respect to $F_1(V) \cup \{R, R_1, R_2\}$, it follows that

5) $h(F_1(W)) \subset F_1(V)$.

Taking closures in both members of 5) we have

6) $h(F_1(Y)) \subset F_1(X)$.

Since $h^{-1} : C(X) \to C(Y)$ is a homeomorphism, by the work we have done, it follows that $h^{-1}(F_1(X)) \subset F_1(Y)$, so $F_1(X) \subset h(F_1(Y))$. Therefore, $h(F_1(Y)) = F_1(X)$ and $Y \approx X$. □

Let $\mathcal{K}$ be the class of compactifications of the real line different from an arc. Recall that compactifications of the real line, different from an arc and with disconnected remainder, have unique hyperspace ([2, Theorem 7]). In [2, Lemma 11] it is shown that if X is a circle, then Y is a continuum such that $C(X) \approx C(Y)$ if and only if Y is either and arc or a circle. Therefore, as a consequence of these results and Theorem 4.4, we have the following theorem.

Theorem 4.5. *The members of $\mathcal{K}$ are C-determined.*

Let $X = V \cup R$ be a compactification of V, with connected and nondegenerate remainder R such that $R_1 \neq R_2$. In [1, Question 1] it is asked if X does not have unique hyperspace. This question remains open.

Note added. The author has recently shown the following results:

(1) If X is a fan with the property of Kelley and infinitely many end-points, then X does not have unique hyperspace.

(2) If X is a smooth fan and Y is a fan such that $C(X) \approx C(Y)$, then $X \approx Y$. In other words, smooth fans have unique hyperspace in the class of fans.

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