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CLOSURES OF DISCRETE SETS OFTEN REFLECT GLOBAL PROPERTIES[1]

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Abstract. We deal with spaces X in which the closure of every discrete subspace has a given property. We prove that, in many cases, the space X has the same property. In particular, if $\mathcal{P} \in$ {perfect normality, tightness$\leq \kappa$, character$\leq \kappa$, countability}, then a compact space X has $\mathcal{P}$ if and only if $\overline{D}$ has $\mathcal{P}$ for every discrete $D \subset X$. We also establish that, under MA+$\neg$CH, if X is compact and the closure of any discrete subspace of X is metrizable, then X is metrizable. On the other hand, under CH, there exists a compact non-metrizable space X in which the closure of any discrete subset is metrizable.

0. Introduction.

A natural way of exploring the properties of a given space is to study its small (in some sense) subspaces and decide whether they reflect the properties of the whole space. For example, to see, whether a countably compact space X is metrizable, it is sufficient to check this for all subsets of X of cardinality $\leq \omega_1$ [Do]. Here the small subspaces of X are the ones of cardinality $\leq \omega_1$ and they reflect the metrizability of X no matter how big the space X is.

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Another kind of small subspace is the one which is nowhere dense. It is well-known that, for any dense-in-itself space X, if all nowhere dense subspaces of X are countable, then X is hereditarily Lindelöf. In this paper we are going to study when a given property $\mathcal{P}$ of the closures of discrete subspaces of X, implies X has $\mathcal{P}$. The first result of this kind was obtained in [Tk], where it was proved that a space X is compact if and only if the closure of every discrete subspace of X is compact. Later, Arhangel'skii and Buzyakova studied the case when the closures of every discrete subspace of X is Lindelöf [AB]. Whether this implies the Lindelöf property in X, is still an open problem.

We call a topological property $\mathcal{P}$ *discretely reflexive in a class* $\mathcal{A}$ if an arbitrary space $X \in \mathcal{A}$ has $\mathcal{P}$ if and only if the closure of every discrete subspace of X has $\mathcal{P}$. We prove that initial κ-compactness is discretely reflexive in Hausdorff spaces. Now, if the property $\mathcal{P}$ is not compact-like, it is usually impossible to prove such theorems even in the class of Tychonoff spaces because in a submaximal Tychonoff space X all discrete subspaces of X are closed and hence the closure of every discrete subspace of X is metrizable. However, there are submaximal spaces with pretty bad properties, which shows that, in such spaces X, the discrete subsets do not reflect any reasonably good property of X.

The picture changes drastically if we consider compact spaces. It turns out that, in this class, quite a few properties are discretely reflexive. For example, a compact space X is pefectly normal if and only if the closure of any discrete subspace of X is perfectly normal. Other examples of discretely reflexive properties in compact spaces are countability, sequentiality, the Fréchet–Urysohn property, character $\leq \kappa$ and tightness $\leq \kappa$. We prove that it is independent of ZFC whether metrizability is discretely reflexive in compact spaces.

1. Notation and terminology.

All spaces under consideration are assumed to be Hausdorff. Given a space X, the family $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\backslash\{\emptyset\}$. If $x \in X$, then $\tau(x, X) = \{U \in \tau(X) : x \in U\}$. A space X is called *discretely generated* if for any $A \subset X$ and any $x \in \overline{A}$, we have $x \in \overline{D}$ for some discrete $D \subset A$. If, for any non-closed $A \subset X$, we have $\overline{D}\backslash A \neq \emptyset$ for some discrete $D \subset A$, the space X

is called *weakly discretely generated*. The space X is *κ-monolithic* if for any $A \subset X$ with $|A| \leq \kappa$, we have $nw(\overline{A}) \leq \kappa$. We say that X is *initially κ-compact* if every open cover of X of cardinality $\leq \kappa$ has a finite subcover.

A Tychonoff space X is *Lindelöf Σ* if it has a compact cover $\mathcal{C}$ and a sequence of closed sets $\mathcal{F} = \{F_n : n \in \omega\}$ such that $\mathcal{F}$ is a network modulo $\mathcal{C}$, i.e., for each $C \in \mathcal{C}$ and any $U \in \tau(X)$ with $U \supset C$ there is an $F \in \mathcal{F}$ such that $C \subset F \subset U$. A space X is $K_{\sigma\delta}$ if X is homeomorphic to a countable intersection of σ-compact subspaces of some space. A continuous image of a $K_{\sigma\delta}$-space is called *a K-analytic* space.

A space X is *linearly Lindelöf* if every open cover of X, linearly ordered by the subset relation, has a countable subcover. The expression $X \vdash \mathcal{P}$ says that the space X has the property $\mathcal{P}$. The formula $ext(X) \leq \kappa$ abbreviates the statement "every closed discrete subset of X has cardinality $\leq \kappa$".

A dense-in-itself uncountable space X is called *Luzin space* if every nowhere dense subspace of X is countable. A space X is called *radial* if, for any $A \subset X$ and any $x \in \overline{A}$, there is a transfinite sequence $s = \{a_\alpha : \alpha < \kappa\} \subset A$ which converges to x, i.e., for any $U \in \tau(x, X)$ there is a $\beta < \kappa$ such that $a_\alpha \in U$ for all $\alpha \geq \beta$. The space X is *pseudoradial* if for any non-closed $A \subset X$ there is a transfinite sequence $s \subset A$ which converges to some point outside of A. Given a Tychonoff space X, a point $x \in \beta X \backslash X$ is called *remote* for X if $x \notin \overline{A}$ for any nowhere dense $A \subset X$. A point x of a space X is *a P-point* if x belongs to the interior of any G_δ-set which contains x.

The rest of the notation is standard and can be found in [En].

2. Discretely reflexive properties in general spaces.

It was proved in [Tk] that compactness is discretely reflexive in Hausdorff spaces. We are going to prove similar statements for various compactness-like properties.

Proposition 2.1. *Let X be an arbitrary space. Suppose that $hl(\overline{D}) \leq \kappa$ for any discrete $D \subset X$. Then $hl(X) \leq \kappa$. In other words, the hereditary Lindelöf number is discretely reflexive.*

Proof: Observe that $hl(X)$ is the supremum of cardinalities of right-separated subspaces of X. If $P \subset X$ is an arbitrary right-separated subspace of X then P is scattered and the set D of isolated points of P is dense in P. Hence $|P| \leq hl(\overline{D}) \leq \kappa$. □

Proposition 2.2. *Let X be an arbitrary space. Suppose that $\overline{D}$ is sequentially compact for each discrete $D \subset X$. Then X is sequentially compact. In other words, the sequential compactness is discretely reflexive.*

Proof: If $S = \{x_n : n \in \omega\} \subset X$ then there exists a subsequence $D = \{x_{n_k} : k \in \omega\}$ of the sequence S such that D is a discrete subspace of X. Thus, by sequential compactness of $\overline{D}$, the sequence D has a convergent subsequence in $\overline{D} \subset X$. □

The following lemma was practically proved in [Tk], but since it was not formulated there explicitly, we give a proof here.

Lemma 2.3. *Let κ be an infinite cardinal. Suppose that $\{F_\alpha : \alpha < \kappa\}$ is a decreasing κ-sequence of non-empty closed subsets of a space X. Then there exists a discrete $D \subset F_0$ such that $\overline{D} \cap F_\alpha \neq \emptyset$ for each $\alpha < \kappa$.*

Proof: Take an arbitrary point $x_0 \in F_0$. Suppose that $1 \leq \beta < \kappa$ and we have chosen points $\{x_\alpha : \alpha < \beta\} \subset F_0$ in such a way that for any $\gamma < \beta$ we have the property

$(*)$ $\overline{D}_\beta \cap F_\gamma \neq \emptyset$, where $D_\beta = \{x_\alpha : \alpha < \beta\}$.

Now, if $\overline{D}_\beta \cap F_\gamma \neq \emptyset$ for any $\gamma < \kappa$, our inductive construction stops. If not, let $\xi(\beta) = \min\{\gamma : \overline{D}_\beta \cap F_\gamma = \emptyset\}$. Observe that $\xi(\beta) \geq \beta$ and choose a point $x_\beta \in F_{\xi(\beta)} \subset F_0$. Clearly, this construction can go on until some $\beta_0 \leq \kappa$ maintaining the property $(*)$. After it is finished, we have $\overline{D}_{\beta_0} \cap F_\gamma \neq \emptyset$ for every $\gamma < \kappa$.

To finish the proof, note that the set $D = D_{\beta_0} \subset F_0$ is discrete. Indeed, for any $\alpha < \beta_0$, we have chosen the point x_α in the set $F_{\xi(\alpha)}$ for which $F_{\xi(\alpha)} \cap \overline{D}_\alpha = \emptyset$. Thus, $x_\alpha \notin \overline{\{x_\gamma : \gamma < \alpha\}}$. On the other hand, $\overline{\{x_\gamma : \gamma > \alpha\}} \subset F_{\xi(\alpha+1)}$ while $x_\alpha \notin F_{\xi(\alpha+1)}$. □

Proposition 2.4. *If the closure of every discrete subspace of X is linearly Lindelöf then X is linearly Lindelöf.*

Proof: It is easy to see that X is linearly Lindelöf if and only if, for any uncountable regular cardinal κ, any decreasing κ-sequence of closed non-empty sets has a non-empty intersection. So, let $\{F_\alpha : \alpha < \kappa\}$ be a decreasing sequence of non-empty closed subsets of X. Apply Lemma 2.3 to find a discrete $D \subset X$ such that $G_\alpha = \overline{D} \cap F_\alpha \neq \emptyset$ for any $\alpha < \kappa$. Now, we have a decreasing sequence $\{G_\alpha : \alpha < \kappa\}$ of non-empty closed sets in the linearly Lindelöf space $\overline{D}$. As a consequence, $\bigcap\{G_\alpha : \alpha < \kappa\} \neq \emptyset$ and therefore $\bigcap\{F_\alpha : \alpha < \kappa\} \neq \emptyset$. □

In [Tk] it was proved that compactness is discretely reflexive. Lemma 2.3 makes it possible to establish a stronger result.

Theorem 2.5. *Let X be an arbitrary Hausdorff space. If the closure of every discrete subspace of X is H-closed then X is compact.*

Proof: Suppose that X is not compact. Let κ be the minimal cardinal such that there is a decreasing family $\mathcal{S} = \{F_\alpha : \alpha < \kappa\}$ of closed non-empty subsets of X with $\bigcap \mathcal{S} = \emptyset$. It is easy to see that X is countably compact and therefore κ is a regular uncountable cardinal.

Apply Lemma 2.3 to find a discrete $D_0 \subset F_0$ such that $\overline{D}_0 \cap F_\beta \neq \emptyset$ for any $\beta < \kappa$. Suppose that, for some $\beta < \kappa$, we have constructed the family $\{D_\alpha : \alpha < \beta\}$ with the following properties:

(i) D_α is a discrete subset of F_α for each $\alpha < \beta$;
(ii) $D_\alpha \subset \overline{D}_\gamma$ whenever $\gamma < \alpha < \beta$;
(iii) $\overline{D}_\alpha \cap F_\nu \neq \emptyset$ for any $\alpha < \beta$ and $\nu < \kappa$.

Fix an ordinal $\nu < \kappa$ and consider the family $\mathcal{F} = \{\overline{D}_\alpha \cap F_\nu : \alpha < \beta\}$. It follows from the properties (i)–(iii) that $\mathcal{F}$ is a decreasing β-sequence of non-empty closed subsets of X. Since $|\beta| < \kappa$, we have $\bigcap \mathcal{F} \neq \emptyset$. As a consequence, the set $G_\nu = \bigcap\{\overline{D}_\alpha : \alpha < \beta\} \cap F_\nu$ is non-empty for any $\nu < \kappa$. Therefore it is possible to apply Lemma 2.3 to the family $\{G_\nu : \beta \leq \nu < \kappa\}$ to obtain a discrete set $D_\beta \subset G_\beta \subset F_\beta$ such that $\overline{D}_\beta \cap G_\nu \neq \emptyset$ for all $\nu \geq \beta,\ \nu < \kappa$. Since $G_\nu \subset F_\nu$ and the family $\{F_\nu : \nu < \kappa\}$ is decreasing, we have $\overline{D}_\beta \cap F_\nu \neq \emptyset$ for all $\nu < \kappa$.

It is easy to see that the properties (i)–(iii) now hold for all $\alpha \leq \beta$ and hence we can construct the sets D_α for all $\alpha < \kappa$ with the properties (i)–(iii) satisfied for all $\alpha < \kappa$.

For each $\alpha < \kappa$, let $P_\alpha = \overline{D}_\alpha$. Then P_α is an H-closed subspace of the H-closed space P_0. In an H-closed space every decreasing sequence of non-empty H-closed subspaces has a non-empty intersection [Ka]. As a consequence, we have $\bigcap \mathcal{S} \supset \bigcap\{P_\alpha : \alpha < \kappa\} \neq \emptyset$ which is a contradiction. □

Theorem 2.6. *Initial κ-compactness is discretely reflexive for any infinite cardinal κ, i.e., if X is a Hausdorff space and the closure of every discrete subspace of X is initially κ-compact then X is initially κ-compact.*

Proof: It is known [St], that X is initially κ-compact if and only if every decreasing κ-sequence $\{F_\alpha : \alpha < \kappa\}$ of closed non-empty subsets of X has a non-empty intersection. Given such a sequence, we can apply Lemma 2.3 to find a discrete $D \subset X$ such that $G_\alpha = \overline{D} \cap F_\alpha \neq \emptyset$ for any $\alpha < \kappa$. Since $\overline{D}$ is initially κ-compact, we have $\emptyset \neq \bigcap\{G_\alpha : \alpha < \kappa\} \subset \{F_\alpha : \alpha < \kappa\}$. □

Recall that a space X is called *ω-bounded* if $\overline{A}$ is compact for any countable $A \subset X$.

Example 2.7. Under CH, there exists a space X which is not ω-bounded, while the closure of any discrete subspace of X is ω-bounded. To see this, take any remote point z of the space $\beta\mathbb{R}$ which at the same time is a P-point in $\beta\mathbb{R}\backslash\mathbb{R}$. Such points exist under CH [Wa, Theorem 4.46]. The space $X = \beta\mathbb{R}\backslash\{z\}$ is not ω-bounded because $\mathbb{Q} \subset X$ and the closure of $\mathbb{Q}$ is not compact.

However, if $D \subset X$ is discrete then $\overline{D}$ is ω-bounded (the closure is taken in X). Indeed, let A be a countable subset of $\overline{D}$. Then $z \notin A$ and hence z is not in the closure in $\beta\mathbb{R}$ of $A \cap (\beta\mathbb{R}\backslash\mathbb{R})$. Thus, $\overline{A \cap (\beta\mathbb{R}\backslash\mathbb{R})}$ is compact. On the other hand, $B = A \cap \mathbb{R}$ is contained in the closure of the discrete set $D \cap \mathbb{R}$ and therefore B is nowhere dense in $\mathbb{R}$. Since z is a remote point, it can not belong to the closure of B in $\beta\mathbb{R}$. Hence $\overline{B}$ is also compact.

Example 2.8. Under CH, neither countability nor σ-compactness are discretely reflexive.

Proof: Under the Continuum Hypothesis, there exists a dense Luzin subspace X of the real line $\mathbb{R}$. Since X has no isolated points, the closure of every discrete $D \subset X$ is nowhere dense and hence countable. However X is not even σ-compact. □

Example 2.9. Under CH, the Lindelöf Σ-property is not discretely reflexive.

Proof: Under the Continuum Hypothesis, there exists a dense Luzin subspace X of the space $\Sigma = \{x \in \mathbb{R}^{\omega_1} : |\{\alpha < \omega_1 : x(\alpha) \neq 0\}| \leq \omega\}$ (see [AmS]). The closure of any discrete subspace of X is nowhere dense and hence countable. Let us show that X is not a Lindelöf Σ-space. If X were Lindelöf Σ, we would have a family $\mathcal{F} = \{F_n : n \in \omega\}$ of closed subspaces of X witnessing the Lindelöf Σ-property of X, i.e, $\mathcal{F}$ would be a network with respect to some compact cover $\mathcal{C}$ of the space X. For each $F_n \in \mathcal{F}$ let $G_n = F_n \backslash \text{Int}(F_n)$. The set $G = \overline{\bigcup\{G_n : n \in \omega\}}$ is countable and hence there is a $C \in \mathcal{C}$ such that $C \backslash G \neq \emptyset$. The compact space C is countable and hence there is an $x \in C \backslash G$ such that x is an isolated point of C. Fix a $U \in \tau(x, X)$ such that $\overline{U} \cap C = \{x\}$. The family $\gamma = \{F \in \mathcal{F} : C \subset F\}$ is a network for C and $x \in \text{Int}(F)$ for any $F \in \gamma$. This shows that the family $\mathcal{B} = \{U \cap \text{Int}(F) : F \in \gamma\}$ is a countable local base at x which is a contradiction. □

In [Ar4], Arhangel'skii calls a space X *strongly discretely Lindelöf* if the closure of any discrete subspace of X is Lindelöf. Lemma 6 of [Ar4] states that any strongly discretely Lindelöf space is linearly Lindelöf. It is also established in [Ar4] that any strongly discretely Lindelöf countably paracompact space is Lindelöf. We are going to prove that the same conclusion holds for a larger class of spaces. Recall that a space X is *countably metacompact* if every open countable cover of X has a point-finite refinement.

Theorem 2.10. *Any linearly Lindelöf countably metacompact space is Lindelöf.*

Proof: Let X be a countably metacompact linearly Lindelöf space. We will need the cardinal $\kappa = \min\{\mu : \mu$ is a cardinal for which there exists an open cover of X of cardinality μ with no countable subcover$\}$. It is easy to see that, for any closed $F \subset X$ and any $\mathcal{U} \subset \tau(X)$ with $F \subset \bigcup \mathcal{U}$ and $|\mathcal{U}| < \kappa$, there is a countable $\mathcal{U}' \subset \mathcal{U}$ such that $F \subset \bigcup \mathcal{U}'$.

Take any $\mathcal{V} \subset \tau(X)$ such that $\bigcup \mathcal{V} = X$, $|\mathcal{V}| = \kappa$ and no countable subfamily of $\mathcal{V}$ covers X. Take an enumeration $\{U_\alpha : \alpha < \kappa\}$ of the family $\mathcal{V}$. For each $\beta < \kappa$, let $W_\beta = \bigcup\{U_\alpha : \alpha < \beta\}$. Since X is linearly Lindelöf, there exists a sequence $\{\beta_n : n \in \omega\} \subset \kappa$ such that $\bigcup\{W_{\beta_n} : n \in \omega\} = X$. Apply countable metacompactness of X to find a sequence $\{F_n : n \in \omega\}$ of closed subsets of X such that $\bigcup\{F_n : n \in \omega\} = X$ and $F_n \subset W_{\beta_n}$ for each $n \in \omega$. The family

$\mathcal{V}_n = \{U_\alpha : \alpha < \beta_n\}$ covers F_n for each $n \in \omega$. Since $|\mathcal{V}_n| < \kappa$, there exists a countable $\mathcal{V}'_n \subset \mathcal{V}_n$ such that $F_n \subset \bigcup \mathcal{V}'_n$. As a consequence, $\mathcal{V}' = \bigcup\{\mathcal{V}'_n : n \in \omega\}$ is a countable subcover of $\mathcal{V}$ which is a contradiction. □

3. Discretely reflexive properties in compact spaces.

We are going to prove that a number of topological properties are discretely reflexive in the class of compact spaces. Recall that, given a cardinal κ, a subspace S of a space X is called *a free sequence of length* κ if $S = \{x_\alpha : \alpha < \kappa\}$ and we have $\overline{\{x_\alpha : \alpha < \beta\}} \cap \overline{\{x_\alpha : \alpha \geq \beta\}} = \emptyset$ for every $\beta < \kappa$.

Proposition 3.1. *Let X be a compact space. Suppose that the closure of every discrete subspace of X has tightness $\leq \kappa$. Then $t(X) \leq \kappa$.*

Proof: Since X is compact, the tightness of X is equal to the supremum of cardinalities of free sequences lying in X [Ar1, Theorem 2.2.13]. Thus, if $t(X) > \kappa$ then there exists a free sequence $F \subset X$ with $|F| = \kappa^+$. Clearly, F is discrete and it is easy to see that $t(\overline{F}) \geq \kappa^+$ which is a contradiction. □

Corollary 3.2. *Let X be a compact space such that $\overline{D}$ is Fréchet–Urysohn (sequential) for any discrete $D \subset X$. Then X is Fréchet–Urysohn (or sequential respectively).*

Proof: Take any $A \subset X$ and any $x \in \overline{A}$. By Propostion 3.1, the space X has countable tightness and hence X is discretely generated [DTTW]. Take a discrete $D \subset A$ such that $x \in \overline{D}$. If the closures of discrete subsets of X are Fréchet–Urysohn then $\overline{D}$ is Fréchet–Urysohn and hence there is a sequence $s \subset D \subset A$ such that $s \to x$.

If the closures of discrete subsets of X are sequential then $\overline{D}$ is sequential and the point x is in the sequential closure of D which is smaller that the sequential closure of A. Therefore x belongs to the sequential closure of A. □

Propostion 3.3. *Let X be a compact space. If the closure of every discrete subspace of X is pseudoradial then X is pseudoradial.*

Proof: Suppose that A is a non-closed subset of X. Since any compact space is weakly discretely generated [DTTW], there is a discrete $D \subset A$ such that $\overline{D} \backslash A \neq \emptyset$. Thus, in the pseudoradial space

$\overline{D}$, the set $A \cap \overline{D}$ is not closed. Therefore there exists a transfinite sequence $s \subset A \cap \overline{D}$ which converges to some point outside A. $\square$

Example 3.4. Under CH, there exists a compact non-radial space X such that the closure of every discrete subspace of X is radial. Thus, it is consistent with ZFC that radiality is not discretely reflexive in compact spaces.

Proof: Let K be the Cantor set with its usual topology. Take any point $a \in K$ and denote the space $K \backslash \{a\}$ by T. Under CH, there exists a point $q \in \beta T \backslash T$ such that q is a P-point in $\beta T \backslash T$ and a remote point for T at the same time [Wa, Corollary 4.24]. By an evident induction we can construct a family $\{F_\alpha : \alpha < \omega_1\}$ with the following properties:

(i) $F_0 = \beta T \backslash T$ and F_α is a clopen subset of $\beta T \backslash T$ for all non-limit $\alpha < \omega_1$;
(ii) $F_\beta = \bigcap \{F_\alpha : \alpha < \beta\}$ if β is a limit ordinal;
(iii) $F_\beta \subset F_\alpha$ and $F_\beta \neq F_\alpha$ if $\beta > \alpha$;
(iv) the family $\{F_\alpha : \alpha$ is a non-limit countable ordinal$\}$ is a local base at q in $\beta T \backslash T$.

Consider the family $\mathcal{D} = \{\{x\} : x \in T\} \cup \{P_\alpha : \alpha < \omega_1\} \cup \{q\}$, where $P_\alpha = F_\alpha \backslash F_{\alpha+1}$ for each $\alpha < \omega_1$. It is clear that $\mathcal{D}$ is a decomposition of the space βT into closed subsets. We leave to the reader the tedious (but standard) verification of the fact that $\mathcal{D}$ is upper semicontinuous and hence the quotient space $X = \beta T / \mathcal{D}$ is Hausdorff. If $r : \beta T \to X$ is the relevant quotient map, let $p = r(q)$ and $\{p_\alpha\} = r(P_\alpha)$ for all $\alpha < \omega_1$. Define the map $h : (\omega_1 + 1) \to P = \{p_\alpha : \alpha < \omega_1\} \cup \{p\}$, by $h(\omega_1) = p$ and $h(\alpha) = p_\alpha$ for all $\alpha < \omega_1$. It is straightforward that h is a homeomorphism.

Denote the set $r(T)$ by S and observe that the compact space $X = S \cup P$ is first countable at every $x \in X \backslash \{p\}$. Take any discrete subspace $D \subset X$ and a set $A \subset \overline{D}$. Given an $x \in \overline{A}$, if $x \neq p$, then $\chi(x, X) = \omega$ and hence there is a sequence in A which converges to x. Now, if $p \in \overline{A}$ then $p \notin \overline{A \cap S}$ because $\overline{A \cap S} \subset \overline{\overline{D} \cap S}$ and $\overline{D} \cap S$ is a nowhere dense subspace of S (remember that q is a remote point for T and $\{q\} \cup T$ is homeomorphic to $\{p\} \cup S$). Therefore $p \in \overline{A \cap P}$ while the space P is homeomorphic to the space $(\omega_1 + 1)$ which is radial. As a consequence, $s \to x$ for some transfinite sequence $s \subset A \cap P \subset A$ and hence $\overline{D}$ is a radial space for any discrete $D \subset X$.

Finally, to see that X is not radial, take a countable dense subset Y of the space S and observe that Y is dense in X and hence $p \in \overline{Y}$. We have mentioned already that p can not belong to the closure of any nowhere dense subset of S and hence no sequence from Y can converge to p. Now, a standard argument shows that no transfinite sequence from Y can converge to p. □

Theorem 3.5. *Let X be a compact space such that, for some infinite cardinal κ, the closure of every discrete subspace of X has character $\leq \kappa$. Then $\chi(X) \leq \kappa$. In other words, the character is discretely reflexive in compact spaces.*

Proof: By Proposition 3.1, the tightness of X is $\leq \kappa$. Suppose that there exists an $x \in X$ such that $\chi(x, X) > \kappa$. Then, for any family $\mathcal{F}$ of closed G_κ-subsets of X, if $|\mathcal{F}| \leq \kappa$, and $x \in \bigcap \mathcal{F}$, then $(\bigcap \mathcal{F})\backslash\{x\} \neq \emptyset$. Choose an $x_0 \in X\backslash\{x\}$ arbitrarily and let $F_0 = X$. Suppose that $\beta < \kappa^+$ and we have constructed points $\{x_\alpha : \alpha < \beta\} \subset X\backslash\{x\}$ and closed G_κ-sets $\{F_\alpha : \alpha < \beta\}$ with the following properties:

(i) $\{x, x_\alpha\} \subset F_\alpha$ for each $\alpha < \beta$;
(ii) $F_\alpha \subset F_\gamma$ if $\gamma < \alpha < \beta$;
(iii) $F_\alpha \cap \overline{\{x_\nu : \nu < \alpha\}} \subset \{x\}$ for each $\alpha < \beta$.

The set $Y_\beta = \{x_\alpha : \alpha < \beta\}$ is discrete. Indeed, for each $\gamma < \beta$, we have $x_\gamma \notin \overline{\{x_\alpha : \alpha < \gamma\}}$ by the properties (i) and (iii). Applying (iii) once more, we can observe that $x_\gamma \notin F_{\gamma+1} \supset \overline{\{x_\alpha : \gamma < \alpha < \beta\}}$ and therefore $x_\gamma \notin \overline{\{x_\alpha : \alpha < \gamma\}} \cup \overline{\{x_\alpha : \gamma < \alpha < \beta\}}$. As a consequence, we obtain $\chi(x, \{x\} \cup \overline{Y}_\beta) \leq \kappa$ and hence there exists a closed G_κ-set $H \ni x$ such that $H \cap \overline{Y}_\beta \subset \{x\}$. Let $F_\beta = \bigcap\{F_\alpha : \alpha < \beta\} \cap H$. Since F_β is a G_κ-set in X, we can choose a point $x_\beta \in F_\beta\backslash\{x\}$. It is straightforward to check that the properties (i)–(iii) hold if $\gamma < \alpha \leq \beta$.

The argument above shows that we can make κ^+ steps in our construction to obtain the set $D = \{x_\alpha : \alpha < \kappa^+\} \subset X\backslash\{x\}$ and the family $\{F_\alpha : \alpha < \kappa^+\}$. The proof of discreteness of Y_β is valid also for $\beta = \kappa^+$, so $D = Y_{\kappa^+}$ is a discrete subset of X. Note that, for each $\beta < \kappa^+$, the properties (i) and (iii) imply that

(∗) $\overline{Y}_\beta \cap \overline{D\backslash Y_\beta} = \overline{\{x_\alpha : \alpha < \beta\}} \cap \overline{\{x_\alpha : \beta \le \alpha < \kappa^+\}} \subset \overline{\{x_\alpha : \alpha < \beta\}} \cap F_\beta \subset \{x\}$.

Consider the non-empty set $F = \bigcap\{\overline{D\backslash Y_\beta} : \beta < \kappa^+\}$. If $y \in F\backslash\{x\}$ then $y \in \overline{D}$ and hence $y \in \overline{Y}_\beta$ for some $\beta < \kappa^+$ (we used the fact that $t(X) \le \kappa$). As a consequence, $y \in \overline{Y}_\beta \cap \overline{D\backslash Y_\beta}$ which contradicts (∗). Thus, $F = \{x\}$ and the family $\{\overline{D\backslash Y_\beta} : \beta < \kappa^+\}$ is a network at the point x. Since the set D is discrete, we have $\chi(x, \overline{D}) \le \kappa$ which makes it possible to find a family $\mathcal{U} = \{U_\alpha : \alpha < \kappa\} \subset \tau(X)$ such that $(\bigcap \mathcal{U}) \cap \overline{D} = \{x\}$. For each $\alpha < \kappa$, fix a $\beta(\alpha) < \kappa^+$ with $\overline{D\backslash Y_{\beta(\alpha)}} \subset U_\alpha$. Then, for any $\beta \in \kappa^+ \backslash \sup\{\beta(\alpha) : \alpha < \kappa\}$, we have $x_\beta \in (\bigcap \mathcal{U}) \cap \overline{D}$, which is a contradiction. □

Proposition 3.6. *Let X be a compact space. Suppose that the closure of every discrete subspace of X is perfectly normal. Then X is perfectly normal. In particular, if the closure of each discrete subset of X is metrizable then X is perfectly normal.*

Proof: It is evident that $s(X) \le \omega$. Fix any closed $F \subset X$. To prove that F is a G_δ-set, it suffices to establish that $X\backslash F$ is Lindelöf. For any $x \in X\backslash F$, take a $V_x \in \tau(x, X)$ such that $\overline{V}_x \cap F = \emptyset$. Apply a lemma of Shapirovsky [Sha1] which says that there is a discrete $D \subset X\backslash F$ and a countable $A \subset X\backslash F$ such that $X\backslash F \subset \bigcup\{V_x : x \in A\} \cup \overline{D}$. Since $\overline{D}$ is perfectly normal, the set $\overline{D}\backslash F$ is σ-compact. Hence $X\backslash F = \bigcup\{\overline{V}_x : x \in A\} \cup (\overline{D}\backslash F)$ is also σ-compact. □

Proposition 3.7. *Let X be a countably compact Tychonoff space.*

1. *If $\kappa < \mathfrak{c}$ and $|\overline{D}| \le \kappa$ for every discrete $D \subset X$ then $|X| \le \kappa$.*
2. *If the closure of every discrete subspace of X has cardinality $< \mathfrak{c}$ then $|X| < \mathfrak{c}$.*
3. *If the closure of every discrete subspace of X is scattered then X is scattered.*

Proof: Consider the following properties: $\mathcal{P}$=“the closure of every discrete subset has cardinality $\le \kappa$”, $\mathcal{Q}$=“the closure of every discrete subset has cardinality $< \mathfrak{c}$” and $\mathcal{R}$ =“the closure of every discrete subset is scattered”. It is straightforward to verify that $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ are invariant under closed continuous onto maps and closed-hereditary.

To prove (3), suppose that $X \vdash \mathcal{R}$ and X is not scattered. Take any closed dense-in-itself subspace $F \subset X$ and note that there exists a continuous onto map $f : F \to [0,1]$ [Sha2]. Now, $F \vdash\!\!- \mathcal{R}$ and the map f is closed so the space $[0,1]$ must have $\mathcal{R}$. However, it is easy to construct a discrete subspace of $[0,1]$ whose closure is not countable and hence not scattered. To do this, note that the complement of the Cantor set is a union of countably many disjoint open intervals. Take the middle point of in each one of them and observe that the closure of the resulting discrete set contains the Cantor set. Thus, $[0,1]$ does not have $\mathcal{R}$ and the resulting contradiction shows that X is scattered whenever it has $\mathcal{R}$.

Observe that $\mathcal{P} \Longrightarrow \mathcal{Q} \Longrightarrow \mathcal{R}$ and hence the space X is scattered if $X \vdash\!\!- \mathcal{Q}$ or $X \vdash\!\!- \mathcal{P}$. But every scattered space coincides with the closure of the discrete set D of its isolated points. Hence $X = \overline{D}$ has cardinality $\leq \kappa$ if X has $\mathcal{P}$, and $|X| < \mathfrak{c}$ if X has $\mathcal{Q}$. □

Remark 3.8. Example 2.8 shows that some compactness-like property in 3.7(1)–(2) is essential, at least, consistently. To see that it is also essential in 3.7(3), take the maximal countable space V constructed in [vD]. The space V has no isolated points while $\overline{D} = D$ is discrete and hence scattered for every discrete subspace $D \subset V$.

Theorem 3.9. *Under MA+¬CH metrizability is discretely reflexive in compact spaces, i.e., if X is a compact space in which the closure of every discrete subset is metrizable then X is metrizable.*

Proof: Suppose that the closure of every discrete subspace of X is metrizable. If X is not metrizable, then it is possible to map X continuously onto a non-metrizable space Y of weight ω_1 (see, for example, [Ar3, IV.8.11]). It is easy to see that the closure of any discrete subspace of Y is metrizable, and hence without loss of generality we can assume that $X = Y$. By Proposition 3.6 the space X is perfectly normal. Any perfectly normal compact space is separable under MA+¬CH [Ro], so let A be a countable dense subspace of X. The space $Z = X \setminus A = \bigcap\{X \setminus \{a\} : a \in A\}$ is $K_{\sigma\delta}$ due to the fact that $X \setminus \{a\}$ is a σ-compact space for each $a \in A$. An evident consequence is that Z is K-analytic. Let K be a compact subspace of Z. Then K is nowhere dense in X and, using a standard MA+¬CH technique, one can construct a discrete $D \subset A$ such that $K \subset \overline{D}$ (see, for example, [Ar3, IV.8.9]). As a consequence, the subspace K is metrizable. Now apply a theorem of Fremlin [Fr] which says that, under MA+$\neg$CH, if any compact subspace of a K-analytic space Z is metrizable then Z has a countable network. Since $X\setminus Z$ is countable, the space X also has a countable network which is a contradiction with $nw(X) = w(X) = \omega_1$. $\square$

Corollary 3.10. *Under MA+$\neg$CH countable weight is discretely reflexive in compact spaces.*

Example 3.11. The Continuum Hypothesis implies that metrizability is not discretely reflexive in compact spaces. Indeed, in [Ku] a compact hereditarily Lindelöf non-separable space Y was constructed under CH. Since $t(Y) = \omega$, the space Y can be continuously and irreducibly mapped onto a subspace X of a Σ-product of real lines [Ar1, Theorem 3.2.4]. It is evident that X is also a perfectly normal non-separable compact space. Since any Σ-product of real lines is ω-monolithic, the closure of every countable (and hence of every discrete) subspace of X is second countable and hence metrizable. This shows an interesting difference between hereditary separability and hereditary Lindelöfness. We have already seen that the hereditary Lindelöf property is discretely reflexive in arbitrary spaces in ZFC. However, (hereditary) separability may not be reflexive even in compact spaces.

Corollary 3.12. *The statements "metrizability is discretely reflexive in compact spaces" and "countable weight is discretely reflexive in compact spaces" are independent of ZFC.*

4. Discrete generability and its applications.

The paper [DTTW] seems to be the first one to study discrete generability in a systematic way. Here we give some new results and applications.

Definition 4.1 ([DTTW]). A space X is *discretely generated* if, for any $A \subset X$ and any $x \in \overline{A}$, we have $x \in \overline{D}$ for some discrete $D \subset A$. The space X is called *weakly discretely generated* if, for any non-closed $A \subset X$, we have $\overline{D}\setminus A \neq \emptyset$ for some discrete $D \subset A$.

Lemma 4.2. *Given a space X and a regular cardinal κ, suppose that $x \in X$, $A \subset X$, $|A| = \kappa$ and, for any $U \in \tau(x, X)$, we have $|A\setminus U| < \kappa$. Then there is a discrete $D \subset A$ such that $x \in \overline{D}$.*

Proof: Observe first that, for any $B \subset A$ with $|B| = \kappa$, we have $x \in \overline{B}$. Now, take an $a_0 \in A$ arbitrarily and let $F_0 = X$. Suppose that $\beta < \kappa$ and we have chosen points $\{a_\alpha : \alpha < \beta\} \subset A$ and non-empty closed sets $\{F_\alpha : \alpha < \beta\}$ so that

(i) $|A \backslash F_\alpha| < \kappa$ for each $\alpha < \beta$;
(ii) $\{x, a_\alpha\} \subset F_\alpha \subset F_\gamma$ for any $\gamma < \alpha < \beta$;
(iii) $\{a_\gamma : \gamma < \alpha\} \cap F_\alpha = \emptyset$ for any $\alpha < \beta$.
(iv) $a_\alpha \notin \overline{\{a_\gamma : \gamma < \alpha\}}$ for every $\alpha < \beta$.

Consider the set $D_\beta = \{a_\alpha : \alpha < \beta\}$ and check whether $x \in \overline{D}_\beta$. If $x \in \overline{D}_\beta$ the inductive construction stops. If not, the set $X \backslash \overline{D}_\beta$ is a neighbourhood of x and therefore $|\overline{D}_\beta \cap A| < \kappa$. For each $\alpha < \beta$ choose an open set $U_\alpha \in \tau(a_\alpha, X)$ such that $x \notin \overline{U}_\alpha$. By regularity of κ, the set $U \cap A$ has cardinality $< \kappa$, where $U = \bigcup\{U_\alpha : \alpha < \beta\}$. Let $F_\beta = (X \backslash U) \cap (\bigcap\{F_\alpha : \alpha < \beta\})$. It is clear that the properties (i)–(iii) hold for all $\alpha \leq \beta$. Since $|\overline{D}_\beta \cap A| < \kappa$ and $|A \backslash F_\beta| < \kappa$, it is possible to choose $a_\beta \in (A \cap F_\beta) \backslash \overline{D}_\beta$. Now, the property (iv) is fulfilled for all $\alpha \leq \beta$ and our inductive construction can go on. Suppose that it stopped after ν steps for some $\nu \leq \kappa$. Once we have the set $D = \{a_\alpha : \alpha < \nu\} = D_\nu \subset A$, note that $x \in \overline{D}$ if $\nu < \kappa$ because this was the reason to stop the induction. If $\nu = \kappa$ we have $x \in \overline{D}$ because $D \subset A$ and $|D| = \kappa$.

We only have to prove that D is discrete. But this follows immediately from the fact that, for any $\alpha < \nu$, the set $V = X \backslash (\overline{\{a_\gamma : \gamma < \alpha\}} \cup F_{\alpha+1})$ is an open neighbourhood of a_α such that $V \cap D = \{a_\alpha\}$. □

Theorem 4.3. *Any radial space is discretely generated and any pseudoradial space is weakly discretely generated.*

Proof: If X is radial and $x \in \overline{B} \backslash B$ for some $B \subset X$, apply a result from [Ar2] to conclude that there exists a regular cardinal κ and a set $A \subset B$ such that $x \in \overline{A}$ and $|U \backslash A| < \kappa$ for any $U \in \tau(x, X)$. By Lemma 4.2, there is a discrete $D \subset A$ such that $x \in \overline{D}$.

Now, if X is pseudoradial and $B \neq \overline{B}$ for some $B \subset X$, apply another result from [Ar2] to conclude that there exists a regular cardinal κ, a set $A \subset B$ with $|A| = \kappa$ and a point $x \in \overline{A} \backslash B$ such that $|A \backslash U| < \kappa$ for any $U \in \tau(x, X)$. By Lemma 4.2 there is a discrete $D \subset A$ such that $x \in \overline{D}$. □

Example 4.4. Under MA+¬CH the space $\{0,1\}^{\omega_1}$ is pseudoradial [Ar2]. However, it was proved in [DTTW] that, under the existence of an L-space, the space $\{0,1\}^{\omega_1}$ is not discretely generated. It is known that L-spaces exist under MA+¬CH [Ro] and therefore it is consistent with the usual axioms of ZFC that there are pseudoradial spaces which are not discretely generated.

Example 4.5. Example 3.4 shows that, under CH, there is a compact pseudoradial space X which is not discretely generated. The same example shows that, under CH, there is a compact space X in which the closure of every discrete subspace is discretely generated while X is not discretely generated.

Proof: It was proved in 3.4 that the closure of any discrete subspace of X is radial and hence discretely generated by Theorem 4.3. This, together with Proposition 3.3, implies that X is pseudoradial. Suppose that X is discretely generated. Then, for any $A \subset X$ and any $x \in \overline{A}$, there is a discrete $D \subset A$ with $x \in \overline{D}$. Since $\overline{D}$ is radial, there is a transfinite sequence $s \subset D$ which converges to x. Therefore X is radial, a contradiction. □

Remark 4.6. It was proved in [DTTW] that every compact space is weakly discretely generated. However, this is not true for H-closed spaces. To see this, let X be the Katětov extension of $\mathbb{R}$. It is known [En, 3.12.6] that X is H-closed. Let ξ be any open ultrafilter which extends the filter of dense open subsets of $\mathbb{R}$. Then $\xi \in X$ and no nowhere dense subset of $\mathbb{R}$ can contain ξ in its closure. In particular, $\xi \notin \overline{D}$ for any discrete $D \subset \mathbb{R}$. Since $X \backslash \mathbb{R}$ is closed and discrete, no discrete subspace of $X \backslash \{\xi\}$ can contain ξ in its closure.

Recall that a space X is called *resolvable* if there exist $A, B \subset X$ such that $A \cap B = \emptyset$ and $\overline{A} = \overline{B} = X$.

Theorem 4.7. *Let X be a hereditarily collectionwise Hausdorff dense-in-itself space. If X is weakly discretely generated then X is resolvable.*

Proof: It was proved in [CLi] that any union of resolvable spaces is resolvable, so it is sufficient to find, for each $x \in X$, a resolvable subspace $H \ni x$. The set $X \backslash \{x\}$ is not closed and therefore there exists a discrete subspace D_0 of $X \backslash \{x\}$ with $x \in \overline{D_0}$. Since X is hereditarily collectionwise Hausdorff, and D_0 is closed discrete in the open set $X\backslash(\overline{D_0}\backslash D_0)$, there exists a disjoint family $\mathcal{U}_0 \subset \tau(X)$ such that $\mathcal{U}_0 = \{U_d^0 : d \in D_0\}$ and $d \in U_d^0$ for any $d \in D_0$. Suppose that we have a discrete set D_n and a disjoint family $\mathcal{U}_n = \{U_d^n : d \in D_n\} \subset \tau(X)$ with $d \in U_d^n$ for each $d \in D_n$. The set $\overline{U_d^n}\backslash\{d\}$ is not closed and hence we can apply weak discrete generability of X to find a discrete $B'_d \subset \overline{U_d^n}\backslash\{d\}$ such that $d \in \overline{B'_d}$. It is evident that the set $B_d = B'_d \cap U_d^n$ is discrete and $d \in \overline{B_d}$. The set $D_{n+1} = \bigcup\{B_d : d \in D_n\}$ is also discrete and hence it is possible to fix a disjoint family $\mathcal{U}_{n+1} = \{U_d^{n+1} : d \in D_{n+1}\} \subset \tau(X)$ such that $d \in U_d^{n+1}$ for each $d \in D_{n+1}$ and $\bigcup\mathcal{U}_{n+1} \subset (\bigcup\mathcal{U}_n)\backslash D_n$.

The family $\{D_n : n \in \omega\}$ we constructed, is disjoint and has the property $D_n \subset \overline{D_{n+1}}$ for each $n \in \omega$. The subspace $H = \{x\} \cup (\bigcup\{D_n : n \in \omega\}$ is the promised resolvable subspace of X because $A = \bigcup\{D_{2n+1} : n \in \omega\}$ and $B = \bigcup\{D_{2n} : n \in \omega\}$ are disjoint dense subspaces of H. □

Corollary 4.8. *Any monotonically normal dense-in-itself space is resolvable. In particular, any stratifiable space is resolvable.*

Proof: It suffices to observe that any monotonically normal space is discretely generated (see [DTTW]), and hereditarily collectionwise normal [Gr, Theorems 5.18 and 5.20]. □

5. Open questions.

The topic discussed in this paper is far from being exhausted. We give below a list of open problems to illustrate this.

Question 5.1. *Is there a ZFC example of a pseudoradial space which is not discretely generated?*

Question 5.2. *Is there a ZFC example of a non-radial compact space in which the closure of every discrete subspace is radial?*

Question 5.3. *Let X be a space in which the closure of any discrete subset is σ-compact. Must X be a Lindelöf space?*

Question 5.4. *Is there a ZFC example of a Tychonoff uncountable space in which the closure of every discrete subspace is countable?*

Question 5.5. *Is there a ZFC example of a Tychonoff space showing that the Lindelöf Σ-property fails to be discretely reflexive?*

Question 5.6. *Is there a ZFC example of a Tychonoff space showing that σ-compactness fails to be discretely reflexive?*

Question 5.7. *Let X be a separable compact space in which the closure of every discrete subspace is metrizable. Is it true in ZFC that X must be metrizable?*

Question 5.8. *Let X be a compact space such that the closure of each discrete subspace of X has cardinality $\leq \mathfrak{c}$. Is it true that $|X| \leq \mathfrak{c}$? (Note that it is an easy consequence of Proposition 2.1 that $|X| \leq 2^{\mathfrak{c}}$).*

Question 5.9. *Let X be a compact space such that the closure of every discrete subspace of X is hereditarily normal. Must X be hereditarily normal?*

Question 5.10. *Let X be a compact space such that the closure of every discrete subspace of X is hereditarily realcompact. Must X be hereditarily realcompact?*

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Tkachuk and Wilson

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