Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 25, Spring 2000
Pages 63-74

TIGHTNESS IN POLYADIC SPACES

M. BELL

Abstract. We prove a theorem whose countable version is that a zero-dimensional polyadic space of countable tightness is a Uniform Eberlein compact space. We prove that if a point p of a polyadic space Y has $\pi\chi(p, Y) = \kappa > \omega$, then there exists $K \subset Y$ such that $p \in K$ and K is homeomorphic to the Cantor cube 2^κ.

Introduction

Let $\alpha\kappa = \kappa \cup \{\infty\}$ be the one point compactification of the discrete space κ where κ is an infinite cardinal. For a cardinal λ, $\alpha\kappa^\lambda$ is the product of λ copies of $\alpha\kappa$ with the product topology. *Polyadic* spaces, introduced by Mrowka [15], are the Hausdorff continuous images of the spaces $\alpha\kappa^\lambda$.

In [9], Gerlits has asked whether a polyadic space of cellularity μ and tightness τ is an image of $\alpha\mu^\tau$. This would be a nice structure theorem that would demonstrate his thesis that the class of polyadic spaces is a 2 cardinal parameter class, i.e., most cardinal functions are functions of these two; as an example, he proved that weight = cellularity $\times$ tightness.

In Section 1 we collect major results from elsewhere that we will use in later sections. Section 2 of this paper consists of essential definitions and facts about the algebra of all clopen subsets of $\alpha\kappa^\lambda$.

2000 *Mathematics Subject Classification.* Primary 54D30, 54B10, 06E15; Secondary 54B15, 54A25.

Key words and phrases. Polyadic, Uniform Eberlein, compact, boolean, tightness, π-character.

Research supported by NSERC of Canada.

The main result of our paper, Theorem 3.1; Corollary 3.2, is that a zero-dimensional polyadic space of cellularity μ and tightness τ is an image of a closed $F \subset \alpha\mu^\tau$. Thus, we are a little closer to the solution of the Gerlits' question. Our result is strong enough to allow us to deduce, rather easily, basic structural properties of zero-dimensional polyadic spaces such as weight = cellularity $\times$ tightness. The zero-dimensional hypothesis seems like a "red herring" but we were unable to eliminate it.

In Section 5 we show that if p is a point in a polyadic space X with $\pi\chi(p, X) = \kappa > \omega$, then there exists a K homeomorphic to 2^κ such that $p \in K \subset X$. This was a natural problem arising from Gerlits' paper [9] and was known for regular cardinals κ but is new for singular cardinals.

All of our spaces are assumed to be Hausdorff. For two spaces X and Y, we say that Y is an image of X denoted $X \twoheadrightarrow Y$ if there exists a continuous surjection $\varphi : X \to Y$. We use $\overline{A}$, $\mathrm{int}(A)$ and $\approx$ to denote closure, interior and "is homeomorphic to", respectively. Cardinals are initial ordinals; κ, λ and τ always denote infinite cardinals; ω is the first infinite cardinal; κ^+ denotes the least cardinal greater than κ; $|X|$ denotes cardinality; $[X]^\tau$ denotes the set of all subsets of X of cardinality τ; $[X]^{<\tau}$ denotes the set of all subsets of X of cardinality $< \tau$; $[X]^{\leq\tau}$ denotes the set of all subsets of X of cardinality $\leq \tau$ and $f|A$ denotes restriction of function.

1. Known Results

Uniform Eberlein spaces, introduced by Benyamini and Starbird [5], are spaces that are homeomorphic to a weakly compact subspace of a Hilbert space. We have the following equivalence.

Result 1.1 (Benyamini, Rudin and Wage [6]). *X is Uniform Eberlein iff X is an image of a closed $F \subset \alpha\kappa^\omega$.*

We denote the clopen algebra of all clopen subsets of X by $\mathrm{CO}(X)$. A boolean space is a compact Hausdorff space such that $\mathrm{CO}(X)$ is a basis. For $\mathcal{C} \subset \mathrm{CO}(X)$ put $\ll \mathcal{C} \gg$ equal to the subalgebra of $\mathrm{CO}(X)$ generated by $\mathcal{C}$. A generating set for $\mathrm{CO}(X)$ is a $\mathcal{C} \subset \mathrm{CO}(X)$ such that $\ll \mathcal{C} \gg = \mathrm{CO}(X)$. A set $\mathcal{C} \subset \mathrm{CO}(X)$ is a (τ, κ) - *disjoint* collection if $\mathcal{C}$ is the union of at most τ many subcollections $\mathcal{C}_\alpha$ where for each α, $\mathcal{C}_\alpha$ is a disjoint collection of cardinality at most κ.

We want a characterization of closed subspaces of $\alpha\kappa^\tau$. The following result was communicated to the author by Petr Simon; for a proof when $\tau = \omega$, see Bell [4]; the proof of the general case is similar.

Result 1.2 (P. Simon)**.** *X is homeomorphic to a closed subspace of $\alpha\kappa^\tau$ iff X is a boolean space such that* $\mathrm{CO}(X)$ *has a* (τ, κ) *- disjoint generating set.*

The cellularity of a space X, denoted by $\mathrm{c}(X)$, is $\sup\{|\mathcal{B}| : \mathcal{B}$ is a disjoint collection of open sets of $X\}$. The following result is credited in [15] to Engelking; for a proof see Gerlits [8].

Result 1.3 (R. Engelking)**.** *If X is an image of $\alpha\kappa^\lambda$, then X is an image of $\alpha\mu^\lambda$ where $\mu = \mathrm{c}(X)$.*

So, the difficulty in Gerlits' question is not with the cellularity. If $A \subset X$ and $p \in \overline{A}$, then $\mathrm{a}(p, A) = \min\{|B| : p \in \overline{B}$ and $B \subset A\}$; $\mathrm{t}(p, X) = \sup\{\mathrm{a}(p, A) : A \subset X$ and $p \in \overline{A}\}$ and the tightness of a space X, denoted by $\mathrm{t}(X)$, is $\sup\{\mathrm{t}(p, X) : p \in X\}$. For compact spaces X, we have by Arhangelskii [1], that X has tightness $\leq \tau$ iff X does not contain a τ^+ - free sequence of points (i.e., a sequence $\{x_\alpha : \alpha < \tau^+\} \subset X$ such that for all $\alpha < \tau^+$, $\overline{\{x_\beta : \beta < \alpha\}} \cap \overline{\{x_\beta : \beta \geq \alpha\}} = \emptyset$). For $x \in \prod_{\alpha \in I} Y_\alpha$ and $C \subset I$, put $[x|C] = \{y \in \prod_{\alpha \in I} Y_\alpha : y$ extends $x|C\}$.

Result 1.4 (Gerlits [9]; Theorem 3, countable version.)**.** *Let $\varphi : Y = \prod_{\alpha \in I} Y_\alpha \twoheadrightarrow X$ where for all $\alpha \in I$, Y_α is a compact space of countable tightness and let $p \in X$. The following are equivalent for $\kappa > \omega$.*

a) *There exists $x \in \varphi^{-1}(p)$ such that for every $A \subset I$ with $|A| < \kappa$, $[x|A] \not\subset \varphi^{-1}(p)$.*

b) *There exists $A \subset X$ with $p \in \overline{A}$ and* $\mathrm{a}(p, A) \geq \kappa$.

c) *There exists $K \subset X$ with $p \in K$ and $K \approx 2^\kappa$.*

2. The clopen algebra of $\alpha\kappa^\lambda$

If B is a boolean algebra, then st(B) is the stone space of all ultrafilters of B. When we say "duality", we refer to the well-known equivalences of boolean spaces and continuous maps and boolean algebras and homomorphisms.

Let X be a boolean space. A subset $\mathcal{C} \subset \mathrm{CO}(X)$ is *centered* if for all finite $\mathcal{F} \subset \mathcal{C}$, $\bigcap \mathcal{F} \neq \emptyset$. If R and S are sets of ordinals, then $R < S$ means that for each $\alpha \in R$ and for each $\beta \in S$, $\alpha < \beta$. A τ - *free* sequence is a $\{b_\alpha : \alpha < \tau\} \subset \mathrm{CO}(X)$ such that whenever R and S are finite subsets of τ with $R < S$, then $\bigcap\{b_\alpha : \alpha \in R\} \setminus \bigcup\{b_\beta : \beta \in S\} \neq \emptyset$. By duality, upon using the previously mentioned result of Arhangelskii, we get that X has tightness $\leq \tau$ iff $\mathrm{CO}(X)$ does not contain a τ^+ - free sequence; for a proof, see van Douwen [7]. A τ - *independent* sequence is a $\{b_\alpha : \alpha < \tau\} \subset \mathrm{CO}(X)$ such that whenever R and S are disjoint finite subsets of τ, then $\bigcap\{b_\alpha : \alpha \in R\} \setminus \bigcup\{b_\beta : \beta \in S\} \neq \emptyset$. By duality, $X \twoheadrightarrow 2^\tau$ iff $\mathrm{CO}(X)$ contains a τ - independent sequence.

Let $\mathcal{B}_\kappa = \{\{\alpha\} : \alpha < \kappa\} \cup \{\alpha\kappa \setminus F : F$ is a non-empty finite subset of $\kappa\}$ be the canonical basis for $\alpha\kappa$. For each $F \subset \lambda$, let π_F denote the projection map of $\alpha\kappa^\lambda$ onto $\alpha\kappa^F$.

An $r \in \mathrm{CO}(\alpha\kappa^\lambda)$ is called a *basic rectangle* if there exists a non-empty finite $F \subset \lambda$ and for each $\alpha \in F$, a $B_\alpha \in \mathcal{B}_\kappa$ such that $r = (\prod_{\alpha \in F} B_\alpha) \times \alpha\kappa^{(\lambda \setminus F)}$. For a basic rectangle r, the just mentioned F and B_α's are unique; we define $\mathrm{supp}(r) = F$ and define $\mathrm{supp}^*(r) = \{\alpha \in \mathrm{supp}(r) : B_\alpha$ is a singleton$\}$. The canonical basis for $\alpha\kappa^\lambda$ is $\{r : r$ is a basic rectangle$\}$. It can be shown that for every $b \in \mathrm{CO}(\alpha\kappa^\lambda) \setminus \{\emptyset, \alpha\kappa^\lambda\}$ there is a smallest (under inclusion) finite $F \subset \lambda$ such that $b = \pi_F^{-1}(\pi_F(b))$ (a fact true in any product space); we denote this smallest set F by $\mathrm{supp}(b)$.

For $C \subset \lambda$, put $\mathrm{Fin}(C, \kappa) = \{s : s$ is a function with domain(s) a non-empty finite subset of C and range$(s) \subset \kappa\}$ and put $\mathrm{Fin}^*(C, \kappa) = \mathrm{Fin}(C, \kappa) \cup \{\emptyset\}$. For $s \in \mathrm{Fin}^*(C, \kappa)$, we put $[s] = \{x \in \alpha\kappa^\lambda : x$ extends $s\}$; note that $[\emptyset] = \alpha\kappa^\lambda$. For $\mathcal{S} \subset \mathrm{Fin}^*(C, \kappa)$, we put $[\mathcal{S}] = \{[s] : s \in \mathcal{S}\}$. For $s \in \mathrm{Fin}(C, \kappa)$, $[s]$ is a special kind of basic rectangle which we call a *finite function rectangle*. If r is a basic rectangle and $H \subset \mathrm{supp}^*(r)$, then $r|H$ denotes the unique $s \in \mathrm{Fin}^*(H, \kappa)$ with $[s] = \pi_H^{-1}(\pi_H(r))$. If C_i, $i < n$, are pairwise disjoint subsets of λ and $s_i \in \mathrm{Fin}(C_i, \kappa)$, then we use juxtaposition $s_0 s_1 \ldots s_{n-1}$ to denote the least common extension of $\{s_i : i < n\}$, i.e., the unique $s \in \mathrm{Fin}(\bigcup_{i<n} C_i, \kappa)$ such that $[s] = \bigcap_{i<n} [s_i]$.

Lemma 2.1. *Let ν be an uncountable regular cardinal. If $\{r_\alpha : \alpha < \nu\}$ is a collection of basic rectangles in $\alpha\kappa^\lambda$ with $\{[r_\alpha | \operatorname{supp}^*(r_\alpha)] : \alpha < \nu\}$ centered, then there exists $A \in [\nu]^\nu$ such that $\{r_\alpha : \alpha \in A\}$ is centered.*

Proof: Get $B \in [\nu]^\nu$ and a finite $R \subset \lambda$ such that for all $\alpha,\beta \in B$, $\operatorname{supp}(r_\alpha) \cap \operatorname{supp}(r_\beta) = R$. Get $A \in [B]^\nu$ and a finite $S \subset R$ such that for all $\alpha \in A$, $\operatorname{supp}^*(r_\alpha) \cap R = S$. Then $\{r_\alpha : \alpha \in A\}$ is centered. □

A crucial fact for us, due to the openness of each π_F is: if $b \in \mathrm{CO}(\alpha\kappa^\lambda)$ and $F \subset \lambda$, then there is a largest (under inclusion) $c \in \mathrm{CO}(\alpha\kappa^\lambda)$ such that $\operatorname{supp}(c) \subset F$ and $c \subset b$. We denote this largest such c by b^F. We refer the reader to Proposition 8.20 in Koppelberg [13] for a proof that $b^F = \alpha\kappa^\lambda \setminus \pi_F^{-1}(\pi_F(\alpha\kappa^\lambda \setminus b))$ (a fact true in any product space).

Lemma 2.2. *Let ν be an uncountable regular cardinal. If $\mathcal{C} \subset \mathrm{CO}(\alpha\kappa^\lambda)$ and $\ll\mathcal{C}\gg$ contains a ν - free sequence, then $\mathcal{C}$ contains a ν - free sequence.*

Proof: Assume $\ll\mathcal{C}\gg$ contains a ν - free sequence. Then, there exists $p \in \mathrm{st}(\ll\mathcal{C}\gg)$ and there exists $A \subset \mathrm{st}(\ll\mathcal{C}\gg)$ such that $p \in \overline{A}$ and $\mathrm{a}(p, A) \geq \nu$. By Result 1.4, $\mathrm{st}(\ll\mathcal{C}\gg)$ contains a subspace homeomorphic to 2^ν. Since $\mathrm{st}(\ll\mathcal{C}\gg)$ is boolean, this implies that $\mathrm{st}(\ll\mathcal{C}\gg)$ maps onto 2^ν and therefore that $\ll\mathcal{C}\gg$ contains a ν - independent sequence. It is a fact about boolean algebras that if $\ll\mathcal{C}\gg$ contains a ν - independent sequence, then $\mathcal{C}$ contains a ν - independent sequence; see Gorelic [10]. Hence $\mathcal{C}$ contains a ν - free sequence. □

Remark. Lemma 2.2 uses the special nature of $\alpha\kappa^\lambda$; it is not true for an arbitrary boolean space X. As an example, let X be the compact ordinal space $\omega_1 + 1$ and for $\alpha < \omega_1$, put $C_\alpha = \{\beta : \beta \leq \alpha\}$. Then $\mathcal{C} = \{C_\alpha : \alpha < \omega_1\}$ does not contain an ω_1 - free sequence but $\{X \setminus C_\alpha\}$ is an ω_1 - free sequence.

Fix an uncountable regular cardinal ν. Let C and D be subsets of λ of cardinality $< \nu$, let $b \in \mathrm{CO}(\alpha\kappa^\lambda)$ and let $s \in \mathrm{Fin}^*(C,\kappa)$. We say that D *pierces* $[s]\cap b$ *beyond* C if $D\cap C = \emptyset$ and whenever r is a basic rectangle with $r \subset ([s]\cap b)\setminus([s]\cap b)^{C\cup D}$ and $r|(\mathrm{supp}^*(r)\cap C) = s$, then $\mathrm{supp}^*(r) \cap D \neq \emptyset$.

Lemma 2.3 (Piercing Lemma)**.** *Let ν be an uncountable regular cardinal, let $\mathcal{B} \subset \mathrm{CO}(\alpha\kappa^\lambda)$ contain no ν - free sequences, let $C \in [\lambda]^{<\nu}$ and let $s \in \mathrm{Fin}^*(C,\kappa)$. Then, there exists $D \in [\lambda]^{<\nu}$ such that for all $b \in \mathcal{B}$, D pierces $[s] \cap b$ beyond C.*

Proof: Assume not, i.e., for all $D \in [\lambda]^{<\nu}$ there exists $b \in \mathcal{B}$ such that D does not pierce $[s] \cap b$ beyond C.

By induction, get $\{b_\alpha : \alpha < \nu\} \subset \mathcal{B}$ such that $D_\alpha = \bigcup_{\beta<\alpha} \mathrm{supp}(b_\beta)\setminus C$ does not pierce $[s] \cap b_\alpha$ beyond C. Put $c_\alpha = [s] \cap b_\alpha$. For each α choose a basic rectangle $r_\alpha \subset c_\alpha\setminus c_\alpha^{C\cup D_\alpha}$ with $r_\alpha|(\mathrm{supp}^*(r_\alpha)\cap C) = s$ such that $\mathrm{supp}^*(r_\alpha) \cap D_\alpha = \emptyset$. Since for each α, $c_\alpha^{C\cup D_\alpha}$ is actually $c_\alpha^{F_\alpha}$ for some finite $F_\alpha \subset C \cup D_\alpha$, we can apply the Pressing Down Lemma to get $B \in [\nu]^\nu$ and a finite $F \subset \lambda$ such that for all $\alpha \in B$, $c_\alpha^{C\cup D_\alpha} = c_\alpha^F$. Apply Lemma 2.1 to get $A \in [B]^\nu$ such that $\{r_\alpha : \alpha \in A\}$ is centered. The set $\{b_\alpha : \alpha \in A\}\cup\{[s]\}\cup\{c_\alpha^F : \alpha \in A\}$ does not contain a ν - free sequence. Therefore, by Lemma 2.2, $\{c_\alpha \setminus c_\alpha^F : \alpha \in A\}$ is not a ν - free sequence (in the induced order on A). Since this collection is centered, there must be finite $R < S < \gamma$, all in A, such that if we put $c = \bigcap_{\beta\in R} (c_\beta \setminus c_\beta^F) \setminus \bigcup_{\beta\in S} (c_\beta \setminus c_\beta^F)$, then $\emptyset \neq c \subset c_\gamma \setminus c_\gamma^F$. Then $\mathrm{supp}(c) \subset C \cup D_\gamma$ and $c \subset c_\gamma$, so $c \subset c_\gamma^{C\cup D_\gamma} = c_\gamma^F$; this contradicts $c \subset c_\gamma \setminus c_\gamma^F$. □

A set $\mathcal{C} \subset \mathrm{CO}(X)$ is a *point - τ* collection if for every $x \in X$, $|\{C \in \mathcal{C} : x \in C\}| \leq \tau$.

Lemma 2.4. *A point - τ collection of finite function rectangles in $\alpha\kappa^\lambda$ is a (τ,κ) - disjoint collection.*

Proof: For $\mathcal{T} \subset \mathrm{Fin}(\lambda,\kappa)$ and for $n \geq 1$ put $\mathcal{T}^n = \{t \in \mathcal{T} : |\mathrm{supp}(t)| \leq n\}$. For $n \geq 1$ let P_n be the statement that whenever $\mathcal{T} \subset \mathcal{T}^n$ and $[\mathcal{T}]$ is point - τ, then $[\mathcal{T}]$ is (τ,κ) - disjoint. It suffices to show that P_n is true for all n. P_1 is true because if $\mathcal{T} \subset \mathcal{T}^1$ and $[\mathcal{T}]$ is point - τ, then $\bigcup_{t\in\mathcal{T}} \mathrm{supp}(t)$ must be a set of cardinality at most τ. Now assume that P_n is true for some n. Take $\mathcal{T} \subset \mathcal{T}^{n+1}$ such that $[\mathcal{T}]$ is point - τ. Apply the Piercing Lemma with $\nu = \tau^+$, $\mathcal{B} = [\mathcal{T}]$, $C = \emptyset$ and $s = \emptyset$ to get a $D \in [\lambda]^{\leq \tau}$ such that for all $t \in \mathcal{T}$, D pierces $[t]$ beyond $\emptyset$. This implies that for all $t \in \mathcal{T}$, $\operatorname{supp}(t) \cap D \neq \emptyset$. For all non-empty finite $F \subset D$ put $\mathcal{T}_F = \{t \in \mathcal{T} : \operatorname{supp}(t) \cap D = F\}$. Since $|D| \leq \tau$ and $\mathcal{T} = \bigcup_{F \in [D]^{<\omega}} \mathcal{T}_F$, it suffices to show that for each F, $[\mathcal{T}_F]$ is (τ, κ) - disjoint. So, fix a non-empty finite $F \subset D$. For each $r \in \operatorname{Fin}(F, \kappa)$ put $\mathcal{T}'_r = \{t \in \mathcal{T}_F : t|(\operatorname{supp}(t) \cap D) = r\}$. Since $\mathcal{T}_F = \bigcup_{r \in \operatorname{Fin}(F,\kappa)} \mathcal{T}'_r$ and $r \neq r'$, $t \in \mathcal{T}'_r$, $t' \in \mathcal{T}'_{r'}$ imply that $[t] \cap [t'] = \emptyset$, it suffices to show that for each $r \in \operatorname{Fin}(F, \kappa)$, $[\mathcal{T}'_r]$ is (τ, κ) - disjoint. Put $\mathcal{T}''_r = \{t|(\operatorname{supp}(t) \setminus D) : t \in \mathcal{T}'_r\}$. Then, $\mathcal{T}''_r \subset \mathcal{T}^n$ and $[\mathcal{T}''_r]$ is point - τ, hence by P_n we have that $[\mathcal{T}''_r]$ is (τ, κ) - disjoint. Thus, $[\mathcal{T}'_r]$ is (τ, κ) - disjoint. □

3. Tightness in Boolean Polyadic Spaces

Theorem 3.1. *If $\mathcal{B} \subset \operatorname{CO}(\alpha\kappa^\lambda)$ does not contain a τ^+ - free sequence, then there exists a point - τ family $\mathcal{C}$ of finite function rectangles in $\alpha\kappa^\lambda$ such that $\mathcal{B} \subset \ll\mathcal{C}\gg$.*

Proof: We will construct $\mathcal{C}$ by induction on the levels of a "tree" of height ω and our inductive steps utilize the Piercing lemma.

By induction on $j \geq 0$, we construct subsets of λ, $C_\emptyset$ and $C_{s_0 s_1 \ldots s_j}$, of cardinality at most τ, such that the following condition holds:

s_0 ranges over all elements of $\operatorname{Fin}(C_\emptyset, \kappa)$, s_j ranges over all elements of $\operatorname{Fin}(C_{s_0 s_1 \ldots s_{j-1}}, \kappa)$ and for all $b \in \mathcal{B}$, $C_\emptyset$ pierces b beyond $\emptyset$ while $C_{s_0 s_1 \ldots s_j}$ pierces $[s_0 s_1 \ldots s_j] \cap b$ beyond $C_\emptyset \cup \bigcup_{k<j} C_{s_0 s_1 \ldots s_k}$.

To begin, apply the Piercing Lemma with $\nu = \tau^+$, $C = \emptyset$ and $s = \emptyset$ to get a $C_\emptyset \in [\lambda]^{\leq \tau}$ such that for all $b \in \mathcal{B}$, $C_\emptyset$ pierces b beyond $\emptyset$. So far, we have s_0 defined where $s_0 \in \operatorname{Fin}(C_\emptyset, \kappa)$. Then, given that we have $s_0 s_1 \ldots s_j$ for all j with $0 \leq j \leq i$, $C_\emptyset$, and $C_{s_0 s_1 \ldots s_{j-1}}$, for all j with $1 \leq j \leq i$, satisfying the above condition, we proceed to stage $i+1$ as follows: for each $s_0 s_1 \ldots s_i$ apply the Piercing Lemma to get a $C_{s_0 s_1 \ldots s_i} \in [\lambda]^{\leq \tau}$ such that for all $b \in \mathcal{B}$, $C_{s_0 s_1 \ldots s_i}$ pierces $[s_0 s_1 \ldots s_i] \cap b$ beyond $C_\emptyset \cup \bigcup_{j<i} C_{s_0 s_1 \ldots s_j}$ and we now have $s_0 s_1 \ldots s_i s_{i+1}$ defined where $s_{i+1} \in \operatorname{Fin}(C_{s_0 s_1 \ldots s_i}, \kappa)$.

Put $\mathcal{S} = \{s_0 s_1 ... s_i : i \geq 0$ and $s_0 \in \mathrm{Fin}(C_\emptyset, \kappa)$ and for all j with $1 \leq j \leq i$, $s_j \in \mathrm{Fin}(C_{s_0 s_1 ... s_{j-1}}, \kappa)\}$. Finally, put $\mathcal{C} = [\mathcal{T}]$ where $\mathcal{T} = \{t :$ there exists $i \geq 0$ and $s_0 s_1 ... s_i \in \mathcal{S}$ such that $s_0 s_1 ... s_i \subset t \in \mathrm{Fin}(C_\emptyset \cup \bigcup_{j<i} C_{s_0 s_1 ... s_j}, \kappa)\}$.

We now show that $\mathcal{C}$ is point - τ. Since, for each $s = s_0 s_1 ... s_i \in \mathcal{S}$, $\{t \in \mathcal{T} : s_0 s_1 ... s_i \subset t \in \mathrm{Fin}(C_\emptyset \cup \bigcup_{j<i} C_{s_0 s_1 ... s_j}, \kappa)\}$ is point - τ, it suffices to show that $[\mathcal{S}]$ is point - τ. Assume not and get $i \geq 0$ and a centered $\{[s_0^\alpha s_1^\alpha ... s_i^\alpha] : \alpha < \tau^+\} \subset [\mathcal{S}]$. Since, for all α, $s_0^\alpha \in \mathrm{Fin}(C_\emptyset, \kappa)$, we can get $A_0 \in [\tau^+]^{\tau^+}$ and a single $s_0 \in \mathrm{Fin}(C_\emptyset, \kappa)$ such that for all $\alpha \in A_0$, $s_0^\alpha = s_0$. Since, for all $\alpha \in A_0$, $s_1^\alpha \in \mathrm{Fin}(C_{s_0}, \kappa)$, we can get $A_1 \in [A_0]^{\tau^+}$ and a single $s_1 \in \mathrm{Fin}(C_{s_0}, \kappa)$ such that for all $\alpha \in A_1$, $s_1^\alpha = s_1$. In this way we continue until stage i whereupon we have that for all $\alpha \in A_i \in [A_{i-1}]^{\tau^+}$, $s_0^\alpha s_1^\alpha ... s_i^\alpha = s_0 s_1 ... s_i$; our contradiction.

We now show that $\mathcal{B} \subset \ll \mathcal{C} \gg$. A basic observation is that if $C \subset \lambda$ and $s \in \mathrm{Fin}^*(C, \kappa)$, then if $b \in \mathrm{CO}(\alpha\kappa^\lambda)$ such that $b \subset [s]$ and $\mathrm{supp}(b) \subset C$, then

$$b \in \ll \{[t] : s \subset t \in \mathrm{Fin}(C, \kappa)\} \gg .$$

This observation implies the following:

Fact. If $b \in \mathcal{B}$, then $b^{C_\emptyset} \in \ll \mathcal{C} \gg$ and if $s = s_0 s_1 ... s_i \in \mathcal{S}$, then

$$([s] \cap b)^{C_\emptyset \cup \bigcup_{j<i} C_{s_0 s_1 ... s_j}} \in \ll \mathcal{C} \gg .$$

We will show that for each $x \in b \in \mathcal{B}$ there exists $c \in \ll \mathcal{C} \gg$ such that $x \in c \subset b$. Then, by compactness of $\alpha\kappa^\lambda$, b will be a finite union of such c's, hence $b \in \ll \mathcal{C} \gg$.

Take $x \in b \in \mathcal{B}$. Choose a basic rectangle r such that $x \in r \subset b$ and $\mathrm{supp}^*(r) = \{\alpha \in \mathrm{supp}(b) : x(\alpha) \neq \infty\}$.

Step 0. If $x \in b^{C_\emptyset}$, then we are done by the above Fact. So, assume that $x \in b \setminus b^{C_\emptyset}$. Choose a basic rectangle r_0 such that $x \in r_0 \subset b \setminus b^{C_\emptyset}$ and $\mathrm{supp}^*(r_0) = \mathrm{supp}^*(r)$. As $r_0 | (\mathrm{supp}^*(r_0) \cap \emptyset) = \emptyset$ and $C_\emptyset$ pierces b beyond $\emptyset$, we have that $\mathrm{supp}^*(r_0) \cap C_\emptyset \neq \emptyset$. Put $s_0 = r_0 | (\mathrm{supp}^*(r_0) \cap C_\emptyset) \in \mathrm{Fin}(C_\emptyset, \kappa)$. Then, $x \in [s_0] \cap b$.

Step 1. If $x \in ([s_0] \cap b)^{C_\emptyset \cup C_{s_0}}$, then we are done by the above Fact. So, assume that $x \in ([s_0] \cap b) \setminus ([s_0] \cap b)^{C_\emptyset \cup C_{s_0}}$. Choose a basic rectangle r_1 such that $x \in r_1 \subset ([s_0] \cap b) \setminus ([s_0] \cap b)^{C_\emptyset \cup C_{s_0}}$

and $\text{supp}^*(r_1) = \text{supp}^*(r)$. As $r_1|(\text{supp}^*(r_1) \cap C_\emptyset) = s_0$ and C_{s_0} pierces $[s_0] \cap b$ beyond $C_\emptyset$, we have that $\text{supp}^*(r_1) \cap C_{s_0} \neq \emptyset$. Put $s_1 = r_1|(\text{supp}^*(r_1) \cap C_{s_0}) \in \text{Fin}(C_{s_0}, \kappa)$. Then, $x \in [s_0 s_1] \cap b$.
Step 2. If $x \in ([s_0 s_1] \cap b)^{C_\emptyset \cup C_{s_0} \cup C_{s_0 s_1}}$, then ... . Continue as in Step 1.

We must stop at some finite step (in which case we are done) because for all $n > 0$, $\text{supp}^*(r_n) = \text{supp}^*(r)$ is a finite set, $\text{supp}^*(r_n) \cap C_{s_0 s_1 \dots s_{n-1}} \neq \emptyset$ and $C_{s_0}, C_{s_0 s_1}, \dots, C_{s_0 s_1 \dots s_{n-1}}$ are pairwise disjoint. □

Corollary 3.2. *A boolean polyadic space of cellularity μ and tightness τ is an image of a closed $F \subset \alpha\mu^\tau$. In particular, a boolean polyadic space of countable tightness is Uniform Eberlein.*

Proof: By Result 1.3, there exists $\varphi : \alpha\mu^\lambda \twoheadrightarrow X$ for some cardinal λ. Then $\mathcal{B} = \{\phi^{-1}(b) : b \in \text{CO}(X)\}$ is a subalgebra of $\text{CO}(\alpha\mu^\lambda)$ which is isomorphic to $\text{CO}(X)$, hence $\mathcal{B}$ does not contain an τ^+ - free sequence. Apply Theorem 3.1 and get a point - τ family $\mathcal{C}$ of finite function rectangles in $\alpha\mu^\lambda$ such that $\mathcal{B} \subset \ll\mathcal{C}\gg$. Lemma 2.4 implies that $\mathcal{C}$ is (τ, μ) - disjoint. Result 1.2 implies that $\text{st}(\ll\mathcal{C}\gg)$ is homeomorphic to a closed subspace of $\alpha\mu^\tau$ and since $\text{st}(\ll\mathcal{C}\gg) \twoheadrightarrow \text{st}(\mathcal{B}) \approx X$ the proof is complete. In particular, when $\tau = \omega$, X will be Uniform Eberlein by Result 1.1. □

Corollary 3.3 (Gerlits [9]). *For boolean polyadic spaces, we have that weight = cellularity $\times$ tightness.*

Proof: Let X be a boolean polyadic space with $\mu = \text{c}(X)$ and $\tau = \text{t}(X)$. If follows from Corollary 3.2 that the weight of X is at most the weight of $\alpha\mu^\tau$ which is $\mu \times \tau$. □

4. π-CHARACTER IN POLYADIC SPACES

For a point p in a space X, a local π-base $\mathcal{B}$ at p is a collection of non-empty open sets of X such that whenever U is a neighbourhood of p, then there exists $B \in \mathcal{B}$ such that $B \subset U$. The π-character of a point p in X, denoted by $\pi\chi(p, X)$ is $\min\{|\mathcal{B}| : \mathcal{B}$ is a local π - base at $p\}$ and we put $\pi\chi(X) = \sup\{\pi\chi(p, X) : p \in X\}$.

We want to show that if X is a polyadic space and $p \in X$ with $\pi\chi(p, X) = \kappa > \omega$, then there exists $K \subset X$ such that $p \in K$ and $K \approx 2^\kappa$. If κ is regular, then it follows from a result of Sapirovski (see page 54 of Juhasz [11]) that there exists $A \subset X$ such that

$p \in \overline{A}$ and $\mathrm{a}(p,A) \geq \kappa$; now applying Result 1.4 gives us $K \subset X$ such that $p \in K$ and $K \approx 2^\kappa$. What about singular κ? Gerlits [9] (part of Theorem 9 there) shows that 2^κ can be embedded as a subspace K of X but not necessarily such that $p \in K$. We address this omission. One consequence of our Theorem 4.1 is that the Arhangelskii problem on page 68 in [2] of whether $\pi\chi(x,X) \leq \mathrm{t}(x,X)$ for any point x in any compact space can not be disproved by a polyadic space.

Remark. Perhaps it is true that in any compact space X, if $p \in X$ and $\pi\chi(p,X) = \kappa$, then there exists $A \subset X$ such that $p \in \overline{A}$ and $\mathrm{a}(p,A) \geq \kappa$; but this is still an open problem for singular κ, [12] notwithstanding, see Bell [3] for a discussion.

Theorem 4.1. *Let X be an image of a product of compact spaces of countable tightness and let $p \in X$. If $\pi\chi(p,X) = \kappa > \omega$, then there exists $K \subset X$ with $p \in K$ and $K \approx 2^\kappa$.*

Proof: Let $\varphi : Y = \prod_{\alpha\in I} Y_\alpha \twoheadrightarrow X$ where for all $\alpha \in I$, Y_α is a compact space of countable tightness. By a basic open set in Y we mean an open set U such that there exists a finite $H \subset I$ and for each $\alpha \in H$, an open U_α in Y_α such that $U = \pi_H^{-1}(\prod_{\alpha\in H} U_\alpha)$. Let $p \in X$ with $\pi\chi(p,X) = \kappa > \omega$. Striving for a contradiction, assume that for all $K \subset X$ such that $K \approx 2^\kappa$, $p \notin K$. Get a closed $F \subset Y$ such that $\varphi|F$ is an irreducible map of F onto X. Choose $x \in F \cap \varphi^{-1}(p)$. Apply Result 1.4 to get $A \subset I$ with $|A| = \lambda < \kappa$ and $[x|A] \subset \varphi^{-1}(p)$. By a result of Malyhin [14], $t(\pi_A(F)) \leq \lambda$ and so by a result of Sapirovski [16], $\pi\chi(\pi_A(F)) \leq t(\pi_A(F)) \leq \lambda$. Let $\mathcal{C}$ be a local π-base at $x|A$ in $\pi_A(F)$ with $|\mathcal{C}| \leq \lambda$. Let us put $p_A = \pi_A|F$. Then $p_A : F \to \prod_{\alpha\in A} Y_\alpha$.

Claim. If U is an open subset of Y and $\varphi^{-1}(p) \subset U$, then there exists $C \in \mathcal{C}$ such that $p_A^{-1}(C) \subset U$.

Proof of Claim : Get basic open sets in Y, $B_1, \ldots, B_n$, such that $\varphi^{-1}(p) \subset \bigcup_{1\leq i\leq n} B_i \subset U$. Let $B = \bigcap\{\pi_A(B_i) : 1 \leq i \leq n$ and $[x|A] \cap B_i \neq \emptyset\}$. Since B is open in $\prod_{\alpha\in A} Y_\alpha$ and $x|A \in B$, get $C \in \mathcal{C}$ such that $C \subset B$.
Take $z \in p_A^{-1}(C)$. As $z \in F$ and $z|A \in C$, we have that $z|A \in B$.

Let $z = (z|A)^\frown w$, the least common extension of $z|A$ and w. Pick i, $1 \leq i \leq n$, such that $(x|A)^\frown w \in B_i$. Since $z|A \in \pi_A(B_i)$, $z|(I \setminus A) = w \in \pi_{I \setminus A}(B_i)$ and B_i is basic open, we have that $z \in B_i$. Hence, $p_A^{-1}(C) \subset \bigcup_{1 \leq i \leq n} B_i \subset U$. $\square_{\text{Claim}}$

As $\varphi|F$ is irreducible, $\{\text{int}(\varphi(p_A^{-1}(C))) : C \in \mathcal{C}\}$ is a π-base at p in X and so $\pi\chi(p, X) \leq |\mathcal{C}| = \lambda < \kappa$; our required contradiction. $\square$

Corollary 4.2. *If p is a point in a polyadic space X with $\pi\chi(p, X) = \kappa > \omega$, then there exists $A \subset X$ such that $p \in \overline{A}$ and $a(p, A) = \kappa$. In particular, $\pi\chi(p, X) \leq t(p, X)$ for all points p in a polyadic space X.*

References

[1] A.Arhangelskii, *On bicompacta hereditarily satisfying Suslin's condition. Tightness and Free sequences*, Soviet Math. Dokl., **1**2:4 (1971), 1253-1257.

[2] ———, *Structure and classification of topological spaces and cardinal invariants*, Russian Math. Surveys, **3**3:6, (1978), 33-96.

[3] M.Bell, *Tightness and π-Character in Centered spaces*, Colloquium Mathematicum, **8**0 (1999), 297-307.

[4] ———, *Universal Uniform Eberlein Compact Spaces*, Proc. Amer. Math. Soc., **128, No. 7** (2000), 2191-2197.

[5] Y.Benyamini and T.Starbird, *Embedding weakly compact sets into Hilbert space*, Israel J. Math., **2**3 (1976), 137-141.

[6] Y.Benyamini, M.E.Rudin and M.Wage, *Continuous Images of Weakly Compact Subsets of Banach Spaces*, Pacific Journal of Mathematics, **7**0 (1977), 309-324.

[7] E.van Douwen, *Cardinal Functions on Boolean Spaces*, in Handbook of Boolean Algebras, Vol. 2, Editors J. Monk with R. Bonnet, (1989), North-Holland Publishing Co.

[8] J.Gerlits, *On a problem of S. Mrowka*, Periodica Mathematica Hungarica, **4 (1)** (1973), 71-79.

[9] ———, *On a Generalization of Dyadicity*, Studia Scientiarum Mathematicarum Hungarica, **13** (1978), 1-17.

[10] I. Gorelic, *A note on Boolean algebras*, Comment. Math. Univ. Carolinae, **35,3** (1994), 523-524.

[11] I. Juhasz, Cardinal Functions in Topology - Ten Years Later, Mathematical Centre Tracts, **123**, Mathematisch Centrum, Amsterdam, (1980).

[12] I. Juhasz and S. Shelah, *$\pi(X) = \delta(X)$ For Compact X*, Topology and its Applications, **32** (1989), 289-294.

[13] S.Koppelberg, *General Theory of Boolean Algebras*, in Handbook of Boolean Algebras, Vol. 1, Editors J. Monk with R. Bonnet, (1989), North-Holland Publishing Co.

[14] V.Malyhin, *On the tightness and the Suslin number in exp X and in product spaces*, Dokl. Akad. Nauk SSSR, **203** (1972), 1001-1003.
[15] S.Mrowka, *Mazur Theorem and* $\mathfrak{m}$ *- adic Spaces*, Bull. Acad. Pol. Sci., **XVIII**, No.6, (1970), 299-305.
[16] B.Sapirovskii, *On π-character and π-weight in compact Hausdorff spaces*, Soviet Math. Dokl., **16** (1975), 999-1004.

Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2
E-mail address: mbell@cc.umanitoba.ca
URL: http://home.cc.umanitoba.ca/~mbell/