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CONNECTIONS BETWEEN CLASSES OF SPACES AND OF MAPPINGS

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ABSTRACT. R. W. Wardle has used a function with hyperspace values to prove some connections between the property of Kelley and confluent mappings. We generalize this idea to define classes of objects and of morphisms in a category, and to investigate some relations between these classes. The obtained results are applied to continuum theory.

1. Introduction

In [27] R. W. Wardle investigated, for a given continuum X, a function $\alpha_X : X \to C^2(X)$, and has shown that continuity of this function is equivalent to the property of Kelley, as well as that commutativity of some natural diagram involving the function α_X (see diagram (3.4) below) is equivalent to confluence of a function f. Thus the function α_X determines a class of continua, namely those having the property of Kelley, and a class of mappings, namely confluent ones.

In this paper the above idea is extended to a more general case. We prove a series of theorems connecting some classes of objects of a category with classes of morphisms. We apply the techniques to continuum theory showing that some results are consequences of this general point of view. Some of them, for example those on C^*-smooth continua and hereditarily weakly confluent mappings, have never been stated before.

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The paper is organized in the following way. After Preliminaries, two examples are studied in Chapter 3. In the next chapter we prove some general results using categorical methods. These results are applied to continuum theory in the rest of the paper, i.e., in Chapters 5-9, where five different functions are investigated.

2. Preliminaries

All spaces considered in this paper are assumed to be topological, and all mappings are continuous functions. By a *continuum* we mean a compact connected Hausdorff space.

We say that a continuum X has the *property of Kelley at a point* $p \in X$ if for any subcontinuum K of X containing p and for any open neighborhood $\mathcal{U}$ of K in the hyperspace $C(X)$ of subcontinua of X there is a neighborhood U of p in X such that if $q \in U$, then there is a continuum $L \in C(X)$ with $q \in L \in \mathcal{U}$. A continuum X has the *property of Kelley* if it has the property of Kelley at each of its points (for the metric case see e.g. [21, Definition 16.10, p. 538]).

The definitions of various types of mappings and their properties can be found e.g. in [13], [17] and [18]. Besides the composition property and the composition factor property of a class $\mathfrak{M}$ of mappings as defined in [18, Chapter 5, Part A, p. 29 and Part B, p. 32], we say that $\mathfrak{M}$ the *weak composition factor property* provided that for every two mappings $f : X \to Y$ and $g : Y \to Z$ with $f \in \mathfrak{M}$, if their composition $g \circ f$ belongs to $\mathfrak{M}$, then the mapping g is in $\mathfrak{M}$.

Obviously the composition factor property implies the weak composition factor property; we will show the the class of hereditarily weakly confluent surjective mappings has the weak composition factor property, while does not have the composition factor property, see below, Corollary 6.2.

2.1. **Categories.**

In the present paper we only will use categories whose objects are sets and whose morphisms are functions. We denote by *Top* the category of topological spaces, by *Comp* the category of Hausdorff compact spaces, and by *Con* the category of Hausdorff continua, the morphisms in all three categories being continuous functions, i.e., mappings. The reader is referred to e.g. [15] for the needed information on categories.

Let a category $\mathcal{C}$ be fixed. Suppose that for every $\lambda \in \Lambda$, where Λ is a set directed by a relation $\leq$, we have an object $X_\lambda \in \mathcal{C}$, and for every $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$, a morphism $f_\lambda^\mu : X_\mu \to X_\lambda$ is defined such that the following two conditions are satisfied:

- $f_\lambda^\mu \circ f_\mu^\nu = f_\lambda^\nu$ for any $\lambda, \mu, \nu \in \Lambda$ satisfying $\lambda \leq \mu \leq \nu$,
- f_λ^λ is the identity on X_λ for each $\lambda \in \Lambda$.

Then the family $\mathbf{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ is called the *inverse system of objects* X_λ with *bonding morphisms* f_λ^μ.

Let $\mathbf{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ be an inverse system. The *inverse limit* $X = \varprojlim \mathbf{S}$ is an object of $\mathcal{C}$ such that for every $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$ there is a morphism $f_\lambda : X \to X_\lambda$ (called the *projection* from the inverse limit X into the λ-th factor X_λ) satisfying $f_\lambda^\mu \circ f_\mu = f_\lambda$, and having the property that if Y is any object of $\mathcal{C}$ such that for each $\lambda \in \Lambda$ there is a morphism $f'_\lambda : Y \to X_\lambda$ satisfying $f_\lambda^\mu \circ f'_\mu = f'_\lambda$ for each $\mu \in \Lambda$ with $\lambda \leq \mu$, then there is a unique morphism $g : Y \to X$ such that $f_\lambda \circ g = f'_\lambda$ for each $\lambda \in \Lambda$ (see [19, Chapter 1, §5, p. 54]). Note that some authors use the term "projective limit" in the same sense.

If $\mathcal{C}$ is a subcategory of the category *Top*, then the inverse limit, if exists, is homeomorphic to the one defined topologically, see e.g. [8, 2.5, p. 98]. In categories *Top, Comp* and *Con* the inverse limits do exist, see [8, p. 98, Theorem 3.2.13, p. 141 and Theorem 6.1.20, p. 355].

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories $\mathcal{C}$ and $\mathcal{D}$. We say that F *preserves inverse limits* if

$$F(\varprojlim \{X_\lambda, f_\lambda^\mu, \Lambda\}) = \varprojlim \{F(X_\lambda), F(f_\lambda^\mu), \Lambda\}.$$

It is known that the hyperspace functors C and 2^* preserve inverse limits, see [21, 1.169, p. 171, and Remark 1.170, p. 174].

We say that a category $\mathcal{C}$ has the *weak composition factor property* provided that for any epimorphism $f : X \to Y$ in $\mathcal{C}$ and for every function $g : Y \to Z$ if $g \circ f$ is a morphism in $\mathcal{C}$, then g is a morphism in $\mathcal{C}$. For example, the categories *Comp* and *Con* have this property, as well as some algebraic categories like groups or fields.

Let two inverse systems $\mathbf{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ and $\mathbf{T} = \{Y_\sigma, g_\sigma^\tau, \Sigma\}$ in a category $\mathcal{C}$ be given. By a *morphism* $\mathbf{h}$ of $\mathbf{S}$ to $\mathbf{T}$ in the category of inverse systems we mean a family $\{\phi, h^\sigma\}$ consisting of a nondecreasing function $\phi : \Sigma \to \Lambda$ such that the set $\phi(\Sigma)$ is cofinal in Λ, and of morphisms $h^\sigma : X_{\phi(\sigma)} \to Y_\sigma$ defined for all $\sigma \in \Sigma$ and such that $g^\tau_\sigma \circ h^\tau = h^\sigma \circ f^{\phi(\tau)}_{\phi(\sigma)}$, i.e., such that the diagram

$$\begin{array}{ccc} X_{\phi(\sigma)} & \xleftarrow{f^{\phi(\tau)}_{\phi(\sigma)}} & X_{\phi(\tau)} \\ {\scriptstyle h^\sigma}\big\downarrow & & \big\downarrow{\scriptstyle h^\tau} \\ Y_\sigma & \xleftarrow[g^\tau_\sigma]{} & Y_\tau \end{array} \tag{2.1.1}$$

is commutative for any $\sigma, \tau \in \Sigma$ satisfying $\sigma \le \tau$. Any morphism $\mathbf{h} : \mathbf{S} \to \mathbf{T}$ induces a morphism of $X = \varprojlim \mathbf{S}$ to $Y = \varprojlim \mathbf{T}$, called the *limit morphism induced by* $\{\phi, h^\sigma\}$, denoted by $h = \varprojlim \{\phi, h^\sigma\} : X \to Y$ and defined as the only morphism $h : X \to Y$ such that the diagram

$$\begin{array}{ccc} X_{\phi(\sigma)} & \xleftarrow{f_{\phi(\sigma)}} & X \\ {\scriptstyle h^\sigma}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ Y_\sigma & \xleftarrow[g_\sigma]{} & Y \end{array} \tag{2.1.2}$$

is commutative for any $\sigma \in \Sigma$ (see [19, p. 57]). For the category *Top* see [8, Section 2.5, p. 101]. The most natural is the case when $\Sigma = \Lambda$ and ϕ is the identity on Λ; then we will write $h = \varprojlim \{h^\lambda\}$.

2.2. **Hyperspaces.** Given a topological space X, we let 2^X denote the hyperspace of all nonempty closed subsets of X equipped with the Vietoris topology (see [21, (0.12), p. 10]). The basis of the Vietoris topology in 2^X consists of sets of the form

$$\langle U_1, \dots, U_n \rangle = \{A \in 2^X : A \subset U_1 \cup \dots \cup U_n \quad \text{and} \quad A \cap U_i \ne \emptyset \quad \text{for each} \quad i \in \{1, \dots, n\}\},$$

where each U_i is open in X (see e.g. [8, 2.7.20, p. 120]). If X is a compact metric space with a metric d, then the topology on 2^X coincides with the one generated by the Hausdorff metric H (see e.g. [21, (0.1), p. 1 and (0.13), p. 10]). Further, we denote by $C(X)$ the hyperspace of all connected elements of 2^X, and we put $C^2(X) = C(C(X))$. If X is a Hausdorff compact space, then $C(X)$ is the hyperspace of all nonempty subcontinua of X. The reader is referred to [13] and [21] for needed information on the structure of hyperspaces.

Given a mapping $f : X \to Y$ between topological spaces X and Y, we consider mappings (called the *induced* ones)

$$2^f : 2^X \to 2^Y \quad \text{and} \quad C(f) : C(X) \to C(Y)$$

defined by

$$2^f(A) = \mathrm{cl} f(A) \text{ for every } A \in 2^X \text{ and } C(f)(A) = \mathrm{cl}\, f(A) \text{ for every } A \in C(X).$$

Thus 2^* and C are functors from *Top* into itself. Note that, if we consider compact spaces only, then $2^f(A) = f(A)$ and $C(f)(A) = f(A)$.

Let X be a continuum. Define $C^* : C(X) \to C^2(X)$ by $C^*(A) = C(A)$. It is known that for any continuum X the function C^* is upper semicontinuous, [21, Theorem 15.2, p. 514], and, in the metric case, it is continuous on a dense G_δ subset of $C(X)$, [21, Corollary 15.3, p. 515]. A continuum X is said to be *C^*-smooth at $A \in C(X)$* provided that the function C^* is continuous at A. A continuum X is said to be *C^*-smooth* provided that the function C^* is continuous on $C(X)$, i.e., at each $A \in C(X)$, see [21, Definition 5.15, p. 517]. For metric continua the following is known. Each arclike continuum is C^*-smooth, [21, Theorem 15.13, p. 525]. C^*-smoothness implies hereditary unicoherence, see [12, Corollary 3.4, p. 203] and [21, Note 1, p. 530]. Thus each arcwise connected C^*-smooth continuum is a dendroid, [21, Theorem 15.19, p. 528]. Further, a locally connected continuum is C^*-smooth if and only if it is a dendrite, [21, Theorem 15.11, p. 522].

Note that if the category *Comp* of Hausdorff compact spaces is under consideration, then C, C^2 and 2^* are functors from *Comp* to *Comp*.

The needed information on nets and nets of sets can be found in [8], [9] and [20].

For a given continuum X the following two functions, α_X and β_X, between some hyperspaces of X, will be used in various places of the paper. Namely the function $\alpha_X : X \to C^2(X)$ is defined by

$$\alpha_X(x) = \{A \in C(X) : x \in A\} \quad \text{for each point} \quad x \in X, \tag{2.2.1}$$

and the function $\beta_X : C(X) \to C^2(X)$ is determined by putting

$$\beta_X(A) = C(A) \quad \text{for each element} \quad A \in C(X). \tag{2.2.2}$$

Three other functions, γ_X, δ_X and ε_X, are considered exclusively in Chapters 7, 8 and 9 (respectively), so their definitions are put in these chapters, correspondingly.

3. Two examples

We begin with recalling two results of R. W. Wardle about a connection between the property of Kelley and confluent mappings. They have been proved in [27] for the metric case (see [27, Theorem 2.2, p. 292 and Theorem 4.2, p. 296]), but their proofs remain valid in the wider sense (sometimes using nets in place of sequences, or using definitions only).

For a given continuum X consider a function $\alpha_X : X \to C^2(X)$ defined above by (2.2.1).

Theorem 3.1. *The function α_X is continuous if and only if X has the property of Kelley.*

Theorem 3.2. *The diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \alpha_X}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_Y} \\ C^2(X) & \xrightarrow[C^2(f)]{} & C^2(Y) \end{array} \tag{3.3}$$

commutes if and only if the mapping f is confluent.

In this way utilizing the function α_X we can define a class of spaces, namely continua having the property of Kelley, and a class of mappings, namely confluent ones.

Using this point of view Wardle proved that the property of Kelley is preserved under confluent mappings, [27, Theorem 4.3, p. 296], and the second named author proved that the function α commutes with the inverse limits, and thus the property of Kelley is preserved under the inverse limit operation provided that the bonding mappings are confluent, [5, Theorem 2, p. 190].

In the present paper we generalize the Wardle's considerations by defining other, similar to α_X, functions. Let us start with another example. Given a continuum X, define a function $\beta_X : C(X) \to$

$C^2(X)$ as in (2.2.2). Continuity of the function β_X means (just by the definition) the C^*-smoothness of the continuum X (see [21, Definition 15.5, p. 513]). Now we will prove an analog of Theorem 3.2 for the function β_X.

Theorem 3.4. *The diagram*

$$\begin{array}{ccc} C(X) & \xrightarrow{C(f)} & C(Y) \\ \beta_X \big\downarrow & & \big\downarrow \beta_Y \\ C^2(X) & \xrightarrow[C^2(f)]{} & C^2(Y) \end{array} \tag{3.5}$$

commutes if and only if the mapping f is hereditarily weakly confluent.

Proof: Let $A \in C(X)$. The inclusion $(C^2(f) \circ \beta_X)(A) \subset (\beta_Y \circ C(f))(A)$ always holds. So take $P \in (\beta_Y \circ C(f))(A)$, i.e., such P which is a subcontinuum of the continuum $f(A)$. The commutativity of diagram (3.5) at A means that P is the image under f of some subcontinuum of A, i.e., that the partial mapping $f|A$ is weakly confluent. Since A was chosen as an arbitrary element of $C(X)$, this means that the mapping f is hereditarily weakly confluent.

Corollary 3.6. *Each hereditarily weakly confluent image of a C^*-smooth continuum is C^*-smooth.*

Proof: Let a continuum X be C^*-smooth and let a surjective mapping $f : X \to Y$ be hereditarily weakly confluent. Then diagram (3.5) commutes and the function β_X is continuous. Thus the composition $\beta_Y \circ C(f)$ is continuous, so is β_Y by the weak composition factor property for the category *Comp*, and thereby the continuum Y is C^*-smooth.

Recall that, by a result of J. Segal (see [25, Theorem 1.1, p. 707] and [21, 1.169, p. 171, and Remark 1.170, p. 174]), if $X = \varprojlim \{X_\lambda, f_\lambda^\mu, \Lambda\}$, where all X_λ are continua, then the hyperspace $C(X)$ is homeomorphic to the inverse limit $\varprojlim \{C(X_\lambda), C(f_\lambda^\mu), \Lambda\}$. Hereafter we will neglect the homeomorphism in matter writing simply $C(X) = \varprojlim \{C(X_\lambda), C(f_\lambda^\mu), \Lambda\}$.

Lemma 3.7. *The function β commutes with the inverse limit operation, i.e., if $X = \varprojlim \{X_\lambda, f_\lambda^\mu, \Lambda\}$, then $\varprojlim \{\beta_{X_\lambda}\} = \beta_X$.*

Proof: Let A and P be arbitrary subcontinua of X. Then $P \in \beta_X(A)$ means that P is a subcontinuum of A, and $P \in \varprojlim \{\beta_{X_\lambda}\}(A)$ means that P is a thread of elements of $\beta_{X_\lambda}(f_\lambda(A))$, where $f_\lambda : X \to X_\lambda$ is the natural projection. The last assertion is equivalent to saying that $P = \varprojlim \{P_\lambda, f_\lambda^\mu | P_\mu, \Lambda\}$ for some $P_\lambda \in \beta_{X_\lambda}(f_\lambda(A))$. To see the equivalence it is enough to put $P_\lambda = f_\lambda(P)$.

Theorem 3.8 *Let $\{X_\lambda, f_\lambda^\mu, \Lambda\}$ be an inverse system of C^*-smooth continua X_λ with hereditarily weakly confluent bonding mappings f_λ^μ. Then $X = \varprojlim \{X_\lambda, f_\lambda^\mu, \Lambda\}$ is a C^*-smooth continuum.*

Proof: Under the assumptions made, the function β_X is the inverse limit of continuous functions β_{X_λ} (see Lemma 3.7), whence it is continuous. Therefore the conclusion follows from the definition of C^*-smoothness.

4. General results

Let $\mathcal{K}$ be an arbitrary category, and let $\mathcal{L}$ be a category whose objects are sets and whose morphisms are functions. Further, let $F : \mathcal{K} \to \mathcal{L}$ and $G : \mathcal{K} \to \mathcal{L}$ be (covariant) functors. For any object X of the category $\mathcal{K}$ let ω_X be a function from $F(X)$ to $G(X)$. We say that $X \in S(\omega)$ provided that ω_X is a morphism of the category $\mathcal{L}$. Further, given a morphism $f : X \to Y$ in $\mathcal{K}$, we say that $f \in M(\omega)$ provided that the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \omega_X}\downarrow & & \downarrow{\scriptstyle \omega_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array} \tag{4.1}$$

To illustrate the introduced concepts note that in the first example of Chapter 3 above $\mathcal{K} = \mathcal{L} = Con$, $\omega_X = \alpha_X$ for any continuum X, F is the identity functor, and $G = C^2$. Then $S(\alpha)$ is the class of all continua having the property of Kelley, and $M(\alpha)$ is the class of confluent mappings. Similarly in the second example of that chapter, $S(\beta)$ is the class of C^*-smooth continua, and $M(\beta)$ is the class of hereditarily weakly confluent mappings.

We start with the following easy theorem.

Theorem 4.2 *If a category $\mathcal{L}$ has the weak composition factor property, $F : \mathcal{K} \to \mathcal{L}$ is a functor, $X \in S(\omega)$, and $f : X \to Y$ is a morphism such that $F(f)$ is an epimorphism, then $Y \in S(\omega)$.*

Proof: By the commutativity of diagram (4.1) we have $\omega_Y \circ F(f) = G(f) \circ \omega_X$. Since ω_X is a morphism of $\mathcal{L}$, the composition $\omega_Y \circ F(f)$ is a morphism of $\mathcal{L}$, and by the weak composition factor property for $\mathcal{L}$ the function ω_Y is a morphism of $\mathcal{L}$, thus $Y \in S(\omega)$.

Theorem 4.3. *The class $M(\omega)$ has the composition property.*

Proof: Consider the diagram

$$
\begin{array}{ccccc}
F(X) & \xrightarrow{F(f)} & F(Y) & \xrightarrow{F(g)} & F(Z) \\
{\scriptstyle \omega_X}\big\downarrow & & \big\downarrow{\scriptstyle \omega_Y} & & \big\downarrow{\scriptstyle \omega_Z} \\
G(X) & \xrightarrow[G(f)]{} & G(Y) & \xrightarrow[G(g)]{} & G(Z)
\end{array}
\tag{4.4}
$$

where $f, g \in M(\omega)$. We have to show that $\omega_Z \circ F(g) \circ F(f) = G(g) \circ G(f) \circ \omega_X$. Really,

$$
\begin{aligned}
\omega_Z \circ F(g) \circ F(f) &= F(f) \circ \omega_Y \circ G(g) && \text{because } g \in M(\omega) \\
&= G(g) \circ G(f) \circ \omega_X && \text{because } f \in M(\omega).
\end{aligned}
$$

Corollary 4.5. *Objects of category $\mathcal{K}$ and morphisms in $M(\omega)$ form a subcategory of $\mathcal{K}$.*

Let $\mathcal{K}(\omega)$ be the category as in Corollary 4.5.

Statement 4.6. *Restrict the functors F and G to the category $\mathcal{K}(\omega)$. Then the function ω is a natural transformation $\omega : F \dot{\to} G$* (in the sense of [15, Chapter I, Section 4, p. 16]).

Theorem 4.7. *If, for each $f \in M(\omega)$, the morphism $F(f)$ is a surjective function, then the class $M(\omega)$ has the weak composition factor property.*

Proof: Consider again diagram (4.4). We have to show that $\omega_Z \circ F(g) = G(g) \circ \omega_Y$. Let $y \in F(y)$. Since $F(f)$ is surjective, there exists a point $x \in F(X)$ such that $F(f)(x) = y$. Thus

$$\begin{aligned}(\omega_Z \circ F(g))(y) &= (\omega_Z \circ F(g) \circ F(f))(x) && \text{by the above equality}\\ &= (G(g) \circ G(f) \circ \omega_X)(x) && \text{by the assumption}\\ &= (G(g) \circ \omega_Y \circ F(f))(x) && \text{since } f \in M(\omega)\\ &= (G(g) \circ \omega_Y)(y) && \text{again by the above equality.}\end{aligned}$$

We say that ω *commutes with inverse limits* provided that for any inverse system $\mathbf{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ with $X = \varprojlim \mathbf{S}$ we have $\varprojlim \{\omega_{X_\lambda}\} = \omega_X$.

The following result is a consequence of the definitions.

Theorem 4.8. *Let $\mathcal{L}$ be a category such that the inverse limit of morphisms of $\mathcal{L}$ always exists. Assume that the functors F and G preserve inverse limits, and that ω commutes with inverse limits. Let $X = \varprojlim \{X_\lambda, f_\lambda^\mu, \Lambda\}$. If for every $\lambda, \mu \in \Lambda$ with $\lambda \le \mu$ we have $X_\lambda \in S(\omega)$ and $f_\lambda^\mu \in M(\omega)$, then $X \in S(\omega)$.*

Theorem 4.9. *Assume that the functors F and G preserve inverse limits, and that ω commutes with inverse limits. Let $\mathbf{h} = \{\phi, h^\sigma\} : \mathbf{S} \to \mathbf{T}$ be a morphism of $\mathbf{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ to $\mathbf{T} = \{Y_\sigma, g_\sigma^\tau, \Sigma\}$ in the category of inverse systems. If $h^\sigma \in M(\omega)$ for each $\sigma \in \Sigma$, and if the limit morphism $h = \varprojlim \mathbf{h}$ does exist, then $h \in M(\omega)$.*

Proof: We have to show that diagram (4.1) commutes with h in place of f. Since $h^\sigma \in M(\omega)$ for each $\sigma \in \Sigma$, the diagram

$$\begin{array}{ccc} F(X_{\phi(\sigma)}) & \xrightarrow{F(h^\sigma)} & F(Y_\sigma) \\ {\scriptstyle \omega_{X_{\phi(\sigma)}}}\big\downarrow & & \big\downarrow{\scriptstyle \omega_{Y_\sigma}} \\ G(X_{\phi(\sigma)}) & \xrightarrow[G(h^\sigma)]{} & G(Y_\sigma) \end{array} \tag{4.10}$$

commutes. Since F and G preserve inverse limits, we can pass to the limit objects and to the limit morphisms in both horizontal parts of diagram (4.10), i.e., we simply can omit the indices σ and $\phi(\sigma)$. Similarly, since ω commutes with inverse limits, we can do the same in both vertical parts of (4.10). In this way the proof is complete.

Let $\mathfrak{M}$ be a subclass of the class of morphisms of a category $\mathcal{K}$. We say that $\mathfrak{M}$ has:
– the *inverse limit projection property* provided that for each inverse system $\mathbf{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$, where X_λ are objects and f_λ^μ are morphisms of $\mathcal{K}$, if $X = \varprojlim \mathbf{S}$ does exist, and if $f_\lambda^\mu \in \mathfrak{M}$ for every $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$, then the projections $f_\lambda : X \to X_\lambda$ are in $\mathfrak{M}$ for each $\lambda \in \Lambda$;
– the *inverse limit property* provided that for every two inverse systems $\mathbf{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ and $\mathbf{T} = \{Y_\sigma, g_\sigma^\tau, \Sigma\}$ and for any morphism $\mathbf{h} = \{\phi, h^\sigma\} : \mathbf{S} \to \mathbf{T}$, where X_λ, Y_σ are objects, and f_λ^μ, g_σ^τ, h^σ are morphisms of $\mathcal{K}$, if $X = \varprojlim \mathbf{S}$, $Y = \varprojlim \mathbf{T}$ and $h = \varprojlim \mathbf{h}$ do exist, and if $h^\sigma \in \mathfrak{M}$ for each $\sigma \in \Sigma$, then $h \in \mathfrak{M}$.

Theorem 4.11. *Let $\mathfrak{M}$ be a subclass of the class of morphisms of a category $\mathcal{K}$. If $\mathfrak{M}$ has the inverse limit property, then it has the inverse limit projection property.*

Proof: Let $\mathbf{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ be an inverse system, with X_λ being objects and f_λ^μ being morphisms of the category $\mathcal{K}$. Fix $\lambda_0 \in \Lambda$, put $\Lambda' = \{\lambda \in \Lambda : \lambda_0 \leq \lambda\}$, and consider two inverse systems: the former, $\mathbf{S}'$, is obtained from $\mathbf{S}$ by restricting the index set Λ to Λ', i.e., $\mathbf{S}' = \{X_\lambda, f_\lambda^\mu, \Lambda'\}$. The latter, $\mathbf{C}$, is a constant one, with all factor objects equal to X_{λ_0} and with the identities as bonding morphisms: $\mathbf{C} = \{Y_\lambda, g_\lambda^\mu, \Lambda'\}$, where $Y_\lambda = X_{\lambda_0}$ and $g_\lambda^\mu = f_{\lambda_0}^{\lambda_0}$ for all $\lambda, \mu \in \Lambda'$ with $\lambda \leq \mu$. Take a morphism $\mathbf{h} : \mathbf{S}' \to \mathbf{C}$ defined by $h^\lambda = f_{\lambda_0}^\lambda : X_\lambda \to X_{\lambda_0}$ for all $\lambda \in \Lambda'$. Put $X = \varprojlim \mathbf{S}$ and note that it is homeomorphic to $\varprojlim \mathbf{S}'$. We will show that $h = f_{\lambda_0}$ is the limit morphism of $\{h^\lambda\}$. To this aim observe that the following diagram commutes

$$\begin{array}{ccc} X_{\lambda_0} & \xleftarrow{f_{\lambda_0}} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle h=f_{\lambda_0}} \\ X_{\lambda_0} & \xleftarrow[i]{} & X_{\lambda_0} \end{array}$$

in which $i = f_{\lambda_0}^{\lambda_0}$ is the identity morphism on X_{λ_0}.

As a consequence of Theorems 4.9. and 4.11 we get the following result.

Corollary 4.12. *Assume that the functors F and G preserve inverse limits, and that ω commutes with inverse limits. If* $\mathbf{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ *is an inverse system with the morphisms* $f_\lambda^\mu \in M(\omega)$, *and if* $X = \varprojlim \mathbf{S}$ *does exist, then the projections* $f_\lambda : X \to X_\lambda$ *are in* $M(\omega)$ *for each* $\lambda \in \Lambda$.

Remark 4.13 One can consider pointed versions of the concepts investigated in this chapter. For example, continuity of the function α_X at a point $x \in X$ is equivalent to the property of Kelley at x (see [27, Theorem 2.2, p. 292]). Similarly, commutativity of diagram (3.3) at a point $x \in X$ is equivalent to the property that the mapping f is confluent relative to x, see [3, Proposition 1, p. 376]. Commutativity of diagram (3.5) at the point $X \in C(X)$ is just the weak confluence of f. Then one can prove pointed versions of theorems of this chapter.

In the rest of the paper we will investigate some examples of the functions ω_X, and we will see some consequences of results of the present chapter. Most of them are well known, sometimes trivial, but we want to stress out that they can be seen as particular cases of much more general theorems proved in Chapter 4.

5. Property of Kelley and confluent mappings

Taking $\mathcal{K} = \mathcal{L} = Con$ and substituting $\omega = \alpha$ (where the function α_X is defined by (2.2.1)) in the theorems of Chapter 4, we get the following known results:

- *the property of Kelley is preserved under confluent mappings* (for the metric case see [27, Theorem 4.3, p. 296]);
- *the property of Kelley is preserved under the inverse limit operation if the bonding mappings are confluent* (for the metric case see [5, Theorem 2, 190]);
- *the class of confluent mappings has the composition property*, [2, III, p. 214], *and the weak composition factor property* (even more, the class has the composition factor property, [2, III, p. 214], but this is not a consequence of theorems of Chapter 4);
- *the class of confluent mappings has the inverse limit property*, [4, Corollary 13, p. 8], *and* consequently (by Theorem 4.11), it *has the inverse limit projection property*, [4, Corollary 7, p. 5].

6. C^*-smoothness and hereditarily weakly confluent mappings

Taking again in the theorems of the fourth chapter $\mathcal{K} = \mathcal{L} = Con$ and substituting $\omega = \beta$ (where β_X is defined by (2.2.2)) we obtain Corollary 3.6, Theorem 3.8 and the following results.

- *The class of hereditarily weakly confluent mappings has the composition property*, [17, Proposition 2.3, p.124].

Corollary 6.1. *The class of hereditarily weakly confluent mappings has the inverse limit property, and* consequently (by Theorem 4.11)*, the inverse limit projection property.*

Taking in the theorems of Chapter 4 the categories $\mathcal{K} = \mathcal{L}$ as the subcategory of *Con* composed of all continua as objects and of surjective mappings only as morphisms we get the following corollary to Theorem 4.7.

Corollary 6.2. *The class of hereditarily weakly confluent surjections has the weak composition factor property.*

Note that the class does not have the composition factor property, as proved in [18, Example 5.26, p. 34].

7. Local connectedness and monotone mappings

We adopt the following definition. A topological space X is said to be *locally connected at a point* $x \in X$ provided that for each open set U containing x there is a connected set V satisfying $x \in \operatorname{int} V \subset V \subset U$ (see [14, §49, I, p. 227]; note that some authors name this property *connectedness im kleinen at* p, e.g. [22, 5.10, p. 75]).

In this section we will use the categories $\mathcal{K} = \mathcal{L} = Comp$. We introduce a new functor W on the category *Comp* into itself, putting $W(X) = 2^X \cup \{\emptyset\}$, and defining $W(f) : W(X) \to W(Y)$ by $W(f)(A) = \operatorname{cl} f(A)$. Here $W(X)$ is understood as the disjoint union of 2^X with the Vietoris topology and of $\{\emptyset\}$ (so $\emptyset$ is an isolated point of $W(X)$).

Further, we consider two other functors on *Comp*. Let F denote the square functor, i.e., $F(X) = X \times X$ and $F(f) = f \times f$, and let a functor G be defined by $G(X) = W(C(X))$ and $G(f) = W(C(f))$.

For a compact Hausdorff space X consider a function $\gamma_X : X \times X \to W(C(X))$ defined by $\gamma_X((x,y)) = \{A \in C(X) : x, y \in A\}$. First of all we have to show that γ_X is well defined, i.e., that for any $x, y \in X$ the value $\gamma_X((x,y))$ is a closed subset of $C(X)$. Really, if $A \in C(X) \setminus \gamma_X((x,y))$, then one of the points x or y, say x, is not in A. Then $\langle X \setminus \{x\}\rangle$ is an open neighborhood of A in $C(X) \setminus \gamma_X((x,y))$.

Proposition 7.1. *If X is a compact Hausdorff space, then γ_X is continuous if and only if X is locally connected.*

Proof: First assume that γ_X is continuous. To prove the local connectedness of X let a point $x \in X$ and an open subset U of X with $x \in U$ be given. Then $\langle\langle U\rangle, C(X)\rangle \cap W(C(X))$ is open in $W(C(X))$, and thus, by continuity of γ_X, we have $\gamma_X^{-1}(\langle\langle U\rangle, C(X)\rangle) \cap W(C(X)))$ is open in $X \times X$. Since $(x,x) \in \gamma_X^{-1}(\langle\langle U\rangle, C(X)\rangle) \cap W(C(X))),$ there is an open set V' in X such that $x \in V'$ and $\gamma_X(V' \times V') \subset \langle\langle U\rangle, C(X)\rangle$. We will show that for each point $y \in V'$ there is a continuum $A(y) \subset U$ such that $x, y \in A(y)$. Then putting $V = \bigcup\{A(y) : y \in V'\}$ we have $x \in V' \subset \operatorname{int} V \subset V \subset U$. So V satisfies the needed condition in the definition of local connectedness of X at x.

To define $A(y)$ note that, since $\gamma_X(V' \times V') \subset \langle\langle U\rangle, C(X)\rangle$, we have $\gamma_X((x,y)) \in \langle\langle U\rangle, C(X)\rangle$ for each $y \in V'$, i.e., there is an element $A(y)$ of $\gamma_X((x,y))$ with $A(y) \in \langle U\rangle$. This means that $A(y)$ is a continuum containing x and y and contained in U, as required. Thus X is locally connected at x, and the proof of one implication is complete.

Second assume that the space X is locally connected. To prove continuity of γ_X we have to show that for every two nets $\{x_\sigma : \sigma \in \Sigma\}$ and $\{y_\sigma : \sigma \in \Sigma\}$, where Σ is any directed set, the conditions $x = \lim_{\sigma\in\Sigma} x_\sigma$ and $y = \lim_{\sigma\in\Sigma} y_\sigma$ imply $\gamma_X((x,y)) = \operatorname{Lim}_{\sigma\in\Sigma}\gamma_X((x_\sigma, y_\sigma))$ (see e.g. [8, Proposition 1.6.6, p. 51]). Denoting by Li and Ls the inferior and the superior topological limit operators, as in [20, p. 237] (compare [9, p. 169]) we see that the above equality is equivalent to the inclusions

(i) $\operatorname{Ls}_{\sigma\in\Sigma}\gamma_X((x_\sigma, y_\sigma)) \subset \gamma_X((x,y))$

and

(ii) $\gamma_X((x,y)) \subset \operatorname{Li}_{\sigma\in\Sigma}\gamma_X((x_\sigma, y_\sigma))$.

The statement (i) (upper semicontinuity) simply states that the limit of continua containing x_σ and y_σ is a continuum containing x and y, and thus is obviously true (even without local connectedness of X). The statement (ii) (lower semicontinuity) says that every continuum containing x and y is the limit of a net of some continua containing x_σ and y_σ. To prove it, take $A \in \gamma_X((x,y))$ and, for every $\sigma \in \Sigma$ let V_σ and W_σ be continua such that $x, x_\sigma \in V_\sigma$ and $y, y_\sigma \in W_\sigma$, and moreover such that $\operatorname{Lim}_{\sigma\in\Sigma} V_\sigma = \{x\}$ and $\operatorname{Lim}_{\sigma\in\Sigma} W_\sigma = \{y\}$. The existence of such continua V_σ and W_σ follows from local connectedness of X at x and at y, respectively. Then putting $A_\sigma = V_\sigma \cup A \cup W_\sigma$ we have $A_\sigma \in \gamma_X((x_\sigma, y_\sigma))$ and $\operatorname{Lim}_{\sigma\in\Sigma} A_\sigma = A$, hence the proof is finished.

Remark 7.2. Compactness of X is essential in Proposition 7.1 because of the following example. In the Euclidean plane let

$$X = (\{0\} \times [0,1]) \cup (\{1\} \times [0,1]) \cup \bigcup\{[0,1] \times \{\frac{1}{n}\} : n \in \mathbb{N}\},$$

and note that X is locally connected. Take $x = (0,0)$ and $y = (1,0)$, and define

$$U = \{0\} \times \bigcup\{\{\frac{t}{n+1} + \frac{1-t}{n} : t \in (\frac{1}{3}, \frac{2}{3})\} : n \in \mathbb{N}\}.$$

Observe that U is open in X, and that $\gamma_X((x,y)) \in \langle\langle U, X\rangle\rangle$, i.e., every element of $\gamma_X((x,y))$ intersects U. For each $n \in \mathbb{N}$ put $x_n = (0, \frac{1}{n})$ and $y_n = (1, \frac{1}{n})$ and note that $\gamma_X((x_n, y_n)) \notin \langle\langle U, X\rangle\rangle$. This shows that γ_X is not continuous at (x,y).

Proposition 7.3. *For every compact Hausdorff space X and a surjective mapping $f : X \to Y$ the diagram*

$$\begin{array}{ccc} X \times X & \xrightarrow{f\times f} & Y \times Y \\ {\scriptstyle\gamma_X}\big\downarrow & & \big\downarrow{\scriptstyle\gamma_Y} \\ W(C(X)) & \xrightarrow[W(C(f))]{} & W(C(Y)) \end{array}$$

commutes if and only if the mapping f is monotone.

Proof: Let $(x,y) \in X \times X$ be fixed. Then the inclusion

$$(W(C(f)) \circ \gamma_X)((x,y)) \subset (\gamma_Y \circ (f \times f))((x,y))$$

always holds. The opposite inclusion is equivalent to the following condition:

(∗) any subcontinuum of Y which contains the points $f(x)$ and $f(y)$ is the image of a subcontinuum of X which contains the points x and y.

Obviously if f is monotone, then condition (∗) holds. We will show the converse implication. Really, if f is not monotone, then there is a subcontinuum Q of Y such that $f^{-1}(Q)$ is not connected. Then it is enough to take x and y in two different components of $f^{-1}(Q)$, and note that Q belong to $(\gamma_Y \circ (f \times f))((x,y))$, but not to $(W(C(f)) \circ \gamma_X)((x,y))$. The proof is complete.

Proposition 7.4. *The function γ commutes with inverse limits.*

Proof: Let $X = \varprojlim \{X_\lambda, f_\lambda^\mu, \Lambda\}$ be the inverse limit of compact Hausdorff spaces X_λ, let $x = \{x_\lambda\}$ and $y = \{y_\lambda\}$ be two threads in X, and let $P \in C(X)$. We have to show that $\gamma_X((x,y)) = (\varprojlim \{\gamma_{X_\lambda}\})((x,y))$. Note that $P \in \gamma_X((x,y))$ means that P is a subcontinuum of X with $x, y \in P$; and $P \in (\varprojlim \{\gamma_{X_\lambda}\})((x_\lambda, y_\lambda))$ means that P is a thread of elements of $\gamma_{X_\lambda}((x_\lambda, y_\lambda))$. The last assertion is equivalent to the equality $P = \varprojlim \{P_\lambda, f_\lambda^\mu | P_\mu, \Lambda\}$ for some $P_\lambda \in \gamma_{X_\lambda}((x\lambda, y\lambda))$. To see the equivalence it is enough to put $P_\lambda = f_\lambda(P)$.

Applying results of Chapter 4 one can obtain the following known assertions:

- *local connectedness of continua is preserved under monotone mappings*;
- *local connectedness of continua is preserved under the inverse limit operation if the bonding mappings are monotone* (see [1, Theorem 4.3, p. 241]);
- *the class of monotone mappings of continua has the composition property*, [18, 5.1, p. 29], *and the weak composition factor property* (even more, the class has the composition factor property, [28, 3.2, p. 140], but this is not a consequence of theorems of Chapter 4);
- *the class of monotone mappings of continua has the inverse limit property* (for countable inverse systems see [10, Theorem 5, p. 58]; compare [23, Theorem 10, p. 69]).

8. Smoothness at a point and mappings monotone relative to a point

We say that a pointed continuum (X, p) is *smooth* provided that the continuum X is smooth at the point p, i.e., if for each subcontinuum L of X which contains p and for each open set V which contains L there exists an open connected set U such that $L \subset U \subset V$ (see [24, p. 563]). See also Theorem 1 of [24, p. 564] for conditions equivalent to smoothness. In the metric case this concept agrees with one in the sense of the definition in [16, p. 81]).

Take $\mathcal{K}$ as the category of pointed continua, and let $\mathcal{L} = Con$. Define $F : \mathcal{K} \to \mathcal{L}$ as the forgetful functor, and let $G : \mathcal{K} \to \mathcal{L}$ be the functor defined by $G((X, p)) = C^2(X)$. For a given pointed continuum (X, p), consider a function $\delta_{(X,p)} : X \to C^2(X)$ defined by $\delta_{(X,p)}(x) = \{K \in C(X) : p, x \in K\}$. The following two theorems have been proved in the metric case in [5, Propositions 2 and 3, p. 185 and 186, respectively], but remain valid for a more general case for Hausdorff continua. In fact, their proofs for the metric case can be transformed to the general case by considering nets in place of sequences.

Theorem 8.1. *The function $\delta_{(X,p)}$ is continuous if and only if the pointed continuum (X, p) is smooth.*

Theorem 8.2. *The diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \delta_{(X,p)} \big\downarrow & & \big\downarrow \delta_{(Y,q)} \\ C^2(X) & \xrightarrow[C^2(f)]{} & C^2(Y) \end{array}$$

commutes if and only if the mapping f is monotone relative to the point p.

Theorem 8.3. *The function δ commutes with inverse limits.*

Proof: The detailed proof of this theorem for the metric case is written as a main part of the proof of Theorem 1 in [5, p. 187]. Its easy transformation to inverse systems in place of inverse sequences is left to the reader.

Applying results of Chapter 4 one can obtain the following assertions:

- *if a mapping* $f : (X, p) \to (Y, q)$ *between pointed continua is monotone relative to* p *and if* (X, p) *is smooth, then* (Y, q) *is smooth* (see [5, Corollary, p. 187] for the metric case);
- *the class of mappings monotone relative to points between pointed continua has the composition property and the weak composition factor property*;
- *if* $\{(X_\lambda, p_\lambda), f^\mu_\lambda, \Lambda\}$ *is an inverse system of pointed smooth continua* (X_λ, p_λ) *with bonding mappings* f^μ_λ *monotone relative to points* p_μ*, then the inverse limit* $(X, p) = \varprojlim \{(X_\lambda, p_\lambda), f^\mu_\lambda, \Lambda\}$ *is a pointed smooth continuum* (see [5, Theorem 1, p. 187] for the metric case);
- *the class of mappings monotone relative to points between pointed continua has the inverse limit property, and* consequently, by Theorem 4.11, it *has the inverse limit projection property* (see [4, Theorems 4 and 1, p. 7 and 4]).

Although the proof of the theorem below does not exploit categorical methods, we intend to prove it for the sake of completeness.

Theorem 8.4. *The class of mappings monotone relative to points between pointed continua has the composition factor property.*

Proof: Let (X, p), (Y, q) and (Z, r) be pointed continua, and let mappings $f : (X, p) \to (Y, q)$ and $g : (Y, q) \to (Z, r)$ be surjective. Assume that $g \circ f$ is a monotone relative to p. Let R be a subcontinuum of Z such that $r \in R$. Then $g^{-1}(R) = f((g \circ f)^{-1}(R))$ so it is connected.

9. Continua and mappings defined by the function ε

In this section we will use the categories $\mathcal{K} = \mathcal{L} = Con$. We will define, for a given continuum X, another function ε_X, and we will investigate the class $S(\varepsilon)$ of continua determined by continuity of ε_X and the class $M(\varepsilon)$ of mappings determined by commutativity of the corresponding diagram. Namely, for a given continuum X, let $\varepsilon_X : X \to C^2(X)$ be defined by $\varepsilon_X(x) = \mathrm{cl}\{K \in C(X) : x \in \mathrm{int}\, K\}$.

Proposition 9.1. *If a continuum* X *is locally connected, then* $\varepsilon_X = \alpha_X$.

Proof: Take any $x \in X$. Then $\varepsilon_X \subset \alpha_X$ by the definitions. To prove the opposite inclusion take any $A \in \alpha_X$ and a set $\mathcal{U}$ open in $C(X)$ such that $A \in \mathcal{U}$. We have to show that there is a continuum $B \in \mathcal{U}$ with $x \in \operatorname{int} B$. By the definition of the Vietoris topology in $C(X)$ there are open sets $U_1, \dots, U_n$ in X such that $A \in \langle U_1, \dots, U_n \rangle \subset \mathcal{U}$. Since $A \subset U_1 \cup \dots \cup U_n$, we may assume that $x \in U_1$. Let V_1 be a continuum such that $x \in \operatorname{int} V_1 \subset V_1 \subset U_1$. Then $B = A \cup V_1$ satisfies the required conditions.

Corollary 9.2. *Each locally connected continuum is in the class $S(\varepsilon)$.*

Proposition 9.1, Theorem 3.2 and the known coincidence of the classes of confluent, quasi-monotone and OM-mappings on locally connected continua (see [2, p. 214] and [18, (6.2), p. 51]) imply the next corollary.

Corollary 9.3. *Let $f : X \to Y$ be a surjective mapping between locally connected continua. Then the following conditions are equivalent:*

(a) *$f \in M(\varepsilon)$;*
(b) *f is confluent;*
(c) *f is quasi-monotone;*
(d) *f is an OM-mapping.*

To prove the next result we need a lemma.

Lemma 9.4. *If a mapping $f : X \to Y$ between continua X and Y is quasi-monotone, then for each continuum $Q \subset Y$, for each $y \in$ int Q, for each $x \in f^{-1}(y)$ there is a continuum $K \subset X$ such that $x \in intK$ and $f(K) = Q$.*

Proof: Take $Q \in C(Y)$, $y \in \operatorname{int} Q$ and $x \in f^{-1}(y)$. Let $C_1, \dots, C_n$ be components of $f^{-1}(Q)$, and assume $x \in C_1$. We will show that $K = C_1$ satisfies the required conditions. Indeed, $f(K) = Q$ since f is quasi-monotone, and $f^{-1}(\operatorname{int} Q) \cap (X \setminus (C_2 \cup \dots \cup C_n))$ is an open set contained in K and containing x.

Proposition 9.5. *Let a mapping $f : X \to Y$ between continua X and Y be quasi-monotone. Then $\varepsilon_Y(f(x)) \subset C^2(f)(\varepsilon_X(x))$ for each point $x \in X$.*

Proof: Take any $A \in \varepsilon_Y(f(x))$. Then there is a net of continua $\{A_\sigma : \sigma \in \Sigma\}$ in Y such that $f(x) \in A_\sigma$ for each $\sigma \in \Sigma$ and $A = \lim_{\sigma\in\Sigma} A_\sigma$. By Lemma 9.4 for each $\sigma \in \Sigma$ there are continua C_σ in X such that $x \in \mathrm{int} C_\sigma$ with $f(C_\sigma) = A_\sigma$. Then by a characterization of compactness in [8, Theorem 3.1.23, p. 128, and Proposition 1.6.1, p. 50] there is a convergent net $\{C_\lambda : \lambda \in \Lambda\}$ finer than $\{C_\sigma : \sigma \in \Sigma\}$. Put $C = \lim_{\lambda\in\Lambda} C_\lambda$. Therefore $C \in \varepsilon_X(x)$, and thus $A = f(C) \in C^2(f)(\varepsilon_X(x))$ as needed.

Proposition 9.6. *Let a mapping $f : X \to Y$ between continua X and Y satisfy the following condition:*

(9.7) *for each $x \in X$ and for each subcontinuum K of X, if $x \in \mathrm{int} K$, then $f(x) \in \mathrm{int} f(K)$.*

Then $C^2(f)(\varepsilon_X(x)) \subset \varepsilon_Y(f(x))$ for each point $x \in X$.

Proof: Take any $A \in C^2(f)(\varepsilon_X(x))$. Then $A = f(C)$ for some $C \in \varepsilon_X(x)$. Therefore there is a net of continua $\{C_\sigma : \sigma \in \Sigma\}$ in X such that for each $\sigma \in \Sigma$ we have $x \in \mathrm{int} C_\sigma$ and $\lim_{\sigma\in\Sigma} C_\sigma = C$. Then $f(x) \in \mathrm{int} f(C_\sigma)$ and $\lim_{\sigma\in\Sigma} f(C_\sigma) = f(C) = A$, whence $A \in \varepsilon_Y(f(x))$, as required.

The next two statements are consequences of the definition of openness of a mapping.

Statement 9.8. *Each open mapping $f : X \to Y$ between continua X and Y satisfy condition* (9.7).

Statement 9.9. *If a continuum X is locally connected, then a mapping $f : X \to Y$ is open if and only if it satisfies condition* (9.7).

Statement 9.10. *Let $f : X \to Y$ be an open mapping between continua X and Y. Then $C^2(f)(\varepsilon_X(x)) \subset \varepsilon_Y(f(x))$ for each point $x \in X$.*

As a consequence of Proposition 9.5 and Corollary 9.10 we get the following result.

Theorem 9.11. *Each open and quasi-monotone mapping between continua is in the class $M(\varepsilon)$.*

We do not know if openness of the mapping is an essential assumption in the above theorem. In other words, we have the following question.

Question 9.12. Is each quasi-monotone mapping between continua in the class $M(\varepsilon)$?

By Corollary 9.3 and Statement 9.9 we get the next statement.

Statement 9.13. *Each confluent and nonopen mapping between locally connected continua is in the class $M(\varepsilon)$, but it does not satisfy condition* (9.7).

The inverse implication to one mentioned in Question 9.12 does not hold. This can be seen from the next example.

Example 9.14. *There exist continua X and Y and a mapping $f : X \to Y$ which is in the class $M(\varepsilon)$ but which is not quasi-monotone.*

Proof: Consider the disjoint union of two copies $P_1 = P \times \{1\}$ and $P_2 = P \times \{2\}$ of a pseudo-arc P with two points $p, q \in P$ distinguished, identify the point $(p, 1) \in P_1$ with $(p, 2) \in P_2$ to get the wedge $W = P_1 \cup P_2$ with $P_1 \cap P_2 = \{(p, 1)\} = \{(p, 2)\}$, and let L be an arc such that $L \cap W = \{(q, 1)\}$. Thus $X = W \cup L$ is a continuum. Let a mapping $f : X \to Y$ glue together each pair of corresponding points of W and be a homeomorphism on L. Thus we can assume that $f|W : W \to P$ is defined by $f((x, 1)) = f((x, 2)) = x \in P$ and that $f(L)$ is an arc in Y such that $f(L) \cap P = \{q\}$ and q is an end point of $f(L)$.

To see that $f \in M(\varepsilon)$ study the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \varepsilon_X}\downarrow & & \downarrow{\scriptstyle \varepsilon_Y} \\ C^2(X) & \xrightarrow[C^2(f)]{} & C^2(Y) \end{array} \tag{9.15}$$

and consider three cases:

(a) if $x \in L \setminus P_1$, then $\varepsilon_X(x) = \alpha_X(x)$ is the family of all subcontinua of X containing x;

(b) if $x \in P_1 \setminus P_2$, then $A \in \varepsilon_X(x)$ if and only if $P_1 \subset A$;

(c) if $x \in P_2 \setminus P_1$, then $A \in \varepsilon_X(x)$ if and only if $P_2 \subset A$.

In any case the above conditions lead to the equality $C^2(f)(\varepsilon_X(x)) = \varepsilon_Y(f(x))$ for each point $x \in X$, i.e., diagram (9.15) commutes. The details are left to the reader.

Finally, to show that f is not quasi-monotone, observe that $f(L)$ is a subcontinuum of Y with the nonempty interior, and that $f^{-1}(f(L))$ has two components: L and $\{(q,1)\}$. The latter one, being degenerate, is not mapped onto $f(L)$.

Contrary to the previously discussed functions α, β, γ and δ, the function ε does not commute with the inverse limit operation. The next example shows this.

Example 9.16. *There is an inverse sequence* $\mathbf{S} = \{X_n, f_m^n, \mathbb{N}\}$ *of trees* X_n *with open bonding mappings* $f_m^n : X_n \to X_m$ *for* $m, n \in \mathbb{N}$ *and* $m \le n$*, such that if* $X = \varprojlim \mathbf{S}$*, then* $\varprojlim \{\varepsilon_{X_n}\} \neq \varepsilon_X$*.*

Proof: In the Euclidean plane $\mathbb{R}^2$, for each $n \in \mathbb{N}$ put $e_0 = (0,0)$ and $e_n = (\frac{1}{n}, 0)$. Define X_n as the cone with the vertex $v = (0,1)$ over the set $\{e_0, e_1, \ldots, e_n\}$. Thus each X_n is a tree, and $X_n \subset X_{n+1}$. Let $f_n^{n+1} : X_{n+1} \to X_n$ be an open retraction that projects the segment ve_n onto the segment ve_0. Thus each f_m^n is open, and $X = \varprojlim \mathbf{S}$ is the harmonic fan, i.e., the cone with vertex v over the set $\{e_0, e_1, \ldots, e_n, \ldots\}$.

Put $x_n = e_0 = (0,0)$ for each $n \in \mathbb{N}$. Then $\{x_n\} \in \varepsilon_{X_n}(x_n)$ for each n, while for $x = \varprojlim x_n = (0,0) \in X$ we have $\{x\} \notin \varepsilon_X(x)$.

Remark 9.17. Note that, in the above example, the continua X_n, as locally connected ones, are in $S(\varepsilon)$ for each n, the bonding mappings f_m^n as open ones are confluent, so they are in $M(\varepsilon)$ according to Corollary 9.3, while $X = \varprojlim \mathbf{S}$ is not in $S(\varepsilon)$.

Let us recall that J. T. Goodykoontz, Jr., studied in [11] a function $g : 2^X \to C(X)$ defined by $g(A) = \bigcap\{K \in C(X) : A \subset \operatorname{int} K\}$ for a hereditarily unicoherent continuum X (i.e., such a continuum X that the intersection of any two of its subcontinua is connected). The function g is also known under the name of the Jones function K, see e.g. [26, p. 373], which is defined by the same formula for arbitrary continua (not necessarily being hereditarily unicoherent). A similar function $G : 2^X \to C^2(X)$ defined by $G(A) = \operatorname{cl}\{K \in C(X) : A \subset \operatorname{int} K\}$ was introduced and investigated by the second named author in [6]. Observe that if $F_1(X) \subset 2^X$ stands for the hyperspace of singletons of X (which is known to be homeomorphic to X), then $G|F_1(X) : X \to C^2(X)$ coincides with ε_X.

A space X is said to be *homogeneous* provided that for every two points $p, q \in X$ there is a homeomorphism $h : X \to X$ such that $h(p) = q$. Since homogeneity of a metric continuum X implies continuity of the function G, see [6, Theorem 6.4, p. 220], we get the following result.

Theorem 9.18. *Each homogeneous metric continuum is in the class $S(\varepsilon)$.*

Metrizability is an essential assumption in Theorem 9.18 be the following example.

Example 9.19. *There is a nonmetric homogeneous continuum X such that ε_X is not continuous.*

Proof: The (nonmetric) continuum X described in [7, Section 3, p. 212] is homogeneous by [7, Corollary 4.7, p. 215]. To show that ε_X is not continuous we will use notation as in [7]. For any $\langle x_0, z_0\rangle \in X$ we have $\varepsilon_X(\langle x_0, z_0\rangle) = \{K \in C(X) : \{x_0\} \times S \subset K\}$ by the definition of the topology on X. Let a sequence $\langle x_n, z_n\rangle$ for $n \in \mathbb{N}$ converge to $\langle x_0, z_0\rangle$ in X, with $x_n \neq x_0$. Then, by [7, Observation 3.2, p. 213], we infer that the sequence x_n tends to x_0, and the sequence $\exp(2\pi i/(x_n - x_0))$ tends to z_0 in S. Thus $\varepsilon_X(\langle x_n, z_n\rangle) = \{K \in C(X) : \{x_n\} \times S \subset K\}$ converges in $C^2(X)$ (again by [7, Observation 3.2, p. 213]) to $\{K \in C(X) : \langle x_0, z_0\rangle \in K\}$, which is strictly larger than $\varepsilon_X(\langle x_0, z_0\rangle)$.

Question 9.20 Can Theorem 9.18 be generalized to open homogeneity? Recall that a continuum is said to be *open homogeneous* if the homeomorphism in the definition of a homogeneous space is replaced by an open mapping.

Remark 9.21. For metric continua the class $S(\varepsilon)$ and the class of continua having the property of Kelley are not related to each other by inclusions. Namely in Example 6.3 of [6, p. 220] a continuum $X \in S(\varepsilon)$ is constructed which does not have the property of Kelley. On the other hand, the continuum X constructed in Example 9.16 (i.e., the harmonic fan) has the property of Kelley while is not in $S(\varepsilon)$ according to Remark 9.17.

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