A CONTINUOUS DECOMPOSITION OF THE
PLANE INTO ACYCLIC CONTINUA EACH OF
WHICH CONTAINS AN ARC

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ABSTRACT. Since 1950 several decompositions of the plane into pseudo-arcs have been described. We construct a continuous decomposition of the plane into nondegenerate acyclic continua none of which is a pseudo-arc. In fact each decomposition element contains a straight line segment.

1. Introduction

In the early 1950’s R. D. Anderson [A1952] constructed a continuous decomposition of the plane into nondegenerate acyclic continua and announced [A1950] that there exists a continuous decomposition of the plane into pseudo-arcs. A pseudo-arc can be characterized as a chainable, hereditarily indecomposable continuum. In his dissertation [B1958], M. Brown describes a construction which when applied to the plane results in a decomposition consisting of a single point and concentric hereditarily indecomposable continua each of which separates the plane. W. Lewis and J. J. Walsh [L1978] used a very geometrical construction to continuously decompose the plane into pseudo-arcs. Recently in [P1998] and in [K2000] Whitney maps have been used to refine continuous decompositions into continuous decompositions made up of pseudo-arcs. For example, in [K2000] M. Brown’s decomposition of the plane is modified and
then refined using Whitney maps to obtain a new continuous decomposition of the plane into pseudo-arcs.

In 1976, Michel Smith [S1976] using results from E. Dyer [D1953] and W. S. Mahavier [M1967] proved that any continuous decomposition of the plane into acyclic continua must have elements that are indecomposable. In [M1989] L. Mohler and L. G. Oversteegen constructed an example to answer in the negative an old question of B. Knaster; namely, whether every continuously irreducible continuum must contain an hereditarily indecomposable tranche. Similar questions about continuous decompositions of the plane naturally arise. This paper addresses the general question of how “nice” the decomposition elements must be; for example, must there always exist a decomposition element that is hereditarily indecomposable? Here we answer this question in the negative by using the techniques of Lewis and Walsh [L1978] to construct a continuous decomposition of plane into chainable continua each of which contains an arc. In fact the geometric nature of these techniques actually allows us to say that each of the decomposition elements contains a straight line segment of a given length. Specifically we will prove the following theorem.

**Theorem 1.1.** Given an \( \varepsilon > 0 \) there exists a continuous decomposition of the plane into chainable continua so that each element of the decomposition contains a straight line segment with length larger than \( \varepsilon \).

Many other questions about continuous decompositions of the plane naturally occur.

**Question 1.2.** Given a continuous decomposition \( G \) of the plane into nondegenerate acyclic continua, must it be the case that given any \( \varepsilon > 0 \) there exists a nondegenerate indecomposable continuum \( C \) and a \( g \in G \) such that \( \text{diam}(C) < \varepsilon \) and \( C \subset g \)?

**Question 1.3.** What can be said about the elements of a continuous decomposition of the plane into acyclic continua where all the the elements are homeomorphic?

**Question 1.4 (Lewis).** Is there a continuous decomposition of the plane into acyclic continua none of which are chainable [L1996]?
Our continuous decomposition will be to define a sequence \( \{P_n\}_{n=1}^{\infty} \) of partitions of the plane into cells with non-overlapping interiors so that the conditions of the following lemma from [L1978] are satisfied. Also see [S1995]. Before stating our version of the lemma we introduce the following notation. If \( \mathcal{P} \) is a collection of sets, then \( \mathcal{P}^* \) denotes the union of members of \( \mathcal{P} \). If \( p \) is a set, then \( \text{st}^1(p, P) = \{ p' \in P : p' \cap p \neq \emptyset \} \) and inductively \( \text{st}^{i+1}(p, P) = \text{st}^i(\text{st}(p, P)^*, P) \). We abbreviate \( \text{st}^i(p, P) \) by \( \text{st}(p, P) \). By \( N_\varepsilon(x) \) we mean the set of points in the plane which are less than an \( \varepsilon \) distance from some point in \( x \).

**Lemma 2.1** (Lewis and Walsh). Let \( X \) be locally compact and \( \{P_n\}_{n=1}^{\infty} \) be a sequence so that for each \( n \in \mathbb{Z}^+ \):

1. The collection \( P_n \) is a locally finite family of non-empty closed subsets of \( X \) with \( P_n^* = X \), with the elements of \( P_n \) having pairwise disjoint interiors, and with \( \text{Cl}((\text{Int}(p_n))) = p_n \) for each \( p_n \in P_n \).
2. For each \( p_n \in P_n \), \( \text{st}^3(p_n, P_{n+1})^* \subset \text{st}(p_n, P_n)^* \).
3. There is a positive number \( L \) such that for each pair \( p_n, p'_n \in P_n \) with \( p_n \cap p'_n \neq \emptyset \), we have that \( p_n \subset N_{L/2n}(p'_n) \).
4. There is a positive number \( K \) such that for each \( p_{n+1} \in P_{n+1} \), there is a \( p_n \in P_n \) with \( p_{n+1} \cap p_n \neq \emptyset \) and \( p_n \subset N_{K/2n+1}(p_{n+1}) \).

Let \( G \) be defined by \( g \in G \) if \( g = \bigcap_{n=1}^{\infty} \text{st}(p_n, P_n)^* \) where \( \bigcap_{n=1}^{\infty} p_n \neq \emptyset \); then \( G \) is a continuous decomposition of \( X \).

In the following construction we follow [L1978]. See also [S1994]. The sequence, \( \{P_n\}_{n=1}^{\infty} \), is defined inductively. Assuming we have already constructed \( \{P_i\}_{i=1}^{n} \), we start stage \( n + 1 \) of the induction given \( \hat{R}_{n+1} \), a division of the plane into either congruent vertical or congruent horizontal strips. Instead of defining the cells \( P_{n+1} \) directly, we first define cells \( Q_{n+1} \) which are much simpler to describe. The cells of \( P_{n+1} \) are the images of the cells of \( Q_{n+1} \) under a homeomorphism \( H_{n+1} \) that is also described inductively. There are three positive rational numbers \( a_{n+1}, b_{n+1}, \) and \( c_{n+1} \) that constrain the construction at stage \( n + 1 \). Given a vertical (resp. horizontal) division, \( \hat{R}_{n+1} \), we refine the division by dividing each strip into finer strips, each with the width \( a_{n+1} \) and infinite length. We denote the
new division by $R_{n+1}$. The construction alternates between working with horizontal strips and vertical strips starting with vertical strips when $n = 1$. Thus when $(n + 1)$ is odd the strips of $R_{n+1}$ run vertically with width $a_{n+1}$.

We partition the plane into a collection of cells $Q_{n+1}$ with non-overlapping interiors by partitioning each strip into cells. The cells $q_{n+1} \in Q_{n+1}$ that partition a vertical strip are exactly as in Figure 1 and are called vertical cells. All the cells $q_{n+1} \in Q_{n+1}$ are identical. The vertical cell $q_{n+1}$ has the width of $a_{n+1}$; has the height of exactly $b_{n+1} + c_{n+1}$; and has the thickness; i.e., vertical transverse thickness, of $b_{n+1}$. Thus the top boundary of a vertical cell is simply the vertical displacement of the bottom boundary by the constant $b_{n+1}$. Note that in our construction $c_{n+1}$ is always much larger than $b_{n+1}$. Each vertical cell is symmetrical about a vertical line; the two identical halves being referred to as chevrons. Each chevron is also vertically symmetrical and consists of two half-chevrons. Thus each vertical cell consists of four congruent half-chevrons. Each half-chevron is a parallelogram. We refer to the short sides of this parallelogram as the side boundaries of the half-chevron. The cells that partition horizontal strips are like those described above except that they are rotated a quarter turn clockwise. They are called horizontal cells.

Once we have the collection $Q_{n+1}$ defined, a homeomorphism $h_{n+1} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined so that $h_{n+1}^{-1}$ “straightens” the boundaries between the cells. When $(n + 1)$ is odd we define $h_{n+1}$ so it maps vertical lines onto themselves by translation so that the preimage of any given vertical cell $q_{n+1} \in Q_{n+1}$ is a rectangle with width $a_{n+1}$ and height $b_{n+1}$. Note that $h_{n+1}$ restricted to the side boundaries of cells is the identity. The set $P_{n+1}$ is defined to be \( \{ H_{n+1}(q_{n+1}) : q_{n+1} \in Q_{n+1} \} \), where $H_{n+1} = H_n \circ h_n = h_1 \circ \cdots \circ h_n$. When $(n + 1)$ is even $h_{n+1}$ maps each horizontal line onto itself by translation and for each $q_{n+1} \in Q_{n+1}$, we have that $h_{n+1}^{-1}(q_{n+1})$ is a rectangle of width $b_{n+1}$ and height $a_{n+1}$. To finish stage $n + 1$ we use $\{ h_{n+1}^{-1}(q_{n+1}) : q_{n+1} \in Q_{n+1} \}$ to define $\hat{R}_{n+2}$, a horizontal (resp. vertical) division of $\mathbb{R}^2$ with strips defined by horizontal (resp. vertical) lines placed at a distance of $b_{n+1}$ apart.

To completely define our construction then we need only to define how to choose $a_1$, $b_1$, and $c_1$, and $h_1$ and how to choose $a_{n+1}$, $b_{n+1}$,
Figure 1. A Cell in $Q_n$.

and $c_{n+1}$, and $h_{n+1}$ when we have completed stage $n$. We can begin, for example, by setting $\delta_1 = 1/4$, $c_1 = 4$, $a_1 = \delta_1/2$, $b_1 = a_1/2$, and $H_1 = Id$, where $Id$ is the identity on $\mathbb{R}^2$. Define $h_1$ as described
above so that $h^{-1}_1$ "straightens" the boundaries of the cells $q_1 \in Q_1$ defined by $a_1$, $b_1$, and $c_1$.

At stage $n + 1$ we proceed as follows:

1. Pick $\delta_{n+1} > 0$ so that $|x - x'| < \delta_{n+1} \implies |H_{n+1}(x) - H_{n+1}(x')| < 1/2^{n+1}$
   for all $x, x' \in \mathbb{R}^2$ where $H_{n+1}(x) = H_n \circ h_n$.
2. Set $c_{n+1} = (3/4)a_n$.
3. Pick $a_{n+1} < \delta_{n+1}/2$ and so that $a_{n+1}$ divides evenly into $(b_n/4)$.
4. Set $b_{n+1} = a_{n+1}/2$.
5. Define $h_{n+1}$ as described above so that $h^{-1}_{n+1}$ "straightens" the boundaries of the cells $q_{n+1} \in Q_{n+1}$ defined by $a_{n+1}$, $b_{n+1}$, and $c_{n+1}$.

The fact that Condition 1 of Lemma 2.1 holds follows immediately. To prove that Condition 2 holds it suffices to show that if $p_n \in P_n$, then

$$st^3(H^{-1}_{n+1}(p_n), H^{-1}_{n+1}(P_{n+1})) \subset st(H^{-1}_{n+1}(p_n), H^{-1}_{n+1}(P_n))^*.$$ 

This follows from the fact that $3a_{n+1} < b_n$ and $c_{n+1} + 3b_{n+1} < a_n$. We turn our attention to Condition 3. If $q_{n+1} \subseteq Q_{n+1}$ and $q_{n+1} \cap q'_{n+1} \neq \emptyset$, then $q_{n+1} \subseteq N_\varepsilon(q'_{n+1})$ where $\varepsilon = a_{n+1} + b_{n+1}$. Since $\varepsilon < \delta_{n+1}$ Condition 3 holds. To show that Condition 4 holds consider $q_n \in Q_{n+1}$. Since $c_{n+1}/2 > \frac{1}{4}a_n$ we know that there is a $q_n \in Q_n$ so that $q_n \cap h_n(q_{n+1}) \neq \emptyset$ and so that $q_n \subseteq N_\varepsilon(h_n(q_{n+1}))$ where $\varepsilon = a_n + b_n$ but $\varepsilon < \delta_n$ and so $H_n(q_n) \subseteq N_K/2^{n+1}(H_{n+1}(q_{n+1}))$ where $K = 2$. Thus Condition 4 holds.

Therefore by Lemma 2.1 our decomposition is continuous. We will denote this decomposition by $G$. As in [L1978] each element of $G$ is chainable.

3. Proof of Theorem 1.1

We extend the notion of half-chevron of a cell to that of the half-chevron of $st(q_n, Q_n)^*$. A half-chevron of $st(q_n, Q_n)^*$ is defined to be a parallelogram that consists of three cell half-chevrons taken from the cells in $st(q_n, Q_n)$. Thus $st(q_n, Q_n)^*$ contains 12 half-chevrons. The side boundaries of a half-chevron of $st(q_n, Q_n)^*$ are the short sides of the parallelogram, which are vertical when $n$ is odd and horizontal when $n$ is even.
We say that \( st(q_{n+1}, Q_n)^* \) crosses a half-chevron of \( st(q_n, Q_n)^* \), say \( q_n^p \), when \( st(h_n(q_{n+1}), h_n(Q_{n+1}))^* \) intersects \( q_n^p \) and extends beyond both the side boundaries of \( q_n^p \) by at least \( 3b_{n+1} \). This definition ensures that if \( st(q_{n+1}, Q_{n+1})^* \) crosses a half-chevron \( q_n^p \) of \( st(q_n, Q_n)^* \) then for any half-chevron \( q_{n+1}^p \) of \( st(q_{n+1}, Q_{n+1})^* \) we have that \( h_n(q_{n+1}) \cap q_n^p \) is a parallelogram that intersects both side boundaries of \( q_n^p \).

Next observe that if \( R \subset q_{n+1}^p \) is a parallelogram that intersects both side boundaries of \( q_{n+1}^p \), then \( h_n(R) \cap q_n^p = R \) is a parallelogram that intersects both side boundaries of \( q_n^p \). This follows from the definition of \( h_n \) and the fact that \( h_n(R) \) must extend at least \( 3b_{n+1} \) beyond each side boundary of \( q_n^p \).

For each given \( n \in \mathbb{Z}^+ \) we will denote the parallelogram \( q_n^p \) by \( R_n \). For \( k \in \{1, 2, ..., n \} \) we define \( R_{n+1}^k = h_k(R_{n+1}^k) \cap q_n^p \). Thus \( R_{n+1}^1 \) is a parallelogram that intersects both side boundaries of \( R_1 = q_1^p \). Since \( R_{n+1}^1 \subset R_n \) we have that in general

\[
R_{n+1}^k = h_k(R_{n+1}^{k+1}) \cap q_n^p \subset h_k(R_n^{k+1}) \cap q_n^p = R_n^k
\]

for all \( k \in \{1, 2, ..., n \} \) and in particular that

\[
R_1^1 \supset R_2^1 \supset \cdots \supset R_n^1 \supset \cdots .
\]

Finally observe that because the lengths of the side boundaries of \( q_n^p \) are \( 3b_n \), the lengths of the short sides of the parallelograms \( R_{n+1}^1 \) approach zero as \( n \) increases. Thus we have that \( A = \cap_{n=1}^{\infty} R_n^1 \) is a line segment that is contained in \( g \) and since \( A \) intersects both side boundaries of \( q_n^p \) it has length greater than \( c_1 - 3b_1 \), which can be made arbitrarily large.

The proof of Theorem 1.1 is complete.

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References


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