CHARACTERIZATIONS OF PRE-$R_0$ AND PRE-$R_1$
TOPOLOGICAL SPACES

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Abstract

In this paper we introduce two new classes of topological spaces called pre-$R_0$ and pre-$R_1$ spaces in terms of the concept of preopen sets and investigate some of their fundamental properties.

1. Introduction

The notion of $R_0$ topological space is introduced by N. A. Shanin [20] in 1943. Later, A. S. Davis [3] rediscovered it and studied some properties of this weak separation axiom. Several topologists (e. g. [6], [9], [10], [19]) further investigated properties of $R_0$ topological spaces and many interesting results have been obtained in various contexts. In the same paper, A. S. Davis also introduced the notion of $R_1$ topological space which is independent of both $T_0$ and $T_1$ but strictly weaker than $T_2$. M. G. Murdeshwar and S. A. Naimpally [18] studied some of the fundamental properties of the class of $R_1$ topological spaces. In 1963, N. Levine [14] offered a new notion to the field of general topology by introducing semi-open sets. He defined this notion by utilizing the known notion of closure of an open set, i.e., a subset of a topological space is semi-open if it is contained in the

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closure of its interior. Since the advent of this notion, several new notions are defined in terms of semi-open sets of which two are semi-$R_0$ and semi-$R_1$ introduced by S. N. Maheshwari and R. Prasad [15] and C. Dorsett [5], respectively. These two notions are defined as natural generalizations of the separation axioms $R_0$ and $R_1$ by replacing the closure operator with the semiclosure operator and openness with semi-openness. In 1982, A. S. Mashhour et al. [17] introduced the notion of preopen set which is also known under the name of locally dense set [2] in the literature. Since then, this notion received wide usage in general topology. In this paper, we continue the study of the above mentioned classes of topological spaces satisfying these axioms by introducing two more notions in terms of preopen sets called pre-$R_0$ and pre-$R_1$. It turns out that pre-$R_0$ and pre-$R_1$ are equivalent with pre-$T_1$ and pre-$T_2$, respectively.

Throughout the paper $(X, \tau)$ (or simply $X$) will always denote a topological space. For a subset $A$ of $X$, the closure, interior and complement of $A$ in $X$ are denoted by $\text{Cl}(A)$, $\text{Int}(A)$ and $X - A$ respectively. By $\text{PO}(X, \tau)$ and $\text{PC}(X, \tau)$ we denote the collection of all preopen sets and the collection of all preclosed sets of $(X, \tau)$, respectively.

2. Preliminaries

Since we shall require the following known definitions, notations and some properties, we recall them in this section.

**Definition 1.** Let $A$ be a subset of a topological space $(X, \tau)$. Then:

1. $A$ is preopen [17], if $A \subset \text{Int}(\text{Cl}(A))$.
2. $A$ is preclosed [17], if $X - A$ is preopen or equivalently if $\text{Cl}(\text{Int}(A)) \subset A$.
3. The intersection of all preclosed sets containing $A$ is called the preclosure of $A$ [8] and is denoted by $\text{pCl}(A)$. 


(4) \((X, \tau)\) is pre-\(T_1\) [13], if to each pair of distinct points \(x\) and \(y\) of \(X\), there exists a pair of preopen sets one containing \(x\) but not \(y\) and the other containing \(y\) but not \(x\).

(5) \((X, \tau)\) is pre-\(T_2\) [13], if to each pair of distinct points \(x\) and \(y\) of \(X\), there exists a pair of disjoint preopen sets, one containing \(x\) and the other containing \(y\).

**Lemma 2.1.** (El-Deeb et al. [8]) Let \((X, \tau)\) be a topological space and \(A, B\) subsets of \(X\). Then the following hold:

1. \(x \in pCl(A)\) if and only if \(A \cap V \neq \emptyset\) for every \(V \in \text{PO}(X, \tau), x \in V\).
2. \(A\) is preclosed in \((X, \tau)\) if and only if \(A = pCl(A)\).
3. \(pCl(A) \subset pCl(B)\) if \(A \subset B\).
4. \(pCl(pCl(A)) = pCl(A)\).

Recall that a subset \(B_x\) of a topological space \(X\) is said to be a pre-neighbourhood of a point \(x \in X\) [13] if there exists a preopen set \(U\) such that \(x \in U \subset B_x\).

**Lemma 2.2.** A subset of a topological space \(X\) is preopen in \(X\) if and only if it is a pre-neighbourhood of each of its points.

### 3. Pre-\(R_0\) Spaces

**Definition 2.** Let \((X, \tau)\) be a topological space and \(A \subset X\). Then the pre-kernel of \(A\) [11], denoted by \(pKer(A)\), is defined to be the set \(pKer(A) = \cap\{G \in \text{PO}(X, \tau) \mid A \subset G\}\).

**Lemma 3.1.** Let \((X, \tau)\) be a topological space and \(x \in X\). Then, \(y \in pKer(\{x\})\) if and only if \(x \in pCl(\{y\})\).

**Proof.** Suppose that \(y \notin pKer(\{x\})\). Then there exists a preopen set \(V\) containing \(x\) such that \(y \notin V\). Therefore, we have \(x \notin pCl(\{y\})\). The converse is similarly shown. \(\square\)
Lemma 3.2. Let \((X, \tau)\) be a topological space and \(A\) a subset of \(X\). Then, \(p\ker(A) = \{x \in X \mid p\cl\{\{x\}\} \cap A \neq \emptyset\}\).

Proof. Let \(x \in p\ker(A)\) and suppose \(p\cl\{\{x\}\} \cap A = \emptyset\). Hence, \(x \notin X - p\cl\{\{x\}\}\) which is a preopen set containing \(A\). This is impossible, since \(x \in p\ker(A)\). Consequently, \(p\cl\{\{x\}\} \cap A \neq \emptyset\).

Next, let \(x \in X\) such that \(p\cl\{\{x\}\} \cap A \neq \emptyset\) and suppose that \(x \notin p\ker(A)\). Then, there exists a preopen set \(U\) containing \(A\) and \(x \notin U\). Let \(y \in p\cl\{\{x\}\} \cap A\). Hence, \(U\) is a pre-neighbourhood of \(y\) which does not contain \(x\). By this contradiction \(x \in p\ker(A)\) and the claim. 

Definition 3. A topological space \((X, \tau)\) is said to be a pre-\(R_0\) space if every preopen set contains the preclosure of each of its singletons.

Lemma 3.3. A topological space \((X, \tau)\) is pre-\(R_0\) if and only if for each \(U \in PO(X, \tau)\), \(x \notin U\) implies \(Cl(\text{Int}\{\{x\}\}) \subset U\).

Proposition 3.4. For a topological space \((X, \tau)\), the following properties are equivalent:

1. \((X, \tau)\) is pre-\(R_0\) space;
2. For any \(F \in PC(X, \tau)\), \(x \notin F\) implies \(F \subset U\) and \(x \notin U\) for some \(U \in PO(X, \tau)\);
3. For any \(F \in PC(X, \tau)\), \(x \notin F\) implies \(F \cap pCl(\{x\}) = \emptyset\);
4. For any distinct points \(x\) and \(y\) of \(X\), either \(pCl(\{x\}) = pCl(\{y\})\) or \(pCl(\{x\}) \cap pCl(\{y\}) = \emptyset\).

Proof. (1) \(\rightarrow\) (2): Let \(F \in PC(X, \tau)\) and \(x \notin F\). Then by (1) \(pCl(\{x\}) \subset X - F\). Set \(U = X - pCl(\{x\})\), then \(U \in PO(X, \tau), F \subset U\) and \(x \notin U\).

(2) \(\rightarrow\) (3): Let \(F \in PC(X, \tau)\) and \(x \notin F\). There exists \(U \in PO(X, \tau)\) such that \(F \subset U\) and \(x \notin U\). Since \(U \in PO(X, \tau), U \cap pCl(\{x\}) = \emptyset\) and \(F \cap pCl(\{x\}) = \emptyset\).

(3) \(\rightarrow\) (4): Suppose that \(pCl(\{x\}) \neq pCl(\{y\})\) for distinct
implies that \( p_{Cl} \) and \( z/ \) \( (4) \). Hence, we have \( x/ \) \( x \). Therefore, we obtain \( p_{Cl}(\{x\}) \cap p_{Cl}(\{y\}) = \emptyset \). The proof for otherwise is similar.

\( (4) \rightarrow (1) \) : Let \( V \in PO(X, \tau) \) and \( x \in V \). For each \( y \notin V, x \neq y \) and \( x \notin p_{Cl}(\{y\}) \). This shows that \( p_{Cl}(\{x\}) \neq p_{Cl}(\{y\}) \). By \( (4) \), \( p_{Cl}(\{x\}) \cap p_{Cl}(\{y\}) = \emptyset \) for each \( y \in X - V \) and hence \( p_{Cl}(\{x\}) \cap ( \bigcup_{y \in X - V} p_{Cl}(\{y\}) ) = \emptyset \). On the other hand, since \( V \in PO(X, \tau) \) and \( y \in X - V \), we have \( p_{Cl}(\{y\}) \subset X - V \) and hence \( X - V = \bigcup_{y \in X - V} p_{Cl}(\{y\}) \). Therefore, we obtain \( (X - V) \cap p_{Cl}(\{x\}) = \emptyset \) and \( p_{Cl}(\{x\}) \subset V \). This shows that \( (X, \tau) \) is a pre-\( R_0 \) space.

**Corollary 3.5.** A topological space \( (X, \tau) \) is a pre-\( R_0 \) space if and only if for any \( x \) and \( y \) in \( X \), \( p_{Cl}(\{x\}) \neq p_{Cl}(\{y\}) \) implies \( p_{Cl}(\{x\}) \cap p_{Cl}(\{y\}) = \emptyset \).

**Proof.** This is an immediate consequence of Proposition 3.4.

**Lemma 3.6.** The following statements are equivalent for any points \( x \) and \( y \) in a topological space \( (X, \tau) \):

1. \( p_{Ker}(\{x\}) \neq p_{Ker}(\{y\}) \);
2. \( p_{Cl}(\{x\}) \neq p_{Cl}(\{y\}) \).

**Proof.** \( (1) \rightarrow (2) \) : Suppose that \( p_{Ker}(\{x\}) \neq p_{Ker}(\{y\}) \), then there exists a point \( z \) in \( X \) such that \( z \in p_{Ker}(\{x\}) \) and \( z \notin p_{Ker}(\{y\}) \). From \( z \in p_{Ker}(\{x\}) \) it follows that \( \{x\} \cap p_{Cl}(\{z\}) = \emptyset \) which implies \( x \in p_{Cl}(\{z\}) \). By \( z \notin p_{Ker}(\{y\}) \), we have \( \{y\} \cap p_{Cl}(\{z\}) = \emptyset \). Since \( x \in p_{Cl}(\{z\}) \), \( p_{Cl}(\{x\}) \subset p_{Cl}(\{z\}) \) and \( \{y\} \cap p_{Cl}(\{x\}) = \emptyset \). Therefore it follows that \( p_{Cl}(\{x\}) \neq p_{Cl}(\{y\}) \). Now \( p_{Ker}(\{x\}) \neq p_{Ker}(\{y\}) \) implies that \( p_{Cl}(\{x\}) \neq p_{Cl}(\{y\}) \).
(2) $\rightarrow$ (1): Suppose that $p\text{Cl}(\{x\}) \neq p\text{Cl}(\{y\})$. Then there exists a point $z$ in $X$ such that $z \in p\text{Cl}(\{x\})$ and $z \notin p\text{Cl}(\{y\})$. It follows that there exists a preopen set containing $z$ and therefore $x$ but not $y$, namely, $y \notin p\text{Ker}(\{x\})$ and thus $p\text{Ker}(\{x\}) \neq p\text{Ker}(\{y\})$.

**Theorem 3.7.** A topological space $(X, \tau)$ is a pre-$R_0$ space if and only if for any points $x$ and $y$ in $X$, $p\text{Ker}(\{x\}) \neq p\text{Ker}(\{y\})$ implies $p\text{Ker}(\{x\}) \cap p\text{Ker}(\{y\}) = \emptyset$.

**Proof.** Suppose that $(X, \tau)$ is a pre-$R_0$ space. Thus by Lemma 3.6, for any points $x$ and $y$ in $X$ if $p\text{Ker}(\{x\}) \neq p\text{Ker}(\{y\})$ then $p\text{Cl}(\{x\}) \neq p\text{Cl}(\{y\})$. Now we prove that $p\text{Ker}(\{x\}) \cap p\text{Ker}(\{y\}) = \emptyset$. Assume that $z \in p\text{Ker}(\{x\}) \cap p\text{Ker}(\{y\})$. By $z \in p\text{Ker}(\{x\})$, it follows that $x \in p\text{Cl}(\{z\})$. Since $x \in p\text{Cl}(\{x\})$, by Corollary 3.5 $p\text{Cl}(\{x\}) = p\text{Cl}(\{z\})$. Similarly, we have $p\text{Cl}(\{y\}) = p\text{Cl}(\{z\}) = p\text{Cl}(\{x\})$. This is a contradiction. Therefore, we have $p\text{Ker}(\{x\}) \cap p\text{Ker}(\{y\}) = \emptyset$.

Conversely, let $(X, \tau)$ be a topological space such that for any points $x$ and $y$ in $X$, $p\text{Ker}(\{x\}) \neq p\text{Ker}(\{y\})$ implies $p\text{Ker}(\{x\}) \cap p\text{Ker}(\{y\}) = \emptyset$. If $p\text{Cl}(\{x\}) \neq p\text{Cl}(\{y\})$, then by Lemma 3.1, $p\text{Ker}(\{x\}) \neq p\text{Ker}(\{y\})$. Therefore $p\text{Ker}(\{x\}) \cap p\text{Ker}(\{y\}) = \emptyset$ which implies $p\text{Cl}(\{x\}) \cap p\text{Cl}(\{y\}) = \emptyset$. Because $z \in p\text{Cl}(\{x\})$ implies that $x \in p\text{Ker}(\{z\})$ and therefore $p\text{Ker}(\{x\}) \cap p\text{Ker}(\{z\}) \neq \emptyset$. By hypothesis, we therefore have $p\text{Ker}(\{x\}) = p\text{Ker}(\{z\})$. Then $z \in p\text{Cl}(\{x\}) \cap p\text{Cl}(\{y\})$ implies that $p\text{Ker}(\{x\}) = p\text{Ker}(\{z\}) = p\text{Ker}(\{y\})$. This is a contradiction. Therefore, $p\text{Cl}(\{x\}) \cap p\text{Cl}(\{y\}) = \emptyset$ and by Corollary 3.5 $(X, \tau)$ is a pre-$R_0$ space.

**Theorem 3.8.** For a topological space $(X, \tau)$, the following properties are equivalent:

1. $(X, \tau)$ is a pre-$R_0$ space;
2. For any nonempty set $A$ and $G \in PO(X, \tau)$ such that $A \cap G \neq \emptyset$, there exists $F \in PC(X, \tau)$ such that $A \cap F \neq \emptyset$ and $F \subset G$;
(3) Any \( G \in PO(X, \tau) \) is \( . G = \cup\{F \in PC(X, \tau) \mid F \subset G\} \);
(4) Any \( F \in PC(X, \tau) \), \( F = \cap\{G \in PO(X, \tau) \mid F \subset G\} \);
(5) For any \( x \in X \), \( pCl\{x\} \subset pKer\{x\} \).

Proof. (1) \( \rightarrow\) (2) : Let \( A \) be a nonempty set of \( X \) and \( G \in PO(X, \tau) \) such that \( A \cap G \neq \emptyset \). There exists \( x \in A \cap G \). Since \( x \in G \in PO(X, \tau) , pCl\{x\} \subset G \). Set \( F = pCl\{x\} \), then \( F \in PC(X, \tau) , F \subset G \) and \( A \cap F \neq \emptyset \).
(2) \( \rightarrow\) (3) : Let \( G \in PO(X, \tau) \), then \( G \supset \cup\{F \in PC(X, \tau) \mid F \subset G\} \). Let \( x \) be any point of \( G \). There exists \( F \in PC(X, \tau) \) such that \( x \in F \) and \( F \subset G \). Therefore, we have \( x \in F \subset \cup\{F \in PC(X, \tau) \mid F \subset G\} \) and hence \( G = \cup\{F \in PC(X, \tau) \mid F \subset G\} \).
(3) \( \rightarrow\) (4) : This is obvious.
(4) \( \rightarrow\) (5) : Let \( x \) be any point of \( X \) and \( y \notin pKer\{x\} \).
There exists \( V \in PO(X, \tau) \) such that \( x \in V \) and \( y \notin V \); hence \( pCl\{y\} \cap V = \emptyset \). By (4) \( (\cap\{G \in PO(X, \tau) \mid pCl\{y\} \subset G\}) \cap V = \emptyset \) and there exists \( G \in PO(X, \tau) \) such that \( x \notin G \) and \( pCl\{y\} \subset G \). Therefore, \( pCl\{x\} \cap G = \emptyset \) and \( y \notin pCl\{x\} \).
Consequently, we obtain \( pCl\{x\} \subset pKer\{x\} \).
(5) \( \rightarrow\) (1) : Let \( G \in PO(X, \tau) \) and \( x \in G \). Let \( y \notin pKer\{x\} \), then \( x \in pCl\{y\} \) and \( y \in G \). This implies that \( pKer\{x\} \subset G \). Therefore, we obtain \( x \in pCl\{x\} \subset pKer\{x\} \subset G \). This shows that \( (X, \tau) \) is a pre-\( R_0 \) space. \( \square \)

Corollary 3.9. For a topological space \( (X, \tau) \), the following properties are equivalent :
(1) \( (X, \tau) \) is a pre-\( R_0 \) space;
(2) \( pCl\{x\} = pKer\{x\} \) for all \( x \in X \).

Proof. (1) \( \rightarrow\) (2) : Suppose that \( (X, \tau) \) is a pre-\( R_0 \) space. By Theorem 3.8, \( pCl\{x\} \subset pKer\{x\} \) for each \( x \in X \). Let \( y \in pKer\{x\} \), then \( x \in pCl\{y\} \) and by Corollary 3.5 \( pCl\{y\} = pCl\{x\} \). Therefore, \( y \in pCl\{x\} \) and hence \( pKer\{x\} \subset pCl\{x\} \). This shows that \( pCl\{x\} = pKer\{x\} \).
(2) \( \rightarrow\) (1) : This is obvious by Theorem 3.8. \( \square \)
The following lemma due to Maki et al. [16] is very useful and important.

Lemma 3.10. In every topological space, each singleton is pre-open or preclosed.

Theorem 3.11. A topological space \((X, \tau)\) is pre-\(R_0\) if and only if it is pre-\(T_1\).

Proof. Necessity. Suppose that \((X, \tau)\) is pre-\(R_0\). For each point \(x \in X\), by Lemma 3.10 \(\{x\}\) is preopen or preclosed in \(X\). If \(\{x\}\) is preopen, then we have \(pCl(\{x\}) \subset \{x\}\) and hence \(\{x\}\) is preclosed by Lemma 2.1. This shows that \((X, \tau)\) is pre-\(T_1\).

Sufficiency. Let \(U\) be any preopen set of \(X\) and \(x \in U\). Since \(\{x\}\) is preclosed, \(pCl(\{x\}) = \{x\} \subset U\). Therefore \((X, \tau)\) is pre-\(R_0\). □

Theorem 3.12. For a topological space \((X, \tau)\), the following properties are equivalent:

1. \((X, \tau)\) is a pre-\(R_0\) space;
2. \(x \in pCl(\{y\})\) if and only if \(y \in pCl(\{x\})\).

Proof. (1) \(\rightarrow\) (2) : This is obvious from Theorem 3.11.

(2) \(\rightarrow\) (1) : Let \(U\) be a preopen set and \(x \in U\). If \(y \notin U\), then \(x \notin pCl(\{y\})\) and hence \(y \notin pCl(\{x\})\). This implies that \(pCl(\{x\}) \subset U\). Hence \((X, \tau)\) is pre-\(R_0\). □

Theorem 3.13. For a topological space \((X, \tau)\), the following properties are equivalent:

1. \((X, \tau)\) is a pre-\(R_0\) space;
2. If \(F\) is preclosed, then \(F = pKer(F)\);
3. If \(F\) is preclosed and \(x \in F\), then \(pKer(\{x\}) \subset F\);
4. If \(x \in X\), then \(pKer(\{x\}) \subset pCl(\{x\})\).
Proof. (1) → (2) : Let $F$ be preclosed and $x \notin F$. Thus $X - F$ is preopen and contains $x$. Since $(X, \tau)$ is pre-$R_0$, $pCl\{\{x\}\} \subseteq X - F$. Thus $pCl\{\{x\}\} \cap F = \emptyset$ and by Lemma 3.2 $x \notin pKer(F)$. Therefore $pKer(F) = F$.

(2) → (3) : In general, $A \subseteq B$ implies $pKer(A) \subseteq pKer(B)$. Therefore, it follows from (2) that $pKer\{\{x\}\} \subseteq pKer(F) = F$.

(3) ↔ (4) : Since $x \in pCl\{\{x\}\}$ and $pCl\{\{x\}\}$ is preclosed, by (3) $pKer\{\{x\}\} \subseteq pCl\{\{x\}\}$. Therefore $pKer\{\{x\}\} = F$.

(4) ↔ (1) : We show the implication by using Theorem 3.12. Let $x \in pCl\{\{y\}\}$. Then by Lemma 3.1 $y \in pKer\{\{x\}\}$. Since $x \in pCl\{\{x\}\}$ and $pCl\{\{x\}\}$ is preclosed, by (4) we obtain $y \in pKer\{\{x\}\} \subseteq pCl\{\{x\}\}$. Therefore $x \in pCl\{\{y\}\}$ implies $y \in pCl\{\{x\}\}$. The converse is obvious and $(X, \tau)$ is pre-$R_0$. \qed

Definition 4. A filterbase $F$ is called $p$-convergent to a point $x$ in $X$ [11], if for any preopen set $U$ of $X$ containing $x$, there exists $B$ in $F$ such that $B$ is a subset of $U$.

Lemma 3.14. Let $(X, \tau)$ be a topological space and let $x$ and $y$ be any two points in $X$ such that every net in $X$ $p$-converging to $y$ $p$-converges to $x$. Then $x \in pCl\{\{y\}\}$.

Proof. Suppose that $x_n = y$ for each $n \in \mathbb{N}$. Then $\{x_n\}_{n \in \mathbb{N}}$ is a net in $pCl\{\{y\}\}$. By the fact that $\{x_n\}_{n \in \mathbb{N}}$ $p$-converges to $y$, then $\{x_n\}_{n \in \mathbb{N}}$ $p$-converges to $x$ and this means that $x \in pCl\{\{y\}\}$. \qed

Theorem 3.15. For a topological space $(X, \tau)$, the following statements are equivalent:

(1) $(X, \tau)$ is a pre-$R_0$ space;

(2) If $x, y \in X$, then $y \in pCl\{\{x\}\}$ if and only if every net in $X$ $p$-converging to $y$ $p$-converges to $x$.

Proof. (1) → (2) : Let $x, y \in X$ such that $y \in pCl\{\{x\}\}$.

Let $\{x_\alpha\}_{\alpha \in \Lambda}$ be a net in $X$ such that $\{x_\alpha\}_{\alpha \in \Lambda}$ $p$-converges to $y$. Since $y \in pCl\{\{x\}\}$, by Corollary 3.5 we have $pCl\{\{x\}\} = pCl\{\{y\}\}$. Therefore $x \in pCl\{\{y\}\}$. This means that $\{x_\alpha\}_{\alpha \in \Lambda}$ $p$-converges to $x$. Conversely, let $x, y \in X$ such that every net
in $X$ $p$-converging to $y$ $p$-converges to $x$. Then $x \in pCl(\{y\})$ by Lemma 3.2. By Corollary 3.5, we have $pCl(\{x\}) = pCl(\{y\})$. Therefore $y \in pCl(\{x\})$.

(2) $\rightarrow$ (1) : Assume that $x$ and $y$ are any two points of $X$ such that $pCl(\{x\}) \cap pCl(\{y\}) \neq \emptyset$. Let $z \in pCl(\{x\}) \cap pCl(\{y\})$. So there exists a net $\{x_\alpha\}_{\alpha \in \Lambda}$ in $pCl(\{x\})$ such that $\{x_\alpha\}_{\alpha \in \Lambda} p$-converges to $z$. Since $z \in pCl(\{y\})$, then $\{x_\alpha\}_{\alpha \in \Lambda} p$-converges to $y$. It follows that $y \in pCl(\{x\})$. By the same token we obtain $x \in pCl(\{y\})$. Therefore $pCl(\{x\}) = pCl(\{y\})$ and by Corollary 3.5 $(X, \tau)$ is pre-$R_0$. □

4. Pre-$R_1$ Spaces

Definition 5. A topological space $(X, \tau)$ is said to be pre-$R_1$ if for $x, y$ in $X$ with $pCl(\{x\}) \neq pCl(\{y\})$, there exist disjoint preopen sets $U$ and $V$ such that $pCl(\{x\})$ is a subset of $U$ and $pCl(\{y\})$ is a subset of $V$.

Proposition 4.1. If $(X, \tau)$ is pre-$R_1$, then $(X, \tau)$ is pre-$R_0$.

Proof. Let $U$ be preopen and $x \in U$. If $y \notin U$, then since $x \notin pCl(\{y\})$, $pCl(\{x\}) \neq pCl(\{y\})$. Hence, there exists a preopen set $V_y$ such that $pCl(\{y\}) \subset V_y$ and $x \notin V_y$ which implies $y \notin pCl(\{x\})$. Thus $pCl(\{x\}) \subset U$. Therefore $(X, \tau)$ is pre-$R_0$. □

Theorem 4.2. For a topological space $(X, \tau)$, the following statements are equivalent :

(1) $(X, \tau)$ is pre-$T_2$ ;
(2) $(X, \tau)$ is pre-$R_1$.

Proof. (1) $\rightarrow$ (2) : Since $X$ is pre-$T_2$, then $X$ is pre-$T_1$. If $x, y \in X$ such that $pCl(\{x\}) \neq pCl(\{y\})$, then $x \neq y$. There exists disjoint preopen sets $U$ and $V$ such that $x \in U$ and $y \in V$; hence $pCl(\{x\}) = \{x\} \subset U$ and $pCl(\{y\}) = \{y\} \subset V$. Hence $X$ is pre-$R_1$. 


(2) → (1) : Suppose that \((X, \tau)\) is pre-\(R_1\). By Proposition 4.1, \((X, \tau)\) is pre-\(R_0\) and hence it is pre-\(T_1\) by Theorem 3.11. Let \(x, y \in X\) such that \(x \neq y\). Since \(pCl(\{x\}) = \{x\} \neq \{y\} = pCl(\{y\})\), there exist disjoint preopen sets \(U\) and \(V\) such that \(x \in U\) and \(y \in V\). Hence \(X\) is pre-\(T_2\). \(\square\)

**Theorem 4.3.** For a topological space \((X, \tau)\), the following statements are equivalent:

1. \((X, \tau)\) is pre-\(R_1\);
2. If \(x, y \in X\) such that \(pCl(\{x\}) \neq pCl(\{y\})\), then there exist preclosed sets \(F_1\) and \(F_2\) such that \(x \in F_1\), \(y \notin F_1\), \(y \in F_2\), \(x \notin F_2\) and \(X = F_1 \cup F_2\).

**Proof.** (1) → (2) : Let \(x, y \in X\) such that \(pCl(\{x\}) \neq pCl(\{y\})\). By Theorem 4.2, \(X\) is pre-\(T_2\) and hence \(x \neq y\). Therefore, there exists disjoint preopen sets \(U_1\) and \(U_2\) such that \(x \in U_1\) and \(y \in U_2\). Then \(F_1 = X - U_2\) and \(F_2 = X - U_1\) are preclosed sets such that \(x \in F_1\), \(y \notin F_1\), \(y \in F_2\), \(x \notin F_2\) and \(X = F_1 \cup F_2\).

(2) → (1) : Suppose that \(x\) and \(y\) are distinct points of \(X\). By Lemma 3.10, there are three cases as follows:

(i) \(\{x\}, \{y\} \in PO(X, \tau)\). Then \(pKer(\{x\}) = \{x\} \neq \{y\} = pKer(\{y\})\). By Lemma 3.6, we have \(pCl(\{x\}) \neq pCl(\{y\})\).

(ii) \(\{x\} \in PO(X, \tau), \{y\} \in PC(X, \tau)\). Then \(pCl(\{y\}) = \{y\}\) and hence \(pCl(\{x\}) \neq pCl(\{y\})\).

(iii) \(\{x\}, \{y\} \in PC(X, \tau)\). Then \(pCl(\{x\}) = \{x\} \neq \{y\} = pCl(\{y\})\) and \(pCl(\{x\}) \neq pCl(\{y\})\).

For each case we have \(pCl(\{x\}) \neq pCl(\{y\})\) and there exist preclosed sets \(F_1\) and \(F_2\) such that \(x \in F_1\), \(y \notin F_1\), \(y \in F_2\), \(x \notin F_2\) and \(X = F_1 \cup F_2\). Now, we set \(U_1 = X - F_2\) and \(U_2 = X - F_1\), then we obtain that \(x \in U_1, y \in U_2, U_1 \cap U_2 = \emptyset\) and \(U_1, U_2\) are preopen. This shows that \((X, \tau)\) is pre-\(T_2\). It follows from Theorem 4.2 that \((X, \tau)\) is pre-\(R_1\). \(\square\)

**Theorem 4.4.** A topological space \((X, \tau)\) is pre-\(R_1\) if and only if for \(x, y \in X\), \(pKer(\{x\}) \neq pKer(\{y\})\), there exist disjoint preopen sets \(U\) and \(V\) such that \(pCl(\{x\}) \subset U\) and \(pCl(\{y\}) \subset V\).
Proof. It follows from Lemma 3.6. \(\square\)

The following notions are due to Dontchev et al. [4]:

A point \(x\) of a topological space \((X, \tau)\) is a pre-\(\theta\)-accumulation point of a subset \(A \subset X\), if for each preopen \(U\) of \(X\) containing \(x\), \(pCl(U) \cap A \neq \emptyset\). The set \(pCl_\theta(A)\) of all pre-\(\theta\)-accumulation points of \(A\) is called the pre-\(\theta\)-closure of \(A\). The set \(A\) is said to be pre-\(\theta\)-closed if \(pCl_\theta(A) = A\). Complement of a pre-\(\theta\)-closed set is said to be pre-\(\theta\)-open.

**Lemma 4.5.** For any subset \(A\) of a topological space \((X, \tau)\), \(pCl(A) \subset pCl_\theta(A)\).

**Lemma 4.6.** Let \(x\) and \(y\) be points in a topological space \((X, \tau)\). Then \(y \in pCl_\theta(\{x\})\) if and only if \(x \in pCl_\theta(\{y\})\).

**Theorem 4.7.** A topological space \((X, \tau)\) is pre-\(R_1\) if and only if for each \(x \in X\), \(pCl(\{x\}) = pCl_\theta(\{x\})\).

Proof. Necessity. Assume that \(X\) is pre-\(R_1\) and \(y \in pCl_\theta(\{x\}) - pCl(\{x\})\). Then there exists a preopen set \(U\) containing \(y\) such that \(pCl(U) \cap \{x\} \neq \emptyset\) but \(U \cap \{x\} = \emptyset\). Thus \(pCl(\{y\}) \subset U\), \(pCl(\{x\}) \cap U = \emptyset\). Hence \(pCl(\{x\}) \neq pCl(\{y\})\). Since \(X\) is pre-\(R_1\), there exist disjoint preopen sets \(U_1\) and \(U_2\) such that \(pCl(\{x\}) \subset U_1\) and \(pCl(\{y\}) \subset U_2\). Therefore \(X - U_1\) is a preclosed pre-neighbourhood at \(y\) which does not contain \(x\). Thus \(y \notin pCl_\theta(\{x\})\). This is a contradiction.

Sufficiency. Suppose that \(pCl(\{x\}) = pCl_\theta(\{x\})\) for each \(x \in X\). We first prove that \(X\) is pre-\(R_0\). Let \(x\) belong to the preopen set \(U\) and \(y \notin U\). Since \(pCl_\theta(\{y\}) = pCl(\{y\}) \subset X - U\), we have \(x \notin pCl_\theta(\{y\})\) and by Lemma 4.6 \(y \notin pCl_\theta(\{x\}) = pCl(\{x\})\). It follows that \(pCl(\{x\}) \subset U\). Therefore \((X, \tau)\) is pre-\(R_0\). Now, let \(a, b \in X\) with \(pCl(\{a\}) \neq pCl(\{b\})\). By Theorem 3.11, \((X, \tau)\) is pre-\(T_1\) and \(a \neq b\). Since \(pCl(\{a\}) = pCl_\theta(\{a\})\) for each \(a \in X\), \(b \notin pCl_\theta(\{a\})\) and hence there exists a preopen
set $U$ containing $b$ such that $a \notin pCl(U)$. Therefore, we obtain $b \in U$, $a \in X - pCl(U)$ and $U \cap (X - pCl(U)) = \emptyset$. This shows that $(X, \tau)$ is pre-$T_2$. It follows from Theorem 4.2 that $(X, \tau)$ is pre-$R_1$. 

References


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