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HEREDITARILY α-NORMAL SPACES AND INFINITE PRODUCTS

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Abstract

Characterizations are given for hereditarily α- normal and hereditarily β-normal spaces. We obtain results related to these spaces and extremally disconnected spaces. Results and a question on infinite products are also given.

1. Introduction

In [AL], two new generalizations of normality were introduced. A space X is called *α-normal* if for any two disjoint closed subsets A and B of X there exist disjoint open subsets U and V of X such that $A \cap U$ is dense in A and $B \cap V$ is dense in B. A space X is called *β-normal* if for any two disjoint closed subsets A and B of X there exist open subsets U and V of X such that $A \cap U$ is dense in A, $B \cap V$ is dense in B, and $\overline{U} \cap \overline{V} = \emptyset$. Clearly, normality implies β-normal and β-normal implies α-normal.

Several results involving extremally disconnected spaces and hereditarily separable spaces were presented in that paper. It was natural to search the literature for such topics and explore the role of α- or β-normality in such spaces. Several of the results in this paper use hereditary α-normality to strengthen Wage's [W] results on extremally disconnected S-spaces. We

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also present several characterizations of hereditarily α- and β-normal spaces.

In 1948, A.H. Stone proved that $\mathbb{N}^{\omega_1}$ is not normal. We show that this space is not α-normal. Several results are given from this example that describe the behavior of hereditary α-normality under uncountable products and characterize certain uncountable products.

2. Hereditarily α-normal

Recall the following well-known characterization of hereditarily normal spaces.

Fact 2.1. For every T_1-space X, the following conditions are equivalent:

(1) The space X is hereditarily normal.

(2) Every open subspace of X is normal.

(3) For every pair of separated sets $A, B \subseteq X$ there exists open sets $U, V \subseteq X$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

We now have a parallel result for α-normality.

Defintition 2.1. A space X is *hereditarily α-normal* if every subspace of X is α-normal.

Theorem 2.1. *For every T_1-space X, the following conditions are equivalent:*

(1) *The space X is hereditarily α-normal.*

(2) *Every open subspace of X is α-normal.*

(3) *For every pair of separated sets, $A, B \subseteq X$ there exists open sets $U, V \subseteq X$ such that $A \cap U$ is dense in A, $B \cap V$ is dense in B and $U \cap V = \emptyset$. (We may call this α-separated.)*

Proof. The implication (1) $\Rightarrow$ (2) is obvious. For (2) $\Rightarrow$ (3), let A and B be separated sets of X. Consider $M = X \setminus (\bar{A} \cap \bar{B})$, an open subspace of X with $A, B \subseteq M$. Since $\mathrm{Cl}_M(A) \cap \mathrm{Cl}_M(B) = \emptyset$ and by hypothesis M is α-normal, there exists open disjoint $U, V \subseteq M$ such that

$$\overline{U \cap \mathrm{Cl}_M(A)}^M = \mathrm{Cl}_M(A) \text{ and } \overline{V \cap \mathrm{Cl}_M(B)}^M = \mathrm{Cl}_M(B)$$

That is, $U \cap A$ is dense on A and $V \cap B$ is dense on B. But M is open in X, thus U, V are open in X as desired. For (3) $\Rightarrow$ (1), let M be a subspace of X and $A, B \subseteq M$ a pair of disjoint closed subsets. Note A and B are separated in X. By hypothesis there exists open disjoint $U, V \subseteq X$ such that $A \cap U$ is dense in A and $B \cap V$ is dense in B. Now $U \cap M$ and $V \cap M$ are open in M and clearly $A \cap (U \cap M)$ is dense in A and $B \cap (V \cap M)$ is dense in B. That is, M is α-normal, that is X is hereditarily α-normal. □

It is curious to note that a parallel to Theorem 2.1 does not hold for the seemingly stronger property of β-normality. Only parts (1) and (2) hold for hereditarily β-normal spaces as the following example and theorem demonstrate.

Example 2.1. Let $X = [0, 1]$ with the usual topology. Clearly, X is metrizable, hence hereditarily normal, thus hereditarily β-normal. Consider $A = [0, \frac{1}{2})$ $B = (\frac{1}{2}, 1]$, two separated subsets of X. There does not exist open disjoint subsets U, V of X such that $A \cap U$ is dense on A, $B \cap V$ is dense on B and $\bar{U} \cap \bar{V} = \emptyset$, since $\{\frac{1}{2}\} \in \bar{U} \cap \bar{V}$ for all such U and V. Thus we see that there is no version of (3) of Theorem 2.1 for β-normal spaces.

Theorem 2.2. *For every T_1-space X, the following conditions are equivalent:*

(1) *The space X is hereditarily β-normal.*

(2) *Every open subspace of X is β-normal.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious. For (2) $\Rightarrow$ (3), let M be a nonempty subspace of X. Let A and B be closed disjoint subsets of M. Clearly $A = \text{Cl}_X(A) \cap M$ and $B = \text{Cl}_X(B) \cap M$. Consider $Y = X \setminus (\text{Cl}_X(A) \cap \text{Cl}_X(B))$ is an open subspace of X with A and B closed disjoint subsets of Y. By hypothesis Y is β-normal, so there exist open $U, V \subseteq Y$ such that $A \cap U$ is dense in A, $B \cap V$ is dense in B and $\text{Cl}_Y(U) \cap \text{Cl}_Y(V) = \emptyset$. Since $M \subseteq Y$, we have $U \cap M$ and $V \cap M$ are open disjoint subsets of M with $A \cap U \cap M$ is dense in A, $B \cap V \cap M$ is dense in B and $\text{Cl}_M(U \cap M) \cap \text{Cl}_M(V \cap M) = \emptyset$. That is, M is β-normal, hence X is hereditarily β-normal. □

It was shown in [AL] that every extremally disconnected α-normal space X is normal and every S-space is α-normal. We now take the natural course and investigate the properties of extremally disconnected hereditarily α-normal spaces. Many of the results in this section strengthen those of Wage by removing the S-space property and inserting the property hereditarily α-normal. The first result shows that for extremally disconnected spaces, hereditarily α-normal is equivalent to hereditarily normal.

Theorem 2.3. *Let X be an extremally disconnected space. X is hereditarily normal if and only if X is hereditarily α-normal.*

Proof. Necessity is clear. For sufficiency, let Y be an open subspace of X. Let A and B be two closed disjoint subsets of Y. Since Y is α-normal, there exists disjoint open subsets U and V of Y such that $A \cap U$ is dense in A and $B \cap V$ is dense in B. But Y is open, hence extremally disconnected. This implies that $\text{Cl}_Y(U)$ and $\text{Cl}_Y(V)$ are disjoint and open in Y. Thus, $A = \text{Cl}_Y(A \cap U) \subseteq \text{Cl}_Y(U)$ and $B = \text{Cl}_Y(B \cap V) \subseteq \text{Cl}_Y(V)$ as desired. □

Corollary 2.1. *Every extremally disconnected hereditarily α-normal space X is hereditarily extremally disconnected.*

Proof. By Theorem 2.3 , X is hereditarily normal. It is an easy exercise to show that every extremally disconnected hereditarily normal space is hereditarily extremally disconnected. □

In [K], Kochinats defined a space X to be *weakly perfect* if every closed subspace A of X contains some subset which is dense in A and is a G_δ in X. Couple this with α-normal and we have the following definition.

Defintition 2.2. A space X is *weakly perfectly α-normal* if X is weakly perfect and α-normal.

Theorem 2.4. *If X is an extremally disconnected, weakly perfectly hereditarily α-normal space, then X is perfect.*

Before proving 2.4, we consider the following lemma.

Lemma 2.1. *Let X be an extremally disconnected hereditarily α-normal space. If A is a closed subset of X and U is an open subset of X such that $\overline{A \cap U} = A$, then $A \cup U$ is an open set in X.*

Proof. Suppose $A \cup U$ is not open. Then there exists $x \in A$ such that $x \in \overline{X \setminus (A \cup U)}$. Since $\overline{A \cap U} = A$ and $\overline{X \setminus (A \cup U)} \subseteq \overline{X \setminus U} = X \setminus U$, we have $(X \setminus (A \cup U)) \cap (\overline{A \cap U}) = \emptyset$ and $(\overline{X \setminus (A \cup U)}) \cap (A \cap U) = \emptyset$ respectively. Hence $X \setminus (A \cup U)$ and $A \cap U$ are separated subsets of X and by Theorem 2.3, X is hereditarily normal. So, there exist open disjoint subsets V and W of X, such that $X \setminus (A \cup U) \subseteq V$ and $A \cap U \subseteq W$. We now have

$$x \in \overline{X \setminus (A \cup U)} \subseteq \overline{V}$$

$$x \in A = \overline{A \cap U} \subseteq \overline{W}.$$

But X is extremally disconnected, so $\overline{V} \cap \overline{W} = \emptyset$, a contradiction. Hence $A \cup U$ is open as desired. □

Proof. [Proof of Theorem 2.4] Let A be a closed subset of X. Since X is weakly perfect, there exists a G_δ, $G = \bigcap_{n\in\omega} G_n$, of X such that $A = \overline{\bigcap_{n\in\omega} G_n}$. Clearly $\overline{G_n \cap A} = A$ for all $n \in \omega$, hence $G_n \cup A$ is an open subset of X for all $n \in \omega$ by Lemma 2.1. Thus $A = \bigcap_{n\in\omega}(G_n \cup A)$ is a G_δ in X. That is, X is perfect. $\square$

It is interesting to note that Wage showed under ♣ that not every extremally disconnected S-space is perfect [W]. In [AL], it was shown that every regular, hereditarily separable space is hereditarily α-normal. Thus, every S-space is hereditarily α-normal. So we see that weakly perfect is a necessary condition for Theorem 2.4.

In his Ph.D. thesis, Wage showed that there are no extremally disconnected hereditarily separable Dowker spaces. Indeed, every extremally disconnected, hereditarily separable, regular space X is normal and countably metacompact, hence countably paracompact. It is unclear at this time if hereditarily separable can be replaced by hereditarily α-normal or even hereditarily normal to obtain the same result.

Question 2.1. *Does there exist a hereditarily normal (α-normal, β-normal), extremally disconnected Dowker space?*

3. Infinite Products

In his 1948 article, A.H. Stone provided a necessary and sufficient condition for the topological product of uncountably many metric spaces to be normal. We now strengthen this result by showing the same holds true for α-normality.

Example 3.1. The product of uncountably many metric spaces may not be α-normal: the product space $\mathbb{N}^{\omega_1}$ of ω_1 copies of the natural numbers is not α-normal.

Proof. For convenience of notation we use ω^{ω_1} instead of $\mathbb{N}^{\omega_1}$. For a contradiction, suppose ω^{ω_1} is α-normal. We will witness two closed disjoint subsets on ω^{ω_1} which cannot be α-separated. Fix $T = \omega \backslash \{0, 1\}$ and define two subsets of ω^{ω_1} as follows:

$$E_0 = \{x \in \omega^{\omega_1} : \exists\, \alpha \in \omega_1 \text{ s.t. } x \restriction_\alpha \text{ is one to one into } T \text{ and } x(\beta) = 0\, \forall\, \beta \geq \alpha\},$$

$$E_1 = \{x \in \omega^{\omega_1} : \exists\, \alpha \in \omega_1 \text{ s.t. } x \restriction_\alpha \text{ is one to one into } T \text{ and } x(\beta) = 1\, \forall\, \beta \geq \alpha\}.$$

It can be easily shown that E_0 and E_1 are indeed disjoint closed subsets of ω^{ω_1}. By assumption, there exists disjoint open subsets U, V of ω^{ω_1} such that $\overline{U \cap E_0} = E_0$ and $\overline{V \cap E_1} = E_1$. Consider the homeomorphism $\phi : \omega^{\omega_1} \to \omega^{\omega_1}$ defined by

$$\phi_x(\alpha) = \begin{cases} x(\alpha) & \text{if } x(\alpha) \in T \\ 1 - x(\alpha) & \text{if } x(\alpha) \in \{0, 1\} \end{cases}$$

Note that $\phi(E_1) = E_0$. Let $V' = \phi(V)$, then $V' \cap E_0$ is an open dense subset of E_0. Hence $U \cap V' \cap E_0$ is an open dense subset of E_0.

Now find an uncountable $\Lambda \subseteq \omega_1$ and corresponding $\mathcal{F} = \{z_\alpha : \alpha \in \Lambda\} \subseteq U \cap V' \cap E_0$ such that $z_\alpha \restriction_\alpha$ is one to one into T and $z_\alpha(\beta) = 0$ for all $\beta \geq \alpha$. For all $\alpha \in \Lambda$, find a finite restriction g_α of z_α such that the basic open set $[g_\alpha] \subseteq U \cap V'$. Consider $h_\alpha = \phi \circ g_\alpha$ and $y_\alpha = \phi \circ z_\alpha$. For each $\alpha \in \Lambda$ we have $y_\alpha \in V \cap E_1$, $[h_\alpha] \subseteq V$, $h_\alpha \restriction_\alpha = g_\alpha \restriction_\alpha$, and $h_\alpha(\beta) = 1$ if $\beta \geq \alpha$. Consider $\mathcal{D} = \{\text{dom}(g_\alpha) : \alpha \in \Lambda\}$. Without loss of generality, we assume

$$\{\text{dom}(g_\alpha) \cap (\omega_1 \backslash \alpha) : \alpha \in \Lambda\} \quad (*)$$

is pairwise disjoint.

By the delta system lemma, there exists an uncountable $\mathcal{D}' \subseteq \mathcal{D}$, indexed by $\Lambda' \subseteq \Lambda$, with a root a. Note that by (*), for each $\alpha \in \Lambda'$, we have $a \subseteq \text{dom}(g_\alpha) \cap \alpha$. Observe that $\{g_\alpha \restriction_a : \alpha \in \Lambda\}$ has only countably many elements. So there exists $\alpha, \gamma \in \Lambda'$, $\alpha < \gamma$, such that $g_\gamma \restriction_a = g_\alpha \restriction_a$. But $g_\alpha \restriction_a = g_\gamma \restriction_a = \phi \circ g_\gamma \restriction_a = h_\gamma \restriction_a$ and $(\text{dom}(g_\alpha) \backslash a) \cap (\text{dom}(h_\gamma) \backslash a) = \emptyset$ by the delta system lemma. We conclude that $[g_\alpha] \cap [h\alpha] \neq \emptyset$. That is, $U \cap V \neq \emptyset$, a contradiction. □

Now we can completely describe the behavior of hereditary α-normality under uncountable products.

Theorem 3.1. *The product of uncountably many spaces containing at least two points is never hereditarily α-normal.*

Proof. Note that such a space contains a copy of $\mathbb{N}^{\omega_1}$. For example see [P]. □

The following result shows that products of uncountably many factors are rarely α-normal.

Theorem 3.2. *If the product space $X = \Pi_{\alpha<\kappa} X_\alpha$ is α-normal, then all spaces, with the exception of at most countably many, are countably compact. In particular, if X^κ is α-normal, then X is countably compact.*

Proof. Suppose that X_α is not countably compact for, say, $\alpha < \omega_1$. Then each X_α, for $\alpha < \omega_1$, contains a closed copy of the discrete space $\mathbb{N}$ of natural numbers. Since α-normality is preserved under closed subspaces, we have $\mathbb{N}^{\omega_1}$ is α-normal. This is a contradiction to Example 3.1. □

Corollary 3.1. *For a family $\{X_\alpha\}_{\alpha<\kappa}$ of metrizable spaces,* [1] *the following conditions are equivalent:*

(1) $\Pi_{\alpha<\kappa} X_\alpha$ *is α-normal.*

(2) $\Pi_{\alpha<\kappa} X_\alpha$ *is paracompact.*

(3) *All spaces, with the exception of at most countably many, are compact.*

Proof. The implication (2) $\Rightarrow$ (1) is obvious. For (1) $\Rightarrow$ (3), by Theorem 3.2 all spaces X_α, with the exception of at most

[1] The same can also be shown for the more general paracompact p-spaces, see [P].

countably many, are countably compact. A countably compact metric space (paracompact space) is compact. For (3) $\Rightarrow$ (2), recall the product of a compact space with a paracompact space is paracompact. Moreover, the product of countably many metric spaces is paracompact. $\square$

In [N], Noble showed that if every power X^κ of a T_1 topological space is normal, then X is compact. This leads to the following question of Arhangel'skii.

Question 3.1. *If every power X^κ of a T_1 topological space is α-(β)-normal, is X compact?*

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