REFLECTING ON COMPACT SPACES

Franklin D. Tall

Abstract

We consider whether, if a topological space reflects via an elementary submodel to a generalized Cantor discontinuum it must in fact equal its reflection. The answers involve large cardinals.

Given an elementary submodel $M$ of some $H(\theta)$ (see [JW, Chapter 24] for a careful elucidation of the implications of this) and a topological space $(X, T) \in M$, we define $X_M$ to be $X \cap M$ with topology generated by $T_M = \{U \cap M : U \in T \cap M\}$. In [JT1] we developed this notion; in [T1] I proved that if $X_M$ is homeomorphic to the Cantor set, then $X = X_M$. I. Juhász (personal communication) asked whether this generalized to arbitrary cardinals, i.e. if $X_M$ is homeomorphic to a generalized Cantor discontinuum $D^\lambda$, where $D$ is the 2-point discrete space, then does $X = X_M$? We shall show that the answer is yes for small $\lambda$, but not necessarily for very huge ones.

The following technical result is the key observation for our work on Juhász’ problem. It and Corollary 2 are due independently to Lucia Junqueira, conversations with whom have been very helpful. Note that when we write “$X_M$”, we are implicitly assuming that $X \in M$. Also note that $D^\lambda \in M$ implies $\lambda \in M$.

* Research supported by NSERC Grant A-7354.

Mathematics Subject Classification: Primary: 54D30; Secondary: 03E35, 03E55, 54B10.

Key words: Elementary submodel, reflection, compact, generalized Cantor discontinuum, huge cardinals.
Theorem 1. Let \( \lambda \) be a set. (Most of the time, it will be a cardinal.) Suppose \((D^\lambda)_M\) is compact. Then \((D^\lambda)_M\) is homeomorphic to \(D^\lambda \cap M\).

Proof. Let \( h : (D^\lambda)_M \to D^\lambda \cap M \) be defined by \( h(f) = f|(|\lambda \cap M)\). Claim \( h \) is a homeomorphism. Since \((D^\lambda)_M\) is compact and also \(T_2\) (since \(D^\lambda\) is \([JT_1]\)), it suffices to show \( h \) is continuous, one-one, and has dense image. Let \([p] = \{ g \in D^\lambda \cap M : g|\text{dom}(p) = p \}\), where \( p \) is a finite partial function from \( \lambda \cap M \) into \( D \). Then \( h^{-1}([p]) = \{ f \in D^\lambda \cap M : f|\text{dom}(p) = p \} \). But this is open in \((D^\lambda)_M\). \( h \) is one-one, since if \( f_1 \neq f_2 \) are in \( D^\lambda \cap M \), \( f_1|(|\lambda \cap M) \neq f_2|(|\lambda \cap M) \) by elementarity. Finally, given any non-empty basic open \([p]\) in \( D^\lambda \cap M \), since \( \text{dom}(p) \subseteq M, p \in M \), so the function \( f \) defined by

i) \( f|\text{dom}(p) = p \),

ii) \( f|(\lambda - \text{dom}(p)) = 0 \),

is in \((D^\lambda)_M\), and \( h(f) \in [p] \). \( \Box \)

Corollary 2. If \( \lambda \subseteq M \) and \((D^\lambda)_M\) is compact, then \((D^\lambda)_M = D^\lambda\).

Proof. In this case, \( h \) is the identity function. \( \Box \)

Corollary 3. Let \( \mu \) be the least ordinal not included in \( M \). Then if \( \lambda \) is a cardinal less than \( \mu \) and \((D^\lambda)_M\) is compact, then \( D^\lambda = (D^\lambda)_M \).

The proof is immediate.

Corollary 4. Let \( \mu \) be the least ordinal not included in \( M \). If \( X_M \) is homeomorphic to \( D^\lambda \), \( \lambda < \mu \), and \( D^\lambda \in M \), then \( X = X_M \).

Proof. By \([J]\), since \( X_M \) is compact, so is \( X \) and there is a continuous map from \( X \) onto \( X_M \). Relativizing, there is a continuous map from \( X_M \) onto \((D^\lambda)_M\). Hence \((D^\lambda)_M\) is compact,
so $D^\lambda$ and hence $2^\lambda \subseteq M$. Therefore $\lambda^+ \subseteq M$. We now do some easy calculation of cardinal functions. See [H] for definitions and theorems. Using a straightforward argument done in detail in [T1], we see that $X$ has no right- or left-separated subspaces of size $\geq \lambda^+$, else $X_M$ would. But $\omega(X_M) = \lambda$. Since $X_M$ and hence $[T_1]$ $X$ is $T_3$, it follows that $|X \cup T| \leq 2^\lambda$, so $X \cup T \subseteq M$, so $X = X_M$.

**Theorem 5.** The first cardinal $\lambda$ — if any — such that $(D^\lambda)_M$ is compact for some $M$ but $\neq D^\lambda$ must be strongly inaccessible.

*Proof.* The first cardinal — if any — for which $(D^\lambda)_M$ is compact but $\neq D^\lambda$ cannot be $\leq 2^\kappa$ for some $\kappa < \lambda$, $\kappa \in M$. By elementarity, we can omit ‘$\kappa \in M$’. The point is that — since $D^\kappa$ is a continuous image of $D^\lambda - (D^\lambda)_M$ compact implies $(D^\kappa)_M$ is compact implies $D^\kappa = (D^\kappa)_M$ implies $2^\kappa \subseteq M$ implies $\lambda \subseteq M$ implies $(D^\lambda)_M = D^\lambda$. The first such cardinal can also not be singular, since $D^\lambda \in M$ implies $\lambda \in M$ implies $cf(\lambda) \in M$ implies $D^{cf(\lambda)} \in M$ (since $D^\lambda \in M$). Then, since $\lambda$ is least and — as before — $(D^{cf(\lambda)})_M$ is compact, $(D^{cf(\lambda)})_M = D^{cf(\lambda)}$. Therefore $D^{cf(\lambda)}$ and a fortiori $cf(\lambda) \subseteq M$. But then there is a set $S$ of cardinals cofinal in $\lambda$ included in $M$. For each $\sigma \in S$, $D^\sigma \in M$ and $(D^\sigma)_M$ is compact, so $D^\sigma = (D^\sigma)_M$ so $D^\sigma$ and hence $\sigma \subseteq M$. But then $\lambda \subseteq M$ and so $D^\lambda = (D^\lambda)_M$, contradiction. Thus $M$ thinks $\lambda$ is strongly inaccessible, so it is.

**Corollary 6.** Suppose $X_M$ is homeomorphic to $D^\lambda \in M$ and $\lambda$ is less than the first strongly inaccessible cardinal. Then $X = X_M$.

*Proof.* As for Corollary 4 above.

Thus if there are no strongly inaccessible cardinals, Juhasz’ problem is solved. A less draconian solution is given by the following two results. $0^#$ is a set of natural numbers, the existence of which has large cardinal strength. See [K]. $V = L$ implies $0^#$ does not exist.
Corollary 7. If $0^\#$ does not exist and $|M| \geq \lambda$ and $(D^\lambda)_M$ is compact, then $(D^\lambda)_M = D^\lambda$.

Proof. This follows immediately from Lemma 8. [KT] If $0^\#$ does not exist and $|M| \geq \lambda$, then $M \supseteq 2^\lambda$.

Corollary 9. If $0^\#$ does not exist and $X_M$ is homeomorphic to $D^\lambda \in M$, then $X = X_M$.

Proof. $|M| \geq |X_M| = 2^\lambda$, so $2^\lambda \subseteq M$, so as in the proof of Corollary 4, $X = X_M$.

By going to very large cardinals, we can find a $\lambda$ such that $(D^\lambda)_M$ is compact but not equal to $D^\lambda$.

Definition. A cardinal $\lambda$ is $\eta$-extendible if there is a $\zeta$ and an elementary embedding $j : V_{\lambda+\eta} \to V_\zeta$, with critical point $\lambda$.

See [K] to find out about such cardinals and about 2-huge ones, which we shall shortly introduce. Here we shall only mention that $\eta$-extendible cardinals are weaker in consistency strength than supercompact cardinals.

Observe that for $\eta \geq 1$,

$$D^{j(\lambda)} \cap j^{\#}V_{\lambda+\eta} = \{j(S) : j(S) \in D^{j(\lambda)} \text{ and } S \in V_{\lambda+\eta}\} = \{j(S) : S \in D^\lambda\} = j^{\#}D^\lambda.$$

Now if we want $D^{j(\lambda)} \in j^{\#}V_{\lambda+\eta}$, we need $\eta \geq 2$, for then $D^\lambda \in V_{\lambda+\eta}$, so $j(D^\lambda) = D^{j(\lambda)} \in j^{\#}V_{\lambda+\eta}$. We would be done if our definition of $X_M$ used “$V_\theta$” instead of “$H(\theta)$” since $j^{\#}V_{\lambda+\eta}$ is an elementary submodel of $V_\zeta$. To get $H(\theta)$, we use the fact that for inaccessible $\theta$, $V_\theta = H(\theta)$, and work with a larger cardinal.

Definition. $\lambda$ is 2-huge if there is an elementary embedding $j : V \to N$, an inner model, with critical point $\lambda$ such that $j(j(\lambda))N \subseteq N$. 


2-hugeness has considerably more consistency strength than supercompactness and assures us that $j^{``}V_{j(\lambda)} \in N$, as is $j^{``}D^\lambda$. $j^{``}V_{j(\lambda)}$ is an elementary submodel of $V_{j(j(\lambda))} = H(j(j(\lambda)))$ (since $j(j(\lambda))$ is inaccessible by elementarity). As before, $D^{j(\lambda)} \in j^{``}V_{j(\lambda)}$ and $D^{j(\lambda)} \cap j^{``}V_{j(\lambda)} = j^{``}D^\lambda$, which is compact $T_2$. $(D^{j(\lambda)})_{j^{``}V_{j(\lambda)}}$ is also a $T_2$ (since $D^{j(\lambda)}$ is and $T_2$ “goes down” [JT1]) topology on $D^{j(\lambda)} \cap j^{``}V_{j(\lambda)}$ that is weaker than the subspace topology and hence the two topologies are equal by compactness. Both $j^{``}V_{j(\lambda)}$ and $V_{j(j(\lambda))}$ are in $N$; the proof that the former is an elementary submodel of the latter can be carried out in $N$. Thus, $N$ thinks there is an elementary submodel $M$ of $H(j(j(\lambda)))$ such that $(D^{j(\lambda)})_M$ is compact $T_2$ but $\neq D^{j(\lambda)}$ (since $j(\lambda) > \lambda$). By elementarity, in $V$ there is an elementary submodel $M'$ of $H_{j(\lambda)}$ such that $(D^\lambda)_{M'}$ is compact $T_2$ but $\neq D^\lambda$. We have proved

**Theorem 8.** If $\lambda$ is 2-huge, then there is an elementary submodel $M$ such that $(D^\lambda)_M$ is compact but $\neq D^\lambda$.

There are several problems that remain:

What is the consistency strength of the existence of a $\lambda$ such that $(D^\lambda)_M$ is compact but $\neq D^\lambda$?

Could such a $\lambda$ be a successor cardinal?

Must the first such $\lambda$ be “larger” than merely ‘strongly inaccessible’?

After this paper was completed, Lucia Junqueira [JT2] proved that the condition that $|M| \geq \lambda$ can be removed from Corollary 7. It follows that the existence of a compact $(D^\lambda)_M \neq D^\lambda$ has consistency strength at least equal to the existence of $0^\#$. In [JT2] we discuss in general when $X_M$ compact implies $X_M = X$. In [T2] we investigate the particular case of when $X_M$ is a dyadic compactum; the results obtained generalize those in this paper. Of course there are simple examples in ZFC of $X$’s which are not equal to $X_M$, even if the latter is compact. For example, let $X$ be the one-point compactification of an uncountable discrete space and let $M$ be countable.
In [Ku], K. Kunen considerably sharpened the large cardinal bounds of Theorems 5 and 8.

References


Department of Mathematics, University of Toronto, Toronto, Ontario M5S 3G3 CANADA

E-mail address: tall@math.utoronto.ca