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ON SOME GENERALIZED LOCAL DEGREES

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Abstract

This note is devoted to the construction of a local degree for a new class of multi-valued mappings. This new considered class of multi-valued mappings satisfies the following condition: the projection onto the domain of a multi-valued mapping induces an epimorphism of the Čech homology groups with compact supports and coefficients in the field of rational numbers Q. The main theorem states that if the generalized local degree is different from 0, then the inclusion $\theta \in \Phi(x)$ has a solution.

1. Introduction

Topological degrees for various classes of multi-valued mappings have been studied by many authors: [1], [2], [3], [4], [7], [8], [11], [15], [17], [18] and others. The purpose of this paper is to build a local degree for a large class of multi-valued mappings Φ which take an open subset U of a n-dimensional oriented manifold M^n onto a n-dimensional vector space R^n. This class of multi-valued mappings satisfies the following condition: the projection onto the domain of multi-valued mapping induces a right-invertible

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homomorphism of the $\check{C}$ech homology groups with compact supports and coefficients in the field of rational numbers Q. The generalized local degree satisfies the usual properties of topological degree for multi-valued mappings, including the existence theorem, which could be formulated as follows: if the generalized local degree is different from 0, then the set of the solutions of the inclusion $\theta \in \Phi(x)$ is nonempty.

This paper is comprised of three sections. In the first and second sections, we give notions, such as the concept of the n-decomposing single-valued maps and the local degree for an n-admissible pair of single-valued maps. In the last section we define and study the generalized local degree for admissible multi-valued vector fields.

2. Preliminaries on n-decomposing Maps

Let G_1 and G_2 be two additive Abelian groups and let $\tau : G_1 \longrightarrow G_2$ be a homomorphism.

Definition 2.1. A homomorphism τ is called an r-homomorphism if τ has a right inverse.

We denote by $\Omega(\tau)$ the set of right-inverse homomorphisms of τ. The following properties are satisfied.

Proposition 2.1. *A homomorphism $\tau : G_1 \longrightarrow G_2$ is an r-homomorphism if and only if the following conditions are satisfied:*

1. *τ is an epimorphism;*
2. *$G_1 = Ker\,\tau \oplus \mathcal{G}$, where $\mathcal{G}$ is a subgroup of G_1.*

Proposition 2.2. *If G_1 and G_2 are two vector spaces over a field K and if $\tau : G_1 \longrightarrow G_2$ is an epimorphism then τ is an r-homomorphism.*

Proposition 2.3. *Let $\tau_1 : G_1 \longrightarrow G_2$ and $\tau_2 : G_2 \longrightarrow G_3$ be two r-homomorphisms. Then their composition $\tau = \tau_2 \circ \tau_1 : G_1 \longrightarrow G_3$ is also an r-homomorphism.*

More properties of r-homomorphisms can be found in [5].

Let X and Y be two Hausdorff topological spaces, and A and B two subsets of X and Y, respectively. Let $f : (X, A) \longrightarrow (Y, B)$ be a continuous, single-valued map of pairs of topological spaces, i.e. $f : X \longrightarrow Y$ and $f(A) \subseteq B$.

Let $H(X, A)$ be the Čech homology groups with compact supports and coefficients in the field of rationals Q. The properties of this groups can be found in [15], [10], [19].

Definition 2.2. We say that the single-valued map f is right-decomposing in the rank $n \geq 0$ on the pair of spaces (Y, B) if the homomorphism $f_* : H_n(X, A) \to H_n(Y, B)$ induced by f in the homology groups is an r-homomorphism.

In this case we say that f is n-decomposing on (Y, B).

The set of the right-inverse homomorphisms of f_* will be denoted by $\Omega(f_*)$.

Let us consider some examples.

Proposition 2.4. *Let $f : (X, A) \longrightarrow (Y, B)$ be a continuous single-valued map which has a continuous right-inverse. Then f is n-decomposing on (Y, B) for every rank $n \geq 0$.*

Proof. Of course, if $(f \circ g) = Id_{(Y,B)}$ then $(f \circ g)_* = (Id_{(Y,B)})_*$ from which we deduce that $f_* \circ g_* = Id_{H_n(Y,B)}$. This equality is true for every $n \geq 0$. □

Proposition 2.5. *Let $f : (X, A) \longrightarrow (Y, B)$ be a Vietoris map, this means that:*

1. *f is continuous, proper, surjective;*
2. *$f^{-1}(B) = A$;*
3. *$f^{-1}(y)$ is Q-acyclic for every $y \in Y$.*

Then the map f is n-decomposing on (Y, B) for every $n \geq 0$.

Proof. From the Vietoris map theorem [15], we know that $f_* : H_n(X, A) \longrightarrow H_n(Y, B)$ is an isomorphism, so that f is n-decomposing on (Y, B) for every $n \geq 0$. □

Proposition 2.6. *If $f : (X, A) \longrightarrow (Y, B)$ is a single-valued continuous map which has a right homotopic inverse $g : (Y, B) \longrightarrow (X, A)$ then f is n-decomposing on (Y, B) for every $n \geq 0$.*

Proof. This is a direct consequence of the fact that $f \circ g \sim Id_{(Y,B)}$. In fact, $(f \circ g)_* = f_* \circ g_* = Id_{H_n(Y,B)}$, hence $g_* \in \Omega(f_*)$. □

Now we shall give some properties of decomposing maps.

Proposition 2.7. *Let $f : (X, A) \longrightarrow (Y, B)$ be a continuous single-valued map. If there exists a continuous single-valued map $g : (Z, C) \longrightarrow (X, A)$ such that the single-valued map $(f \circ g)$ is n-decomposing on (Y, B) then f is also n-decomposing on (Y, B).*

Proof. Let us consider the following commutative diagram:

$$\begin{array}{ccc} & (Z, C) & \\ g \swarrow & & \searrow (f \circ g) \\ (X, A) & \xrightarrow{f} & (Y, B) \end{array}$$

This diagram induces the following commutative diagram in homology:

$$\begin{array}{ccc} & H_n(Z, C) & \\ g_* \swarrow & & \searrow (f \circ g)_* = f_* \circ g_* \\ H_n(X, A) & \xrightarrow{f_*} & H_n(Y, B) \end{array}$$

Let $\tau \in \Omega\left((f \circ g)_*\right)$, then we have the following commutative diagram:

$$\begin{array}{ccccc} H_n(Y, B) & \xrightarrow{\tau} & H_n(Z, C) & \xrightarrow{f_* \circ g_*} & H_n(Y, B) \\ & & g_* \downarrow & & \Updownarrow \\ & & H_n(X, A) & \xrightarrow{f_*} & H_n(Y, B) \end{array}$$

From the above diagram we obtain the equalities

$$f_* \circ (g_* \circ \tau) = (f_* \circ g_*) \circ \tau = (f \circ g)_* \circ \tau = Id_{H_n(Y,B)},$$

yielding $(g_* \circ \tau) \in \Omega(f_*)$. □

Corollary 2.8. *Let $f : (X, A) \longrightarrow (Y, B)$ be a continuous single-valued map and $(Z, C) \subseteq (X, A)$. If the restriction of f to (Z, C) is n-decomposing on (Y, B) then f is also n-decomposing on (Y, B).*

Proof. The proof is a consequence of the above proposition and the fact that $f \circ i = \tilde{f}$ where $i : (Z, C) \longrightarrow (X, A)$ is the natural injection. □

Proposition 2.9. *Let $f : (X, A) \longrightarrow (Y, B)$ be a single-valued map which is n-decomposing on (Y, B) and $g : (Y, B) \longrightarrow (Z, C)$ be a single-valued map which is n-decomposing on (Z, C). Then the single-valued map $(g \circ f)$ is n-decomposing on (Z, C).*

The proof is obvious.

3. Generalized Local Degree for Pairs of Single-valued Maps

We consider the following:

1. M^n is a n-dimensional oriented manifold with a fixed orientation $O \in \Gamma(M^n, Q)$;
2. U is an open subset of M^n;
3. R^n is a n-dimensional topological vector space with origin θ and a fixed orientation;
4. X is an arbitrary Hausdorff topological space.

The basic properties of oriented manifolds and of fundamental classes of compacta can be found in [8].

Let (f, g) be a pair of single-valued maps defined in the following way:

$$U \xleftarrow{f} X \xrightarrow{g} R^n.$$

Then we say that the pair (f, g) is acting from U to R^n.

Let K be a compact subset of U which contains $f \circ g^{-1}(\theta)$, then we can consider the following diagram:

$$(U, U \backslash K) \xleftarrow{f} (X, X \backslash f^{-1}(K)) \xrightarrow{g} (R^n, R^n \backslash \theta). \qquad (3.1)$$

Definition 3.1. A pair of single-valued maps (f, g) from U to R^n is called n-admissible on $(U, U \backslash K)$ if f is n-decomposing on $(U, U \setminus K)$.

The set of all n-admissible pairs on $(U, U \backslash K)$ is denoted by $\mathcal{AP}(U, U \setminus K)$.

We shall give a construction of a local degree for this class of pairs of single-valued maps.

From diagram (3.1) we obtain the following diagram:

$$H_n(U, U \backslash K) \xleftarrow{f_*} H_n(X, X \backslash f^{-1}(K)) \xrightarrow{g_*} H_n(R^n, R^n \backslash \theta)$$

Let $\Omega\,(f_*, U, K)$ be the set of the right-inverse homomorphisms of the r-homomorphism f_*.

For $\sigma \in \Omega\,(f_*, U, K)$ we obtain the following diagram:

$$\begin{array}{ccccc} H_n(U, U \backslash K) & \xleftarrow{f_*} & H_n(X, X \backslash f^{-1}(K)) & \xrightarrow{g_*} & H_n(R^n, R^n \backslash \theta) \\ & \sigma \searrow & \| & & \\ & & H_n(X, X \backslash f^{-1}(K)) & & \end{array}$$

Let O_K and O_θ be the fundamental classes of the compacta K and $\{\theta\}$ respectively. Then we have:

$$g_* \circ \sigma(O_K) = k \cdot O_\theta$$

where $k \in Q$.

Definition 3.2. The rational number k is called the local degree of (f, g) associated to $\sigma \in \Omega\,(f_*, U, K)$ and it will be denoted by $\gamma_\sigma\,[(f, g), U, K]$.

If there is no confusion about which spaces we are considering, we will simply write $\gamma_\sigma(f, g)$.

Definition 3.3. The set of rationals

$$\nabla\,[(f, g), U, K] = \{\gamma_\sigma(f, g) \mid \sigma \in \Omega\,(f_*, U, K)\}$$

is called the generalized local degree of the pair

$$(f, g) \in \mathcal{AP}(U, U \setminus K).$$

We shall study some properties of the generalized local degree.
Suppose that:

1) U and V are two open subsets of M^n;

2) K and K_1 are two compact subsets of M^n;

3) $K \subset K_1 \subset V \subset \overline{V} \subset U$;

4) X, X_1 are Hausdorff topological spaces.

Suppose futher that we have two pairs (f, g) and (f_1, g_1) of single-valued maps defined in the following way:

$$(U, U\backslash K) \xleftarrow{f} (X, X\backslash f^{-1}(K)) \xrightarrow{g} (R^n, R^n\backslash\theta)$$

and

$$(V, V\backslash K_1) \xleftarrow{f_1} (X_1, X_1\backslash f_1^{-1}(K_1)) \xrightarrow{g_1} (R^n, R^n\backslash\theta)$$

Proposition 3.1. *Let* $\alpha : (X_1, X_1\backslash f_1^{-1}(K_1)) \longrightarrow (X, X\backslash f^{-1}(K))$ *be a single-valued map such that the following diagram is commutative:*

$$\begin{array}{ccccc}
(V, V\backslash K_1) & \xleftarrow{f_1} & (X_1, X_1\backslash f_1^{-1}(K_1)) & \xrightarrow{g_1} & (R^n, R^n\backslash\theta) \\
i\downarrow & & \downarrow\alpha & & \| \\
(U, U\backslash K) & \xleftarrow{f} & (X, X\backslash f^{-1}(K)) & \xrightarrow{g} & (R^n, R^n\backslash\theta)
\end{array}$$

where i is the natural injection. If the pair (f_1, g_1) is n-admissible on $(V, V\backslash K_1)$, then the pair (f, g) is n-admissible on the pair $(U, U\backslash K)$ and we have the inclusion

$$\nabla\left[(f,g),U,K\right] \supset \nabla\left[(f_1,g_1),V,K_1\right].$$

Proof. Consider the commutative diagram:

$$\begin{array}{ccccc} H_n(V,V\backslash K_1) & \stackrel{f_{1*}}{\longleftarrow} & H_n(X_1,X_1\backslash f_1^{-1}(K_1)) & \stackrel{g_{1*}}{\longrightarrow} & H_n(R^n,R^n\backslash\theta) \\ i_*\downarrow & & \downarrow\alpha_* & & \| \\ H_n(U,U\backslash K) & \stackrel{f_*}{\longleftarrow} & H_n(X,X\backslash f^{-1}(K)) & \stackrel{g_*}{\longrightarrow} & H_n(R^n,R^n\backslash\theta) \end{array}$$

From the excision theorem [19] it follows that the homomorphism i_* is an isomorphism. Therefore, $i_*(O_{K_1}) = O_K$. Let $\sigma \in \Omega\left(f_{1_*}, V, K_1\right)$, in this case the homomorphism $\sigma_1 = \alpha_* \circ \sigma \circ i_*^{-1}$ is a right-inverse of the homomorphism f_*. This means that $(f,g) \in \Omega\left(f_*, U, K\right)$.

Now let us prove the inclusion $\nabla[(f,g),U,K] \supset \nabla\left[(f_1,g_1),V,K_1\right]$. Let $\sigma \in \Omega\left(f_{1_*}, V, K_1\right), \gamma_\sigma(f,g) \in \nabla\left[(f_1,g_1),V,K_1\right]$ then we know that

$$\sigma_1 = \alpha_* \circ \sigma \circ i_*^{-1} \in \Omega\left(f_*, U, K\right),$$

and we have the equalities:

$$g_*(\sigma_1(O_K)) = g_*(\alpha_*(\sigma(O_{K_1}))) = g_{1_*}(\sigma(O_{K_1})) = \gamma_\sigma(f_1,g_1)\cdot O_\theta. \quad \square$$

Consider the following diagram $U \stackrel{f}{\longleftarrow} X \stackrel{g}{\longrightarrow} R^n$. Let K be a compact subset of U containing $f \circ g^{-1}(\theta)$ and let $h : (R^n, R^n\backslash\theta) \longrightarrow (R^n, R^n\backslash\theta)$ be a continuous single-valued map. For h we can define $\deg h \in Q$, where $h_n^*(O_\theta) = \deg h \cdot O_\theta$. Notice that in this case $f \circ (h\circ g)^{-1}(\theta) \subseteq K$.

Proposition 3.2. *If $(f,g) \in \mathcal{AP}(U, U\setminus K)$ then $(f, h\circ g) \in \mathcal{AP}(U, U\setminus K)$ and*

$$\nabla\left((f,h\circ g),U,K\right) = \deg h \cdot \nabla\left((f,g),U,K\right).$$

Proof. We have the following diagram:

$$(U, U\backslash K) \xleftarrow{f} (X, X\backslash f^{-1}(K)) \xrightarrow{g} (R^n, R^n\backslash\theta) \xrightarrow{h} (R^n, R^n\backslash\theta),$$

from which we deduce:

$$\begin{array}{ccccc} H_n(U, U\backslash K) & \xleftarrow{f_*} & H_n(X, X\backslash f^{-1}(K)) & \xrightarrow{g_*} & H_n(R^n, R^n\backslash\theta) \\ & & & & \downarrow h_* \\ & & & & H_n(R^n, R^n\backslash\theta) \end{array}$$

Let $\sigma \in \Omega\,(f_*, U, K)$, then:

$$\gamma_\sigma(f, h\circ g)\cdot O_\theta = (h\circ g)_* \circ \sigma(O_K) = h_* \circ g_* \circ \sigma(O_K) =$$
$$= h_*(\gamma_\sigma(f,g)\cdot O_\theta) = \gamma_\sigma(f,g)\cdot \deg h \cdot O_\theta.$$

Hence $\gamma_\sigma(f, h\circ g) = \deg h\,\cdot\,\gamma_\sigma(f,g)$. This yields:

$$\nabla\,[(f, h\circ g), U, K] = \{\gamma_\sigma(f, h\circ g) \mid \sigma \in \Omega\,(f_*, U, K)\} =$$
$$= \{\deg h\cdot\gamma_\sigma(f,g) \mid \sigma\in\Omega\,(f_*, U, K)\} = \deg h\cdot\nabla\,[(f,g), U, K]. \quad \square$$

Proposition 3.3. *Let* (f, g) *be an element of* $\mathcal{AP}(U, U\setminus K)$ *such that* $f\circ g^{-1}(\theta) = \emptyset$. *Then* $\nabla\,[(f,g), U, K] = \{0\}$.

Proof. Under the above conditions we can consider the commutative diagram:

$$\begin{array}{ccccc} (U, U\backslash K) & \xleftarrow{f} & (X, X\backslash f^{-1}(K)) & \xrightarrow{g} & (R^n, R^n\backslash\theta) \\ & & & \widetilde{g} \searrow & \uparrow i \\ & & & & (R^n\backslash\theta, R^n\backslash\theta) \end{array}$$

where $\widetilde{g} = g$. Then

$$\gamma_\sigma(f,g)\cdot O_\theta = g_*\circ\sigma(O_K) = i_*\circ\widetilde{g}_*\circ\sigma(O_K).$$

Using the fact that $\widetilde{g}_*$ is the trivial homomorphism, we obtain that $\gamma_\sigma(f,g) = 0$ for every $\sigma\in\Omega\,(f_*, U, K)$. $\square$

Corollary 3.4. *If* $(f,g) \in \mathcal{AP}(U, U \setminus K)$ *such that*

$$\nabla\left[(f,g),U,K)\right] \not= \{0\} \text{ then } f \circ g^{-1}(\theta) \not= \emptyset.$$

Definition 3.4. Two n-admissible pairs (f_i, g_i) $(i = 0,1)$ on $(U, U \backslash K)$ with:

$$(U, U\backslash K) \xleftarrow{f_i} (X, X\backslash K) \xrightarrow{g_i} (R^n, R^n\backslash\theta) \qquad (3.2)$$

are called homotopic if there exists a pair of continuous single-valued maps (φ, ψ) such that:

$$(U, U\backslash K) \xleftarrow{\varphi} (X \times [0,1], K \times [0,1]) \xrightarrow{\psi} (R^n, R^n\backslash\theta)$$

satisfying the equalities:

$$\varphi(x,0) = f_0(x), \psi(x,0) = g_0(x); \varphi(x,1) = f_1(x); \psi(x,1) = g_1(x).$$

We will denote this property by $(f_0, g_0) \overset{K}{\sim} (f_1, g_1)$.

Proposition 3.5. *If* $(f_0, g_0) \overset{K}{\sim} (f_1, g_1)$ *then* $\nabla\left[(f_0,g_0),U,K\right] = \nabla\left[(f_1,g_1),U,K\right]$.

Proof. This is a consequence of the equalities $f_{0*} = f_{1*}$and $g_{0*} = g_{1*}$. □

4. Generalized Local Degree for Multi-valued Mappings

Let X and Y be two topological spaces. A multi-valued mapping taking X to Y is a relation F which associates to each element $x \in X$ a nonempty subset $F(x) \subset Y$.

Let $K(Y)$ be the collection of all nonempty compact subsets of Y; the notation $F : X \longrightarrow K(Y)$ expresses the fact that F associates to each element of X a nonempty compact subset of Y.

The set $\Gamma_X(F) = \{(x,y) \mid x \in X, y \in F(x)\} \subset X \times Y$ is called the graph of the multi-valued mapping F on X. In this case we can define two natural projections: $t_F : \Gamma_X(F) \longrightarrow X, t(x,y) = x$; $r_F : \Gamma_X(F) \longrightarrow Y,\ r(x,y) = y$ for every $(x,y) \in \Gamma_X(F)$.

For each element $x \in X$ we have the equality $F(x) = r_F(t_F^{-1}(x))$. This means that a multi-valued mapping is well defined by the quintuple $(X, Y, \Gamma_X(F), t_F, r_F)$.

Let M^n be a n-dimensional oriented manifold, U an open subset of M^n and R^n a n-dimensional topological vector space with origin θ and a fixed orientation. Let $F : U \longrightarrow K(R^n)$ be an upper semi-continuous mapping.

Definition 4.1. A multi-valued mapping $F : U \longrightarrow K(R^n)$ is called an admissible multi-valued vector field in a neighborhood of a compactum $K \subset U$ if the pair of projections (t_F, r_F):

$$U \xleftarrow{t_F} \Gamma_U(F) \xrightarrow{r_F} R^n$$

satisfies the following conditions:

1. $K \supset \{x \in U \mid \theta \in F(x)\}$;
2. the pair (t_F, r_F) is n-admissible on $(U, U \backslash K)$.

The set of all admissible multi-valued vector fields on a neighborhood U of the compactum K will be denoted by $\mathcal{A}dm(U, K)$.

Let M^n be an oriented manifold with a fixed orientation $O \in \Gamma(M^n, Q)$, and U an open subset of M^n. Let $\Phi : U \longrightarrow K(R^n)$ be an admissible multi-valued mapping in a neighborhood U of a compactum K.

Definition 4.2. The generalized local degree of an admissible multi-valued mapping Φ in the neighborhood U of K is defined to be the following set of rationals:

$$\begin{aligned} Deg(\Phi, U, K) &= \nabla\left[(t_\Phi, r_\Phi), U, K\right] \\ &= \{\gamma_\sigma(t_\Phi, r_\Phi) \mid \sigma \in \Omega\left((t_\Phi)_*, U, K\right)\}, \end{aligned}$$

where t_Φ and r_Φ are the natural projections of Φ:

$$(U, U \backslash K) \xleftarrow{t_\Phi} (\Gamma_U(F); \Gamma_{U\backslash K}(\Phi)) \xrightarrow{r_\Phi} (R^n, R^n \backslash \theta).$$

Let us describe some properties of this generalized local degree.

Theorem 4.1. *If* $Deg(\Phi, U, K) \;\; / =\{0\}$ *then the inclusion* $\theta \in \Phi(x)$ *admits at least one solution in* K.

Proof. This is a consequence of the corresponding property of the local degree for n-admissible pairs (see Corollary 3.4). □

Let Z be a Hausdorff topological space, $f : Z \longrightarrow U$ a surjective, proper, continuous, single-valued map, and $g : Z \longrightarrow R^n$ a continuous single-valued map.

In the case when we have $g \circ f^{-1}(x) = \Phi(x)$ for every $x \in U$, we say that the quintuple $\mu = (U, R^n, Z, f, g)$ is a representation of the multi-valued mapping Φ.

We say that the representation μ is admissible if (f, g) is a n-admissible pair on $(U, U\backslash K)$.

Let $K_1 \subset U$ be a compact subset of U, $K \subset K_1$. Let (f, g) be a n-admissible pair on $(U, U\backslash K_1)$. Then, we can consider also the diagram:

$$(U, U\backslash K_1) \xleftarrow{f} (Z, Z\backslash f^{-1}(K_1)) \xrightarrow{g} (R^n, R^n\backslash\theta)$$

Let $\Omega\,(f_*, U, K_1)$ be the set of the right-inverse homomorphisms of f_*,

$$f_* : H_n(Z, Z\backslash f^{-1}(K_1)) \longrightarrow H_n(U, U\backslash K_1).$$

We can define:

$$Deg_\mu(\Phi, U, K_1) = \nabla\,[(f, g), U, K_1] = \{\gamma_\sigma(f, g) \mid \sigma \in \Omega\,(f_*, U, K_1)\}.$$

Proposition 4.2. $Deg_\mu(\Phi, U, K_1) \subseteq Deg(\Phi, U, K)$.

Proof. We have the commutative diagram:

$$\begin{array}{ccccc}
(U, U\backslash K_1) & \xleftarrow{f} & (Z, Z\backslash f^{-1}(K_1)) & \xrightarrow{g} & (R^n, R^n\backslash\theta) \\
\downarrow i & & \downarrow \alpha & & || \\
(U, U\backslash K) & \xleftarrow{t_\Phi} & (\Gamma_U(F); \Gamma_{U\backslash K}(\Phi)) & \xrightarrow{r_\Phi} & (R^n, R^n\backslash\theta)
\end{array}$$

where $\alpha(z) = (f(z); g(z))$ for each $z \in Z$. We finish the proof by applying proposition 3.1. □

Corollary 4.3. *If* $\Phi : U \to K(R^n)$ *has* $\mu = (U, R^n, Z, f, g)$ *as an admissible representation on* $(U, U\backslash K_1)$ *and if* $Deg_\mu(\Phi, U, K_1) \neq \{0\}$ *then there exists* $x_0 \in U$ *such that* $\theta \in \Phi(x_0)$.

Let $\Phi : U \to K(R^n)$ be a multi-valued vector field admissible in a neighborhood of a compactum K. Let $g : (R^n, R^n\backslash\theta) \longrightarrow (R^n, R^n\backslash\theta)$ be a continuous single-valued map.

Let $\Psi : U \to K(R^n)$ be the multi-valued mapping defined as the composition $g \circ \Phi = \Psi$.

Proposition 4.4. *The multi-valued mapping* $\Psi : U \to K(R^n)$ *is also admissible on* $(U, U\backslash K)$ *and we have the inclusion:*

$$(\deg g) \cdot Deg(\Phi, U, K) \subseteq Deg(\Psi, U, K).$$

Proof. Let us consider the following commutative diagram:

$$\begin{array}{ccccc} & & (\Gamma_U(\Phi); \Gamma_{U\backslash K}(\Phi)) & \xrightarrow{r_\Phi} & (R^n, R^n\backslash\theta) \\ & t_\Phi \swarrow & & & \\ (U, U\backslash K) & & \downarrow \alpha & & \downarrow g \\ & t_\Psi \nwarrow & & & \\ & & (\Gamma_U(\Psi); \Gamma_{U\backslash K}(\Psi)) & \xrightarrow{r_\Psi} & (R^n, R^n\backslash\theta) \end{array}$$

where $\alpha(x, y) = (x, g(y))$ for every $(x, y) \in \Gamma_U(\Phi)$.

It is obvious that the quintuple $\mu = (U, R^n, \Gamma_U(\Phi), t_\Phi, g \circ r_\Phi)$, is an admissible representation of the multi-valued vector field Ψ. So Ψ is an admissible multi-valued vector field and $Deg_\mu(\Psi, U, K) \subseteq Deg(\Psi, U, K)$. From the definition of the generalized local degree $Deg_\mu(\Psi, U, K)$ we obtain that $Deg_\mu(\Psi, U, K) = (\deg\ g) \cdot Deg(\Phi, U, K)$. Thus,

$$(\deg\ g) \cdot Deg(\Phi, U, K) \subset Deg(\Psi, U, K) \qquad \square$$

Proposition 4.5. *Let* Φ_0 *and* $\Phi_1 : U \to K(R^n)$ *be two multi-valued mappings, which satisfy the following conditions:*

1. $\Phi_1(x) \subset \Phi_0(x)$ *for every element* $x \in U$;

2. *the set* $K = \{x \in U \mid \theta \in \Phi_0(x)\}$ *is compact in* U *and* (t_{Φ_1}, r_{Φ_1}) *is a* n*-admissible pair on* $(U, U\backslash K)$;

3. Φ_0 *is admissible on the pair of topological spaces* (U, K).

Then, Φ_1 *is also an admissible multi-valued vector fields and*

$$Deg(\Phi_1, U, K) \subseteq Deg(\Phi_0, U, K).$$

Proof. We can consider the following commutative diagram:

$$\begin{array}{ccccc} (U, U\backslash K) & \xleftarrow{t_{\Phi_1}} & (\Gamma_U(\Phi_1); \Gamma_{U\backslash K}(\Phi_1)) & \xrightarrow{r_{\Phi_1}} & (R^n, R^n\backslash\theta) \\ || & & \downarrow i & & || \\ (U, U\backslash K) & \xleftarrow{t_{\Phi_0}} & (\Gamma_U(\Phi_0); \Gamma_{U\backslash K}(\Phi_0)) & \xrightarrow{r_{\Phi_0}} & (R^n, R^n\backslash\theta) \end{array}$$

where i is the natural injection. The statement then follows from the proposition 3.1. □

Theorem 4.6. *Let* $\Phi : U \longrightarrow K(R^n)$ *be an upper semi-continuous multi-valued vector field admissible in a neighborhood of a compactum* K. *Suppose that the following conditions are satisfied:*

1. *there exists an open subset* V *of* M^n *such that* $K \subset V \subset \overline{V} \subset U$;

2. *for every element* $x \in V$, $\Phi(x)$ *is acyclic.*

Then, $Deg(\Phi, U, K) = \{k\}$, *where* k *is defined by the equality:*

$$r_{\Phi*} \circ t_{\Phi*}^{-1}(O_K) = k \cdot O_\theta$$

Proof. This is a consequence of the definition of the generalized local degree and the Vietoris Maps Theorem, (see [15]). □

In the case when $\Phi : U \longrightarrow K(R^n)$ satisfies the conditions of Theorem 4.6 we say that it is acyclic in the neighborhood of the compactum K.

Corollary 4.7. *Let $\Phi_0, \Phi : U \longrightarrow K(R^n)$ be two multi-valued mappings satisfying the following conditions:*

1. $\Phi(x) \subset \Phi_0(x)$ *for every* $x \in U$;
2. *the set* $K = \{x \in U \mid \theta \in \Phi_0(x)\}$ *is compact in* U *and the pair* (t_Φ, r_Φ) *is* n*-admissible on* $(U, U\backslash K)$;
3. Φ_0 *is a multi-valued vector field acyclic in a neighborhood of the compactum* K.

Then $Deg(\Phi, U, K) = \{k\}$, *where* k *is defined by the equality:*

$$r_{\Phi_0 *} \circ t^{-1}_{\Phi_0 *}(O_K) = k \cdot O_\theta.$$

References

[1] Borisovich Yu.G., Gel'man B.D., Myshkis A.D., Obukhovskii V.V., *Topological methods in the theory of fixed points of multi-valued mappings*, Russian Math. Surveys **35** 1980, 65-143.

[2] Borisovich Yu.G., Gel'man B.D., Myshkis A.D., Obukhovskii V.V., *Multi-valued mappings*, J.Soviet Math. **24** 1984, 719-791.

[3] Borisovich Yu.G., *A modern approach to the theory of topological characteristics of non-linear operators. I, II*; *Lecture Notes in Mathematics: Global Analysis - Studies and Appl. III, IV*, Springer-Verlag, **1334** 1988, 199-220; **1453** 1990, 21-50.

[4] Borisovich Yu.G., *Topological characteristics and the investigation of solvability for nonlinear problems*, Izvestiya VUZ'ov, Mathematics N. 2, 1997, 3-23 (in Russian).

[5] Borsuk K., *Theory of retracts*, Monographie Matematyczne PAN, PWN, Warszawa, 1967.

[6] Benkafadar N.M., Gel'man B.D., *On a local degree of one class of multivalued vector fields in infinite-dimensional Banach spaces*, Abstract And Applied Analysis **1**, N. 4 1996, 381-396.

[7] Dzedzej Z., *Fixed point index theory for a class of nonacyclic multivalued maps*, Rospr. Math. N. 253, Warszawa, 1985.

[8] Dold A., *Lectures on Algebraic Topology*, Springer-Verlag, 1972.

[9] Eilenberg S., Montgomery D., *Fixed point theorems for multi-valued transformations*, Amer. J. Math. **58** 1946, 214-222.

[10] Eilenberg S., Steenrod N., *Foundations of Algebraic Topology*, Princeton, 1952.

[11] Granas A., *Sur la notion du degré topologique pour une certaine classe de transformations multivalentes dans des espaces de Banach*, Bull. Acad. Polon. Sci. **7** 1959, 181-194.

[12] Granas A., Jaworowski J.W., *Some theorems on multi-valued maps of subsets of the Euclidean space*, Bull. Acad. Polon. Sci. No6 (1965), 277-283.

[13] Dugundji J., Granas A., *Fixed point theory*, Monographie Matematyczne PAN, PWN, Warszawa, **1** 1982.

[14] Gel 'man B.D., *Generalized degree for multivalued mappings*, Lecture Notes in Math., Springer-Verlag, **1520** 1992, 174-192.

[15] Gorniewicz L., *Homological methods in fixed point theory of multivalued maps*, Dissertations Math. **129**, Warszawa, 1979.

[16] Kryszewski W., *Topological and approximation methods of degree theory of set-valued maps*, Dissert. Math. **336**, Warszawa, 1994.

[17] Kucharski Z., *A coincidence index*, Bull. Acad. Polon. Sci., Ser. Sci. Math., Astron. et Phys. **24** , N. 4, 1976, 245-252.

[18] Siegberg Z., Skordev G., *Fixed point index and chain approximation*, Pacific J. Math. **102**, N. 2, 1982, 455-486.

[19] Spanier E.H., *Algebraic Topology*, McGraw-Hill, 1966.

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