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ON SOME COVERING PROPERTIES OF CONTINUA

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Abstract

In this survey paper various old and new results related to some covering properties of continua are recalled and discussed. The considered properties and concepts are: the covering property, the covering property hereditarily, the strong covering property, as well as the minimal closed covers of continua. Many relations between them are studied.

A family $\mathcal{F}$ of nonempty closed subsets of a continuum X is said to *cover* X (or to be a *cover of* X) provided that the union of $\mathcal{F}$ is X. The *hyperspace of* (nonempty) *subcontinua* of X equipped with the Hausdorff metric (see [5, 2, p. 9] and [9, 0.1, p. 1]) is denoted by $C(X)$, and $F_1(X) \subset C(X)$ stands for the hyperspace of singletons. A *Whitney map* for $C(X)$ means a mapping $\mu : C(X) \to [0, \infty)$ such that $\mu(\{x\}) = 0$ for each $x \in X$, and if $A \subsetneq B$, then $\mu(A) < \mu(B)$. For each $t \in [0, \mu(X)]$ the preimage $\mu^{-1}(t)$ is called a *Whitney level* for $C(X)$. The reader is referred to [5] and [9] for these and other concepts used in this paper.

Since the paper collects results due to various authors (sometimes in one theorem), it is hard to attach any particular author's name to a quoted result (whose parts can be due to different authors). Thus, as a rule, no authors' names are joined to the theorems. Instead, we cite the sources of the quoted results very carefully.

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Let X be a continuum and let μ be a given Whitney map for $C(X)$. Then X is said to have the *covering property relative to* μ, writing $X \in CP(\mu)$, provided that for each $t \in [0, \mu(X)]$ no proper subcontinuum of $\mu^{-1}(t)$ covers X. A continuum X is said to have the *covering property*, writing $X \in CP$, provided that $X \in CP(\mu)$ for each Whitney map $\mu : C(X) \to [0, \infty)$. It is shown in [11, Proposition 13, p. 159] that the covering property is independent of the choice of any Whitney map μ for $C(X)$, i.e., the following result holds.

Theorem 1. *Let X be a continuum. If $X \in CP(\mu)$ for some μ, then $X \in CP$.*

A continuum X is said to have the *covering property hereditarily*, writing $X \in CPH$, provided that each of nondegenerate subcontinua of X has the covering property, i.e., if $K \in C(X) \setminus F_1(X)$, then $K \in CP$ (see [11, property (2), p. 159] and compare [9, p. 486]). The property $CPH(\mu)$, for a fixed Whitney map μ, is defined similarly to $CP(\mu)$. Then Theorem 1 implies a corollary (see [11, Corollary 14, p. 160]).

Corollary 2. *Let X be a continuum and let μ be a Whitney map for $C(X)$. Then $X \in CPH(\mu)$ implies $X \in CPH$.*

Obviously, for each continuum X, if $X \in CPH$ then $X \in CP$ but not conversely, as an example shows of the union of a circle and a ray spiraling down on it, [11, p. 160].

In [9, Theorem 14.73.3, p. 482] it is shown that if X is a continuum, then $X \in CP$ if and only if for each Whitney map each Whitney level is an irreducible continuum. As consequences of this result and of [6, Section 6, 2., p. 179] we get the following corollaries (see [9, Theorems 14.73.1, p. 478, and 14.14.1, p. 418]).

Corollary 3. *Let X be a continuum. If $X \in CP$, then X is irreducible and unicoherent.*

Corollary 4. *Let X be a continuum. If $X \in CPH$, then X is hereditarily irreducible and hereditarily unicoherent.*

A further progress on continua having the covering property can be found in [9, Theorem 14.73.21, p. 497], [4, Theorem 3.2, p. 178], [3, Theorem 2.2, p. 199], and [12, Theorem, p. 294]. These results tie several conditions defined in very different ways, so they are of a special importance. To formulate them it is needed to recall some definitions.

Let X be a continuum. Define a function $C^* : C(X) \to C(C(X))$ by $C^*(A) = C(A)$. It is known that for any continuum X the function C^* is upper semicontinuous, [9, Theorem 15.2, p. 514], and it is continuous on a dense G_δ subset of $C(X)$, [9, Corollary 15.3, p. 515]. A continuum X is said to be *C^*-smooth at* $A \in C(X)$ provided that the function C^* is continuous at A. A continuum X is said to be *C^*-smooth* provided that the function C^* is continuous on $C(X)$, i.e., at each $A \in C(X)$, see [5, Section 35, p. 253] and [9, Definition 5.15, p. 517].

A continuum X is said to be *absolutely C^*-smooth* provided that if X is a subcontinuum of a continuum Z, then the function $C^* : C(Z) \to C(C(X))$ is continuous at X, see [5, Section 35, p. 253] and [4, p. 178]. Each arc-like continuum is C^*-smooth, [9, Theorem 15.13, p. 525]. C^*-smoothness implies hereditary unicoherence, see [3, Corollary 3.4, p. 203] and [9, Note 1, p. 530]. Thus each arcwise connected C^*-smooth continuum is a dendroid, [9, Theorem 15.19, p. 528]. Further, a locally connected continuum is C^*-smooth if and only if it is a dendrite, [9, Theorem 15.11, p. 522].

A continuum Y is defined to be in *Class (W)* provided that for each continuum X and each surjective mapping $f : X \to Y$ the following condition is satisfied:

($*$) for each continuum $B \subset Y$ there is a continuum $A \subset X$ such that $f(A) = B$.

Mappings $f : X \to Y$ satisfying ($*$) are called *weakly confluent*. See [5, Remark, p. 198], [9, p. 497] and [9, Chapter 13, Section 6, p. 299-305 and Exercise 13.71, p. 310].

A *compactification* of a space S is a compact space Z containing a homeomorphic copy S' of S as a dense subset. The set $Z \setminus S'$ is called a *remainder* of the compactification.

The above mentioned results can be summarized as follows (see [5, conditions (a)-(d), p. 254, and Theorem 67.1, p. 320], where a proof of the equivalences is presented).

Theorem 5. *For a continuum X the following statements are equivalent:*

(5.1) *X is absolutely C^*-smooth;*

(5.2) *$X \in Class(W)$;*

(5.3) *each compactification Y of $[0,1)$ with X as the remainder has the property that $C(Y)$ is a compactification of $C([0,1))$;*

(5.4) *$X \in CP$.*

For continua satisfying some additional conditions the following early results are known (see [6, Theorem 4.2, p. 171 and Section 6, p. 179]; [11, Propositions 18, 19 and 20, p. 162 and 163]; [9, Lemma 14.13.1, p. 415, and Theorems 14.73.12, p. 490, and 14.73.17, p. 493]).

Theorem 6. *Each member of the following classes of continua has the covering property hereditarily:*

(6.1) *arc-like,*

(6.2) *hereditarily indecomposable,*

(6.3) *nonplanar circle-like (thus each nonplanar solenoid,* i.e., each solenoid distinct from a circle*).*

Furthermore, we have the following three characterizations (see [9, Theorem 14.73.19 and 14.73.20, p. 496], [2, Proposition 4, p. 387] and [5, Exercise 67.17, p. 325]).

Theorem 7. *Let a continuum X be hereditarily decomposable. Then $X \in CPH$ if and only if X is arc-like.*

Theorem 8. *Let a continuum X be arcwise connected. Then $X \in CPH$ if and only if it is an arc.*

Theorem 9. *Let X be a (metric) compactification of the half-open interval $[0,1)$. Then $X \in CP$ if and only if $C(X) = \mathrm{cl}_{C(X)} C([0,1))$ (i.e., if and only if $[0,1)$ approximates each subcontinuum of the remainder).*

To formulate the next variant related to coverings of continua let us come back to the definition of the covering property. The condition of this definition which claims that no proper *subcontinuum* of any Whitney level covers the continuum X has been relaxed in [8] by considering certain special covers of X. Namely, the condition has been replaced by demanding that no proper *compact subset* of $\mu^{-1}(t)$ covers X. The following concept is defined in [8, p. 191]. A subset $\mathcal{A}$ of $C(X)$ is said to be a *minimal closed cover of* X provided that

(a) $\mathcal{A}$ covers X, i.e., $\cup\mathcal{A} = X$;

(b) $\mathcal{A}$ is a closed subset of $C(X)$;

(c) each element of $\mathcal{A}$ is a nondegenerate proper subcontinuum of X, i.e., $\mathcal{A} \subset C(X) \setminus (F_1(X) \cup \{X\})$;

(d) no proper closed subset of $\mathcal{A}$ covers X.

Examples and basic properties of this concept are given in [8]. In particular the following existence theorem is shown in [8, Theorem 3, p. 195].

Theorem 10. *Each collection of nondegenerate proper subcontinua of a continuum X which is closed in $C(X)$ and covers X contains a minimal closed cover of X.*

It is proved in [8, Theorem 4, p. 196] that any minimal closed cover of an arc is finite. This fact is used to show much more general result, namely the following characterization of graphs, see [8, Theorem 5, p. 198]. Recall that a *(linear) graph* is defined as a continuum which is the union of a finite number of arcs pairwise disjoint except for their end points. In [5, Section 5, p. 33, and Section 65, p. 305] the term of a *finite graph* is used in the same meaning.

Theorem 11. *A continuum X is a graph if and only if each minimal closed cover of X is finite.* The above result can be considered as a starting point of a study of relations between structures of some covers of a given continuum X and the structure of X itself. In other words we are interested in the following problem, that can be seen as a research program rather than a particular question.

Problem 12. Given a continuum X, find connection between properties of some special (closed) covers of X by its subcontinua and properties of X.

The next result (see [8, Theorem 7, p. 202]) is also related to the above problem.

Theorem 13. *If all minimal closed covers of a continuum X are countable, then X is hereditarily locally connected.*

To see that the converse implication is not true consider the Gehman dendrite, i.e., a dendrite G having the set $E(G)$ of its end points homeomorphic to the Cantor ternary set in $[0, 1]$, such that all ramification points of G are of order 3. For each point $p \in G \setminus E(G)$ the dendrite G has an uncountable minimal closed cover $\mathcal{A}(p)$ which consists of all the arcs joining the point p with end points of G, see [8, Figure 5, p. 199]. For each $p \in G \setminus E(G)$ the cover $\mathcal{A}(p)$ is homeomorphic to the Cantor set, thus it is totally disconnected.

The above construction can be generalized to smooth dendroids. Recall that a *dendroid* means an arcwise connected and hereditarily unicoherent continuum. If a dendroid X contains a point $v \in X$ (called an *initial point of X*) such that for each sequence of points x_n tending to a point x the sequence of arcs vx_n tends to the limit arc vx, then X is said to be *smooth.* A point x of a dendroid X is called an *end point* provided that x is an end point of each arc in X containing x.

Theorem 14. *Let a dendroid X different from an arc be smooth and have its set $E(X)$ of end points closed. Then for each initial point $v \in X \setminus E(X)$ there exists a minimal closed cover $\mathcal{A}(v)$ which is homeomorphic to $E(X)$.*

Proof. Put $\mathcal{A}(v) = \{vx : x \in E(X)\}$. Obviously $X = \bigcup\{vx : x \in E(X)\}$, so (a) follows. Smoothness of X implies that $\mathcal{A}(v)$ is closed. Since X is not an arc, each element of $\mathcal{A}(v)$ is distinct from the whole X, and since $v \in X \setminus E(X)$, the arcs vx are all nondegenerate. So (c) is satisfied. Finally, if $\mathcal{B} \subset \mathcal{A}(v)$ and $\mathcal{B} \neq \mathcal{A}(v)$, then there is an end point $e \in E(X)$ such that ve is not in $\mathcal{B}$. Thus $e \in (\cup\mathcal{A}(v)) \setminus (\cup\mathcal{B})$, and therefore $\mathcal{B}$ does not cover X. Hence $\mathcal{A}(v)$ is a minimal closed cover of X.

Let $h : E(X) \to \mathcal{A}(v)$ be a function defined by $h(x) = vx$ for each $x \in E(X)$. Let $e_n \in E(X)$ be a sequence of end points of X converging to an end point e. Then the sequence of arcs ve_n tends to the arc ve by smoothness of X, so h is continuous. It is one-to-one by its definition, so it is a homeomorphism by the compactness of $E(X)$. The proof is complete. □

Since each dendrite X is a smooth dendroid (with any of its points as an initial one), and since the set $E(X)$ of the end points of X is totally disconnected (equivalently: 0-dimensional, see [7, §51, V, Theorem 2, p. 292]) we conclude the following.

Corollary 15. *Let a dendrite X have the set $E(X)$ of its end points closed. Then for each point $v \in X \setminus E(X)$ there exists a totally disconnected minimal closed cover $\mathcal{A}(v)$ composed of subarcs of X such that $\mathcal{A}(v)$ is homeomorphic to $E(X)$.*

Relations between total disconnectedness of each of minimal closed covers of a continuum X and hereditary local connectedness of X are discussed in [8] and [1]. Answering a question in [8, p. 204] it is shown that hereditary local connectedness of a continuum does not imply total disconnectedness of any of its closed covers. Namely a dendrite X is constructed in [1, Example 5] having the set $E(X)$ of all its end points countable (but not closed) and such that a minimal closed cover of X contains a subarc of $C(X)$. The inverse implication (from total disconnectedness of each of minimal closed covers of a continuum X to hereditary local connectedness of X) remains an open question, [1, Question 6]. Further, it is known that the Gehman dendrite has a minimal closed cover which is an arc, [1, Example 9]. Thus the next questions are natural.

Questions 16. *Let a dendroid X have the set $E(X)$ of all its end points countable and closed. Is then each minimal closed cover of X countable? If not, is an answer positive provided that X is smooth?*

In the light of the above mentioned results and questions the following problems are of some interest.

Problems 17. Characterize continua (hereditarily locally connected continua, in particular dendrites) all minimal closed covers of which are a) countable, b) totally disconnected.

A continuum X is said to have the *strong covering property* (writing $X \in SCP$) provided that for each Whitney map $\mu : C(X) \to [0, \infty)$ and for each $t \in [0, \mu(X)]$ the Whitney level $\mu^{-1}(t)$ for $C(X)$ is a minimal closed cover of X. Thus, $X \in SCP$ implies $X \in CP$ for each continuum X. The inverse implication is not true, because an arc has the covering property (even hereditarily, see (6.1) of Theorem 6), while it does not have the strong covering property because each minimal closed cover of an arc is finite by Theorem 11, so it cannot be equal to any Whitney level which obviously is infinite.

A subcontinuum K of a continuum X is said to be *terminal in X* provided that for each subcontinuum L of X the condition $K \cap L \neq \emptyset$ implies either $K \subset L$ or $L \subset K$. The family of all terminal subcontinua of a given continuum X will be denoted by $\mathrm{Ter}\,(X)$. In the next theorem (see [1, Theorem 21]) connections are shown between the strong covering property and various related properties of continua.

Theorem 18. *Consider the following conditions a continuum X may satisfy:*

(18.1) *X is hereditarily indecomposable;*

(18.2) *for each Whitney map $\mu : C(X) \to [0, \infty)$ and for each $t \in (0, \mu(X))$ the set $\mathrm{Ter}\,(X) \cap \mu^{-1}(t)$ is dense in $\mu^{-1}(t)$;*

(18.3) *$\mathrm{Ter}\,(X)$ is a dense subset of $C(X)$;*

(18.4) *$\{X\}$ is an accumulation point of $\mathrm{Ter}\,(X)$;*

(18.5) *X is indecomposable;*

(18.6) *$X \in SCP$;*

(18.7) *$X \in CP$;*

(18.8) *X is unicoherent and irreducible.*

Then the following implications hold:

$$\begin{array}{ccccccccc} (18.1) & \Rightarrow & (18.2) & \Rightarrow & (18.3) & \Rightarrow & (18.4) & \Rightarrow & (18.5) \\ & & \Downarrow & & & & & & \\ & & (18.6) & \Rightarrow & (18.7) & \Rightarrow & (18.8) & & \end{array}$$

Examples are presented in [1, Remarks 23] showing that all the implications in the above diagram cannot be reversed except the two, from (18.2), that remain open questions.

References

[1] J. J. Charatonik and W. J. Charatonik, *On the strong covering property of continua*, Topology Proc. **25** (2000), 75-84.

[2] W. J. Charatonik, *Some counterexamples concerning Whitney levels*, Bull. Polish Acad. Sci. Math. **31** (1983), 385–391.

[3] J. Grispolakis, S. B. Nadler, Jr. and E. D. Tymchatyn, *Some properties of hyperspaces with applications to continua theory*, Canad. J. Math. **31** (1979), 197–210 .

[4] J. Grispolakis and E. D. Tymchatyn, *Weakly confluent mappings and the covering property of hyperspaces*, Proc. Amer. Math. Soc. **74** (1979), 177–182.

[5] A. Illanes and S. B. Nadler, Jr., *Hyperspaces*, M. Dekker, New York and Basel, 1999.

[6] J. Krasinkiewicz and S. B. Nadler, Jr., *Whitney properties*, Fund. Math. **98** (1978), 165–180.

[7] K. Kuratowski, *Topology* **2**, Academic Press and PWN, New York, London and Warszawa, 1968.

[8] S. Macías, *Coverings of continua* , Topology Proc. **20** (1995), 191–205.

[9] S. B. Nadler, Jr., *Hyperspaces of sets*, M. Dekker, New York and Basel, 1978.

[10] S. B. Nadler, Jr., *Continuum theory*, M. Dekker, New York, Basel and Hong Kong, 1992.

[11] A. Petrus, *Whitney maps and Whitney properties of $C(X)$*, Topology Proc. **1** (1976), 147–172.

[12] C. W. Proctor, *A characterization of absolutely C^*-smooth continua*, Proc. Amer. Math. Soc. **92** (1984), 293–296.

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