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FAMILIES OF RANK ONE IN SCATTERED SPACES

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Abstract

In this paper we study the properties of spaces with open-and-closed families of countable rank with T_0-separates points. In particular, we prove that every separable scattered compact space with such family is metrizable.

1. Introduction

All considered spaces are assumed to be regular. We shall use the terminology from [8], $|Y|$ denotes the cardinality of a set Y, $cl_X A$ or clA denotes the closure of a set A in a space X, $N = \{1, 2, ...\}$ and we consider the discrete topology on N, $\psi(x, X)$ denotes the pseudocharacter of a point x in a space X, $\psi(X) = \sup\{\psi(x, X) : x \in X\}$ denotes the pseudocharacter of a space X, $c(X) = \sup\{|\gamma| : \gamma$ is a family of pairwise disjoint non-empty open subsets of $X\}$ - the Souslin number of X, $d(X) = \min\{|Y| : Y$ is dense in $X\}$ - the density of X, $l(X) = \min\{\tau :$ every open cover of X contains a subcover of cardinality $\leq \tau\}$ - the Lindelöf number of X, $hl(X) = \sup\{l(Y) : Y \subseteq X\}$ - the hereditary Lindelöf number of X.

A space is called scattered if it contains no non-empty dense in itself subspace (see [9, 14]).

Let X be a scattered space. Denote $X^{(0)} = X$, $X_\alpha = \{x \in X^{(\alpha)} : x$ is an isolated point in $X^{(\alpha)}\}$, $X^{(\alpha+1)} = X^{(\alpha)} \setminus X_\alpha$ for every $\alpha \geq 0$ and $X^{(\alpha)} = \bigcap\{X^{(\beta)} : \beta < \alpha\}$ for each limit

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ordinal α. If $x \in X_\alpha$, then we put $is(x, X) = \alpha$. The ordinal number $is(X) = \min\{\alpha : X_\alpha = \emptyset\}$ is called the index of scatteredness of X. For every $\alpha < is(X)$ the set $\bigcup\{X_\beta : \beta < \alpha\}$ is open in X and X_α is a non-empty discrete subspace of X.

A space X is called τ-metalindelöf if every open cover γ of X has an open refinement ω for which $ord(x, \omega) = |\{U \in \omega : x \in U\}| \leq \tau$ for each $x \in X$. An $\aleph_0$-metalindelöf space is called metalindelöf. Denote $ml(X) = \min\{\tau : X \text{ is } \tau\text{-metalindelöf}\}$ - the metalindelöf degree of X.

Let B be a family of subsets of a space X. The family B T_0-separates points of X if for every pair of distinct points $x, y \in X$ there exists an element $U \in B$ such that $U \bigcap \{x, y\}$ is a singleton set. A subfamily $B' \subseteq B$ has the rank one if for every two sets $U, V \in B'$ we have $U \bigcap V = \emptyset$, or $U \subseteq V$, or $V \subseteq U$. We say that the rank $r(B) \leq \tau$, where τ is a finite or infinite cardinal, if $B = \bigcup\{B_\mu : \mu \in M\}$, where $|M| \leq \tau$ and B_μ is a family of rank one for each $\mu \in M$.

The *weak rank* of a space X is the cardinal $wr(X) = \min\{r(B) : B$ is a family of open-and-closed subsets which T_0-separates points of $X\}$.

The *rank* of a space X is the cardinal $r(X) = \min\{r(B) : B$ is an open base of $X\}$.

A family γ of subsets of a space X is independent if $\bigcap \gamma \neq \emptyset$ and $V \setminus W \neq \emptyset$, $W \setminus V \neq \emptyset$ for every distinct elements $V, W \in \gamma$.

Let B be a family of subsets of X. The cardinal $sr(B) = \sup\{|\gamma| : \gamma \subseteq B$ and γ is independent$\}$ is called the small rank of B. If every independent subfamily $\gamma \subseteq B$ is finite, then the small rank $sr(B)$ of B is subinfinite.

The *weak small rank* of a space X is the cardinal $wsr(X) = \min\{sr(B) : B$ is a family of open-and-closed subsets which T_0-separates points of $X\}$ and the *small rank* of X is the cardinal $sr(X) = \min\{sr(B) : B$ is an open base of $X\}$.

The spaces with bases of finite rank were studied in [2, 5, 6, 7, 8, 11]. G. Gruenhage and P. Nyikos [11] showed that if a compactum has a countable base of finite small rank, then it is metrizable.

S. Troyanski raised the following question.

Question 1.1. *Let X be a scattered compact space. Is it true that $wr(X) = 1$?*

The following questions were formulated during the discussion of the Question 1.1 with the Professors P. Kenderov, S. Nedev and S. Troyanski in the autumn of 1989.

Question 1.2. *Is it true that the weak rank of every scattered compact space is finite or countable ?*

Question 1.3. *Is it true that every scattered compact space is a continuous image of a compact scattered space of weak rank one or of countable weak rank ?*

In the present article we give the negative answers to all above questions.

2. Preliminary Results

Some statements of this section are obvious or wellknows and we omit its proofs.

A transfinite sequence of sets is a family of sets $\{H_\mu : \mu \in M\}$, where M is a well-ordered set.

A transfinite sequence $\{H_\mu : \mu \in M\}$ is called:

– increasing if $H_\alpha \subseteq H_\beta$ and $H_\beta \setminus H_\alpha \neq \emptyset$ for $\alpha < \beta$ and $\alpha, \beta \in M$;

– decreasing if $H_\beta \subseteq H_\alpha$ and $H_\alpha \setminus H_\beta \neq \emptyset$ for $\alpha < \beta$ and $\alpha, \beta \in M$;

– monotone if it is increasing or decreasing.

If $<$ is a linear order on a set Z and $x \in Z$, then we put $Z(x) = \{y \in Z : y < x\}$ and $Z^+(x) = \{y \in Z : y \leq x\}$.

Proposition 2.1. *Let $\gamma = \{H_\mu : \mu \in M\}$ be a monotone transfinite sequence of open-and-closed sets of a space X. Then $|M| \leq c(X)$.*

A union of τ closed subsets of X is called an F_τ-set. A union of a countable family of closed subsets is called an F_σ-set.

For a cardinal τ by τ^+ denote the smallest cardinal greater than τ.

Proposition 2.2. *Let $\gamma = \{H_\mu : \mu \in M\}$ be a transfinite increasing family of open sets of X and $|H_\mu| < \tau$ for each $\mu \in M$. Then $|\bigcup\{H_\mu : \mu \in M\}| \leq \tau$ and $|M| \leq \tau$.*

Let X be a space and $x \in X$. The set $Q(x, X) = \bigcap\{U : x \in U$ and U is open-and-closed in $X\}$ is called the quasi-component of a point x in X.

If X is an infinite connected space, then the weak rank $wr(X)$ is not determined. In this case we consider that $wr(X) = \infty$ and $\tau < \infty$ for every cardinal τ.

Proposition 2.3. *The weak rank $wr(X)$ of a space X is determined if and only if $Q(x, X) = \{x\}$ for each point $x \in X$.*

The class of spaces with open T_0-separating families of rank one is quite broad.

A linear order $<$ on a space X is called a (strongly) α-left order if for each point $x \in X$ the set $X^+(x)$ is closed (respectively is open-and-closed) in X. A space with a (strongly) α-left order is called a (strongly) α-left space (see [4]).

Proposition 2.4. *Let X be a space. Then:*

1. If X is an α-left space, then X has an open T_0-separating family of rank one.

2. If X is a strongly α-left space, then $wr(X) = 1$.

Proposition 2.5. (see [4]) *A space X is scattered if and only if on X there exists a well-order such that the set $X(x)$ is open for every $x \in X$. In particular, every scattered space is an α-left space.*

Probably the appearance of the Question 1.1 is motived by the next fact which follows from Propositions 2.4 and 2.5.

Corollary 2.6. *Every scattered space has an open T_0-separating family of rank one.*

Proposition 2.7. *Let X be a collectionwise normal scattered space, τ be an infinite cardinal and $\tau \geq d(X)$. Then:*

1. If $is(X) \leq 2$, then $wr(X) = 1$.

2. If $is(X) \leq 3$ and $|X| \leq \tau^+$, then X has a T_0-separating family of rank one of open F_τ-sets.

Proof. Let $X_2 \neq \emptyset$ and $X_3 = \emptyset$. The set X_2 is discrete and closed in X and the set X_0 is dense and open in X. The subspace X_0 is discrete and $|X_0| = d(X) = \tau$. There exists a discrete family $\{U_x : x \in X_2\}$ of open-and-closed subsets of X such that $X = \bigcup\{U_x : x \in X_2\}$ and $U_x \bigcap X_2 = \{x\}$ for each $x \in X_2$. Put $V_x = U_x \bigcap X_0$ and $Y_x = U_x \bigcap X_1$. On Y_x there exists some well-order $<$ such that $|\{y \in Y_x : y < z\}| \leq \tau$ for every $z \in Y_x$. Now we put $B_x = \{U_x\} \bigcup \{\{z\} : z \in V_x\} \bigcup \{V_x \bigcup \{y \in Y_x : y < z\} : z \in Y_x\}$ and $B = \bigcup\{B_x : x \in X_2\}$. The family B is open, T_0-separates points of the space X and $|V| \leq \tau$ for each $V \in B$.

Let $X_2 = \emptyset$ and $X_1 \neq \emptyset$. There exists an open-and-closed family $\{H_x : x \in X\}$ of X such that $H_x \bigcap X = \{x\}$ for every $X \in X_1$. Then $B = \{\{x\} : x \in X_0\} \bigcup \{H_x : x \in X_1\}$ is an open-and-closed family of rank one which T_0-separates the points of X. In this case $wr(X) = 1$.

Let $X_1 = \emptyset$. In this case X is a discrete space. The proof is complete. □

Proposition 2.8. *$l(X) \leq d(X) + ml(X)$.*

Proposition 2.9. *$c(X) = d(X)$ for every scattered space X.*

Theorem 2.10. *Let X be a scattered space. Then $|X| = hl(X) = \psi(X) + l(X)$.*

Proof. For every space Z we have $\psi(Z) \leq hl(Z) \leq |Z|$ and $l(Z) \leq hl(Z)$. Fix a scattered space X. It is sufficient to prove that $|X| \leq \psi(X) + l(X)$.

We put $\tau = \psi(X) + l(X)$. For every $\alpha < is(X)$ and each point $x \in X_\alpha$ we fix an open subset U_x of X such that $x \in U_x \subseteq cl_X U_x \subseteq \bigcup\{X_\beta : \beta \leq \alpha\}$ and $cl_X U_x \bigcap X_\alpha = \{x\}$. We prove that $|U_x| \leq \tau$ for each $x \in X$. If $\alpha = 0$ and $x \in X_\alpha$, then $U_x = \{x\}$ and $|U_x| \leq \tau$. Suppose that $\alpha > 0$, $x \in X_\alpha$ and $|U_y| \leq \tau$ for every $y \in \bigcup\{X_\beta : \beta < \alpha\}$. The subspace $Y_x = cl_X U_x \setminus \{x\}$ is an F_τ-subset of X. Hence $l(Y_x) \leq \tau$ and there exists a subset

Z_x of Y_x such that $|Z_x| \leq \tau$ and $Y_x \subseteq \bigcup\{U_y : y \in Z_x\}$. Therefore $|U_x| \leq |\bigcup\{U_y : y \in Z_x\}| \leq \tau$.

The open cover $\{U_x : x \in X\}$ of X contains a subcover $\{U_x : x \in Y\}$, where $Y \subseteq X$ and $|Y| = l(X) \leq \tau$. By construction $|X| \leq |\bigcup\{U_x : y \in Y\}| \leq \tau$. The proof is complete. □

3. The Rank and the Pseudocharacter of Spaces

Theorem 3.1. *$\psi(X) \leq c(X) + wr(X)$ for every space X.*

Proof. The assertion is trivial if $wr(X) = \infty$. Assume now that $c(X) + wr(X) = \tau < \infty$. If a cardinal τ is finite, then X is discrete and $\psi(X) = 1 \leq |X| = c(X)$. Suppose that the cardinal τ is infinite.

There exists the family $B = \{B_\mu : \mu \in M\}$ of rank one of open-and-closed subsets of X for which $|M| \leq \tau$ and B T_0-separates the points of X.

Fix a point $b \in X$. Suppose that $\psi(b, X) > \tau$. We put $P = \bigcap\{U \in B : b \in U\}$. Two cases are possible.

Case 1. $\psi(b, P) \leq \tau$.
Since $\psi(b, X) > \tau$ and $\psi(b, P) \leq \tau$, for every ordinal $\alpha < \tau^+$ there exist a point $y_\alpha \in X \setminus P$ and a set $U_\alpha \in B$ such that $y_0 \notin U_0$, $y_\alpha \in \bigcap\{U_\beta : \beta < \alpha\} \setminus U_\alpha$ for each $\alpha < \tau^+$ and $b \in \bigcap\{U_\alpha : \alpha < \tau^+\}$. Denote $H_\mu = \{\alpha < \tau^+ : U_\alpha \in B_\mu\}$. By construction, $\{U_\alpha : \alpha \in H_\mu\}$ is a decreasing transfinite sequence of open-and-closed sets. By virtue of Proposition 2.1, $|H_\mu| \leq c(X) \leq \tau$ for every $\mu \in M$. Since $|M| \leq \tau$ and $\bigcup\{H_\mu : \mu \in M\} = \{\alpha < \tau^+\}$, we have $|H_\mu| = \tau^+$ for some $\mu \in M$. This is a contradiction. Hence the case 1 is impossible.

Case 2. $\psi(b, P) > \tau$.
For every ordinal $\alpha < \tau^+$ there exist a point $y_\alpha \in P$ and a set $U_\alpha \in B$ such that $b \notin U_\alpha$, $y_0 \in U_0$ and $y_\alpha \in U_\alpha \setminus \bigcup\{U_\beta : \beta < \alpha\}$ for $\alpha \geq 1$.

Let $y_0 \in P \setminus \{b\}$. Then for some $U_0 \in B$ we have $U_0 \bigcap \{y_0, b\} = \{y_0\}$. Suppose that $\tau^+ > \alpha > 0$ and the elements $\{y_\beta, U_\beta : \beta < \alpha\}$ are constructed. Since $\psi(b, P) > \tau$, there exists a point $y_\alpha \in P \setminus \bigcup\{U_\beta : \beta < \alpha\}$ such that $y_\alpha \neq b$. For some $U_\alpha \in B$ we have $U_\alpha \bigcap \{y_\alpha, b\} = \{y_\alpha\}$.

Fix $\mu \in M$. We put $H_\mu = \{\alpha < \tau^+ : U_\alpha \in B_\mu\}$. If $\alpha, \beta \in H_\mu$ and $\alpha < \mu$, then $U_\alpha \subseteq U_\beta$ or $U_\alpha \bigcap U_\beta = \emptyset$, i. e. $U_\beta \setminus U_\alpha$ is an open non-empty subset. For $\alpha \in H_\mu$ by $g(\alpha)$ denote the smallest element of H_μ greater than α. Assume that $|H_\mu| = \tau^+$. Then $\{V_\alpha = U_{g(\alpha)} \setminus U_\alpha : \alpha \in H_\mu\}$ is a family of pairwise disjoint non-empty open subsets. Therefore $|H_\mu| \leq c(X) \leq \tau$. This is a contradiction. Hence the case 2 is impossible, too. The proof is complete. □

From Theorem 3.1, Proposition 2.9 and Theorem 2.10 it follows

Corollary 3.2. $|X| \leq d(X) + l(X) + wr(X) = d(X) + wr(X) + ml(X)$ *for every scattered space* X.

Corollary 3.3. *Let* X *be an metalindelöf scattered space of countable weak rank. Then* $|X| = d(X) = c(X)$.

Corollary 3.4. *Let* X *be a Lindelöf scattered space of countable weak rank. Then:*

1. *If* X *is separable, then* X *is countable.*
2. *If* X *is separable and locally compact, then* X *is metrizable.*

Corrolary 3.5. *Every compact separable scattered space of countable weak rank is metrizable.*

Theorem 3.6. *If* X *is a hereditarilly paracompact scattered space, then* $wr(X) = 1$.

Proof. By Telgarski's theorem [9, 14], $\dim Y = 0$ for every subspace Y of X. For every $\alpha < is(X)$ and every point $x \in X_\alpha$ we fix an open-and-closed subset U_x of X such that $x \in U_x \subseteq \bigcup\{X_\beta : \beta \leq \alpha\}$ and $U_x \bigcap X_\alpha = \{x\}$. We affirm that $wr(U_x) = 1$ for every $x \in X$. This assertion is trivial if $\alpha = 0$

and $x \in X_0$. Assume that $\alpha \geq 1$ and $wr(U_y) = 1$ for each $y \in \{X_\beta : \beta < \alpha\}$. Fix $b \in X_\alpha$. Since $V_b = U_b \setminus \{b\}$ is a zero-dimensional paracompact space, then there exists a disjoint family $\{W_\mu : \mu \in M\}$ of open-and-closed subsets of X such that $V_b = \bigcup\{W_\mu : \mu \in M\}$ and for every $\mu \in M$ there is a point $x(\mu) \in V_b$ such that $W_\mu \subseteq U_{x(\mu)}$. Hence $wr(W_\mu) = 1$ for every $\mu \in M$. For every $\mu \in M$ we fix a family B_μ of open-and-closed subsets of W_μ which T_0-separates points of W_μ and $wr(B_\mu) = 1$. Denote $B = \bigcup\{\{W_\mu\} \bigcup B_\mu : \mu \in M\}$. Then $r(B) = 1$ and B T_0-separates points of U_b. Hence $wr(U_b) = 1$. The following assertion completes the proof. □

Theorem 3.7. *Let X be a paracompact space,* $\dim X = 0$ *and for every point $x \in X$ there is an open set U_x of X such that $x \in U_x$ and $wr(U_x) = 1$. Then $wr(X) = 1$.*

Proof. There exist a subset Y of X and a discrete cover $\{V_y : y \in Y\}$ of X such that $V_y \subseteq U_x$ for each $y \in Y$. Since $wr(V_y) = wr(U_x) = 1$, there exists an open-and-closed system B_y of X such that $\bigcup\{W : W \in B_y\} \subseteq V_y$ and B_y T_0-separates points of V_y. Denote $B = \{B_y \bigcup\{V_y\} : y \in Y\}$. Then $r(B) = 1$ and B T_0-separates points of X. The proof is complete. □

4. The Generalized Rank

Let m be a finite or infinite cardinal and X be a space. The cardinal $r_m(X) = \min\{r(B) : B$ is a family of open F_m-sets which T_0-separates points of $X\}$ is called the m-rank of X. If m is finite, then $r_m(X) = r_1(X) = wr(X)$.

Theorem 4.1. *Let X be a scattered locally compact space and $is(X) \leq 3$. Then $\psi(X) \leq d(X) + r_m(X) + m^+$ for every cardinal m.*

Proof. For finite m the assertion follows from Theorem 3.1. Assume that m is infinite. Then $r_m(X) < \infty$. We put $\tau = d(X) + r_m(X) + m^+$.

Since X is scattered, then $\dim F = 0$ for every compact subspace F of X (see [7, 11]). Hence for every $i \leq 2$ and every point $x \in X_i$ there exists an open compact subset U_x of X such that $x \in U_x \subseteq \bigcup\{X_j : j \leq i\}$ and $U_x \bigcap X_i = \{x\}$. If $x \in X_0 \bigcup X_1$, then $|U_x| \leq d(X) \leq \tau$ and $\psi(x, X) \leq \tau$.

Fix $b \in X_2$. Suppose that $\psi(b, X) > \tau$. Denote $V_b = U_b \setminus X_0$. Since $|X_0| \leq \tau$, $\psi(b, V_b) = \psi(b, X) > \tau$. By construction, V_x is the one-point Alexandroff compactification of the discrete space $V_b \setminus \{b\}$ (see [1, 8]).

Let $\{B_\mu : \mu \in M\}$ be families of rank one of open F_m-sets, $|M| \leq r_m(X) \leq \tau$ and the family $B = \bigcup\{B_\mu : \mu \in M\}$ T_0-separates points of X.

Denote $P = \bigcap\{V_b \bigcap U : b \in U, U \in B\}$. We have two possible cases.

Case 1. $\psi(b, P) \leq \tau$.
In this case for every ordinal $\alpha < \tau^+$ there exist a point $y_\alpha \in V_b \setminus P$ and a set $U_\alpha \in B$ such that $b \in U_\alpha$, $y_0 \notin U_0$ and $y_\alpha \in \bigcap\{U_\beta : \beta < \alpha\} \setminus U_\alpha$ if $\alpha \leq 1$.

We put $H_\mu = \{\alpha < \tau^+ : U_\alpha \in B_\mu\}$. By construction, $\{W_\alpha = V_b \bigcap U_\alpha : \alpha \in H_\mu\}$ is a decreasing transfinite sequence of open F_m-sets of V_b. If $\alpha \in H_\mu$ and the set $\{\beta \in H_\mu : \beta < \alpha\}$ is infinite, then the set $W_\alpha \setminus \{y_\beta \in V_b : \beta < \alpha\}$ is not open in V_b. Hence the set $\{\beta \in H_\mu : \beta < \alpha\}$ is finite for each $\alpha \in H_\mu$ and the set H_μ is countable. This is a contradiction by virtue of the conditions $|M| \leq \tau$ and $|\bigcup\{H_\mu : \mu \in M\}| = \tau^+$.

Case 2. $\psi(b, P) > \tau$.
For every ordinal $\alpha < \tau^+$ there exist a point $y_\alpha \in P$ and a set $U_\alpha \in B$ such that $b \notin U_\alpha$, $y_0 \in U_0$ and $y_\alpha \in U_\alpha \setminus \bigcup\{U_\beta : \beta < \alpha\}$ for $\alpha \geq 1$.

We put $H_\mu = \{\alpha < \tau^+ : U_\alpha \in B_\mu\}$. For some $\mu \in M$ we have $|H_\mu| = \tau^+$.

Since X_0 is dense in X and $|X_0| \leq \tau$, then there is a point $x \in X$ such that $|\{\alpha \in H_\mu : x \in U_\mu\}| = \tau^+$. Denote $W_\mu = \{\alpha \in H_\mu : x \in U_\mu\}$. By construction, $\{U_\mu : \mu \in W_\mu\}$ is an increasing transfinite sequence of F_m-sets. Since

$|U_\alpha \bigcap V_b| \leq m$ for every $\alpha < \tau^+$, then by Proposition 2.2 we have $|\bigcup\{U_\alpha \bigcap V_b : \alpha \in W_\mu\}| \leq m^+ \leq \tau$. By construction, $\tau^+ = |\{y_\alpha : \alpha \in W_\mu\}| \leq |\bigcup\{U_\alpha \bigcap V_b : \alpha \in W_\mu\}|$, a contradiction. Hence the case 2 is impossible, too. The proof is complete. □

By Theorem 4.1, Proposition 2.9 and Theorem 2.10 it follows

Corollary 4.2. *Let X be a locally compact scattered space and $is(X) \leq 3$. Then:*

1. $|X| \leq d(X) + l(X) + r_m(X) + m^+$ for every cardinal m.

2. If $d(X) + l(X) \leq \aleph_1$ and $r_{\aleph_0}(X) = \aleph_0$, then $|X| \leq \aleph_1$.

5. Mappings and Cardinality of Scattered Spaces

Theorem 5.1. *Let $f : X \longrightarrow Y$ be a closed continuous mapping of a scattered space X onto a space Y. Then:*

1. There exists a closed subspace Z of X such that $f(Z) = Y$ and $d(Z) = d(Y)$.

2. $|Y| \leq d(Y) + wr(X) + ml(X)$.

Proof. Let S be a dense subset of the space Y and $|S| = d(Y)$. For every point $y \in S$ we fix some point $x(y) \in f^{-1}(y)$. Put $P = \{x(y) : y \in S\}$ and $Z = cl_X P$. Then $f(P) = S$ and $d(Z) = d(Y) = |S|$. Since f is a closed mapping, $f(Z) = Y$. The assertion 1 is proved. ¿From Corollary 3.2 it follows that $|Z| \leq d(Z) + wr(Z) + ml(Z) \leq d(Y) + wr(X) + ml(X)$. Hence $|Y| \leq |Z| \leq d(Y) + wr(X) + ml(X)$. The proof is complete. □

Corollary 5.2. *Let $f : X \longrightarrow Y$ be a continuous closed mapping of a paracompact scattered space X of countable rank onto a locally compact separable space Y. Then Y is a countable metrizable space.*

A space X is called a space of pointwise countable type if every point $x \in X$ is contained in some compact subset of the countable character. Spaces of pointwise countable type were introduced by A. V. Arhangel‘skii (see [3, 10]).

Corollary 5.3. *Let $f : X \longrightarrow Y$ be a closed continuous mapping of a metalindelöf scattered space of countable rank onto a separable space Y of pointwise countable type. Then Y is a countable metrizable space.*

Corollary 5.4. *Let $f : X \longrightarrow Y$ be a closed continuous mapping of a metalindelöf scattered space X onto a space Y. If $d(Y) < \psi(Y)$, then $wr(X) \geq \psi(Y)$.*

Theorem 5.5. *Let Y be a scattered space. Then there exist a perfectly normal paracompact scattered space X of weak rank one and an open continuous mapping $f : X \longrightarrow Y$ onto Y.*

Proof. For every $\alpha < is(Y)$ and every point $y \in Y_\alpha$ we fix an open set U_y of Y such that $y \in U_y = (\bigcup\{Y_\beta : \beta < \alpha\}) \bigcup\{y\}$. We affirm that for every $y \in Y$ there exist a perfectly normal paracompact scattered space X_y and an open continuous mapping f_y of X_y onto U_y.

Let $y \in Y_0$. Then $U_y = \{y\}$. In this case we put $X_y = \{y\}$ and f_y is the identity mapping.

Suppose that $\alpha \geq 1$ and X_y, f_y are constructed for every $y \in \bigcup\{Y_\beta : \beta < \alpha\}$. Fix $b \in Y_\alpha$. Denote by Z the discrete sum $+\{X_y : y \in U_b \setminus \{b\}\}$ of the spaces $\{X_y : y \in U_b \setminus \{b\}\}$.

Put $X_b = \{b\} \bigcup (Z \times N)$, $f_b(b) = b$ and $f_b(z, i) = f_y(z)$ for every $y \in U_b \setminus \{b\}$, every $i \in N$ and every $x \in X_y$. On a set X_b we consider the topology relatively to which the topological product $Z \times N$ is an open subspace and the neighbourhoods of the point b in X_b are the form $f_b^{-1}(V) \bigcap (Z \times \{i \in N : i \geq n\})$, where $n \in N$ and V is an open neighbourhood of the point b in Y. The space X_b is paracompact, perfectly normal, scattered and the mapping f_b is open and continuous. Now we put $X = +\{X_y : y \in Y\}$ and $f(x) = f_y(x)$ for every $y \in Y$ and every $x \in X_y$. The proof is complete. □

A family B of subsets of a space X is called a pseudobase for a space X if B is a family of open subsets of X and $\{x\} = \bigcap\{U \in B : x \in U\}$ for each $x \in X$.

Lemma 5.6. *Let B be a pseudobase of a space $X, \emptyset \in B$ and $r(B) = 1$. Then:*

1. *Every set $U \in B$ is open-and-closed in X.*
2. *If $U, V \in B$, then $U \bigcap V \in B$.*
3. *$wr(X) = 1$.*

Proof. Fix $U \in B$. Suppose that U is non-empty and $b \in U$. For every $x \in X \setminus U$ there exists a set $V_x \in B$ such that $x \in V_x$ and $b \notin V_x$. Since $r(B) = 1$, $U \bigcap V_x = \emptyset$. Hence $X \setminus U = \bigcup\{U_x : x \in X \setminus U\}$ is open-and-closed in X. The assertions 2 and 3 are obvious. The proof is complete. □

Proposition 5.7. *If X is a space with a pseudobase of rank one, then there exists a continuous bijection $f : X \longrightarrow Y$ onto a hereditarilly paracompact space Y with a base of rank one.*

Proof. Let B be a pseudobase of X of rank one. Denote by Y the set X with the topology generated by the base B. Every space with a base of rank one is hereditarilly paracompact (see [5], Corollary 1). The proof is complete. □

Proposition 5.8. *Let X be a paracompact space with a G_δ-diagonal. If $\dim X = 0$, then X has a pseudobase of rank one.*

Proof. It is obvious. □

Corollary 5.9. *For every T_0-space X there exist a paracompact perfectly normal σ-discrete space Z with a pseudobase of rank one and a continuous open mapping $f : Z \longrightarrow X$ onto X.*

Proof. From J. R. Isbell‘s theorem [12, 13] there exist a σ-discrete perfectly normal paracompact space Z and a continuous open mapping f of Z onto X. Proposition 5.8 completes the proof. □

6. Rank of Spaces and Cartesian Product

Proposition 6.1. $wr(X \times Y) \leq wr(X) + wr(Y)$.

Proof. Let B_1 be a family of open-and-closed subsets of X which T_0-separates points of X and B_2 be a family of open-and-closed subsets of Y which T_0-separates points of Y. We put $B = \{U \times Y, X \times V : U \in B_1, V \in B_2\}$. Then $r(B) = r(B_1) + r(B_2)$ and B T_0-separates points of $X \times Y$. The proof is complete. □

Corollary 6.2. *Let $\{X_\mu : \mu \in M\}$ be a family of spaces. Then $wr(\prod\{X_\mu : \mu \in M\}) \leq \sum\{r(X_\mu) : \mu \in M\}$.*

Lemma 6.3. *$wr(X) \geq wsr(X)$ for every space X. If $sr(X) = 1$, then $r(X) = 1$.*

Proof. Obvious. □

Let X be a space. We put $ic(X) = \infty$ if X is discrete and $ic(X) = \min\{|L| : L$ is not closed subset of $X\}$ if X is not discrete.

Theorem 6.4. *Let X and Y be spaces and $ic(Y) < \psi(X)$. Then $wsr(X \times Y) \geq 2$.*

Proof. Let $ic(Y) = \tau$. Fix in Y a subset L of cardinality τ with an accumulation point $c \in Y \setminus L$. Assume that $Y = L \bigcup \{c\}$. There is a point $b \in X$ such that $\psi(b, X) > \tau$.

Let B be a system of open-and-closed subsets of the space $X \times Y$ which T_0-separates points of $X \times Y$.

We put $H = \bigcap\{U \in B : (b, c) \in U\}$ and $Z = \{y \in L : (b, y) \in H\}$.

Case 1. $c \in cl_Y Z$.

If $y, z \in Z$, then we consider $y < z$ if $y \neq z$ and $U \bigcap \{(b, y), (b, z)\} = \{(b, y)\}$ for some $U \in B$.

If $y, z \in Z$ and $y < z$, then we fix $U(y, z) \in B$ such that $U(y, z) \bigcap \{(b, y), (b, z)\} = \{(b, y)\}$. By construction, if $y < z$, then $(b, c) \notin U(y, z)$. For every pair $y, z \in Z$, where $y < z$, there exists an open-and-closed subset $V(y, z)$ of X such that $(b, y) \in V(y, z) \times \{y\} \subseteq U(y, z)$ and $U(y, z) \bigcap (V(y, z) \times \{c\}) = \emptyset$. We put $A = \bigcap\{V(y, z) : y, z \in Z, y < z\}$. Since $|Z| = \tau$ and $\psi(b, X) > \tau$, we have $|A| > \tau$. Fix $x \in A \setminus \{b\}$ and $U \in B$ such that $|\{(x, c), (b, c)\} \bigcap U| = 1$. There exists an open subset W of Y such that $c \in W$ and for every $y \in W \bigcap Z$ we have $|\{(x, y), (b, y)\} \bigcap U| = 1$. Fix $y, z \in W \bigcap Z$ for which $y < z$ and

$y \neq z$. Then, by construction, $U \bigcap U(y,z) \neq \emptyset$, $U \setminus U(y,z) \neq \emptyset$ and $U(y,z) \setminus U \neq \emptyset$. Therefore, $sr(B) \geq 2$.
Case 2. $c \in cl_Y(L \setminus Z)$.
In this case we assume that $Z = \emptyset$. For every $y \in L$ we fix $U_y \in B$ such that $(b,c) \in U_y$ and $(b,y) \notin U_y$. For every $y \in L$ there exists an open-and-closed subset V_y of X such that $(V_y \times \{y\}) \bigcap U_y = \emptyset$ and $V_y \times \{c\} \subseteq U_y$. We put $A = \bigcap\{V_y : y \in L\}$. Since $|L| = \tau$ and $\psi(b,X) > \tau$, we have $|A| > \tau$. Fix $x \in A \setminus \{b\}$ and $U \in B$ such that $|\{x,c),(b,c)\} \bigcap U| = 1$. There exists an open subset W of Y such that $c \in W$ and for every $y \in W \bigcap L$ we have $|\{(x,y),(b,y)\} \bigcap U\}| = 1$. If $y \in W \bigcap L$, then $U_y \bigcap U \neq \emptyset$, $U_y \setminus U \neq \emptyset$ and $U \setminus U_y \neq \emptyset$. Hence $wsr(B) \geq 2$. The proof is complete. □

Theorem 6.5. *Let X and Y be spaces, $ic(Y) = \tau$ be an infinite cardinal, $b \in X$, $n \in N$ and if $b \in Z$ and Z be a closed G_τ-subset of X, then $wsr(Z) \geq n$. Then $wsr(X \times Y) \geq n+1$.*

Proof. If $n = 1$, then $\psi(b,X) > \tau$. By Theorem 6.4, we have $wsr(X \times Y) \geq 2 = n+1$.

Let $n \geq 2$. Fix in Y a subset L of cardinality τ with an accumulation point $c \in Y \setminus L$. Assume that $Y = L \bigcup \{c\}$. Let B be a system of open-and-closed subsets of $X \times Y$ which T_0-separates points of $X \times Y$. We put $H = \bigcap\{U \in B : (b,c) \in H\}$ and $Z = \{y \in L : (b,c) \in H\}$.
Case 1. $c \in cl_Y Z$.
If $y,z \in Z$ we consider that $y < z$ if $y \neq z$ and $U \bigcap \{(b,y),(b,z)\} = \{(b,y)\}$ for some $U \in B$.

If $y,z \in Z$ and $y < z$, then we fix $U(y,z) \in B$ for which $U(y,z) \bigcap \{(b,y),(b,z)\} = \{(b,y)\}$. By construction, $(b,c) \notin U(y,z)$. For every pair $y,z \in Z$, where $y < z$, there exists an open-and-closed subset $V(y,z)$ of X such that $(b,y) \in V(y,z) \times \{y\} \subseteq U(y,z)$ and $U(y,z) \bigcap (V(y,z) \times \{c\}) = \emptyset$. We put $A = \bigcap\{V(y,z) : y,z \in Z, y < z\}$. Since $b \in A$ and $|Z| = \tau$, A is a closed G_τ-subset of X and $sr(A) \leq n$. Hence there exist n elements $W_1, W_2, ..., W_n \in B$ such that the family

$\{P_i = W_i \bigcap (A \times \{c\}) : i \leq n\}$ is independent. For every pair $1 \leq i < j \leq n$ we fix the points $x_{ij1}, x_{ij2}, x_{ij3} \in A$ such that $(x_{ij1}, c) \in W_i \setminus W_j$, $(x_{ij2}, c) \in W_j \setminus W_i$, and $(x_{ij3}, c) \in W_i \bigcap W_j$. There exists an open subset W of Y such that $c \in W$ and $\{x_{ij1}\} \times W \subseteq W_i \setminus W_j$, $\{x_{ij2}\} \times W \subseteq W_j \setminus W_i$, $\{x_{ij3}\} \times W \subseteq W_i \bigcap W_j$ for every $1 \leq i < j \leq n$. Fix $y, z \in W \bigcap Z$ for which $y < z$. If $1 \leq i < j \leq n$, then $(x_{ij3}, y) \in U(y, z) \bigcap W_i \bigcap W_j$, $(x_{ij1}, y) \in U(y, z) \setminus W_j$, $(x_{ij2}, y) \in U(y, z) \setminus W_i$, $(x_{ij3}, z) \in (W_i \bigcap W_j) \setminus U(y, z)$. Hence the family $\{U(y, z), W_1, ..., W_n\}$ is independent and $sr(B) \geq n + 1$.

Case 2. $c \in cl_Y(L \setminus Z)$.

In this case we assume that $Z = \emptyset$. For every $y \in L$ we fix $U_y \in B$ and an open subset V_y of X such that $(b, c) \in U_y$, $(b, y) \notin U_y$, $(b, c) \in V_y \times \{c\} \subseteq U_y$ and $U_y \bigcap (V_y \times \{y\}) = \emptyset$. We put $A = \bigcap \{V_y : y \in L\}$. Since $b \in A$ and $|L| = \tau$, we have $sr(A) \geq n$. There exist n elements $W_1, W_2, ..., W_n \in B$ such that the family $\{P_i = W_i \bigcap (A \times \{c\}) : i \leq n\}$ is independent. For every $1 \leq i < j \leq n$ we fix the points $x_{ij1}, x_{ij2}, x_{ij3} \in A$ such that $(x_{ij1}, c) \in W_i \setminus W_j$, $(x_{ij2}, c) \in W_j \setminus W_i$ and $x_{ij3}, c) \in W_i \bigcap W_j$. There exists an open subset W of Y such that $c \in W$ and $\{x_{ij1}\} \times W \subseteq W_i \setminus W_j$, $\{x_{ij2}\} \times W \subseteq W_j \setminus W_i$ and $\{x_{ij3}\} \times W \subseteq W_i \bigcap W_j$. Fix $y \in W$. Then for every $1 \leq i < j \leq n$ we have $(x_{ij3}, c) \in U_y \bigcap W_i \bigcap W_j$, $(x_{ij1}, y) \in W_i \setminus U_y$, $(x_{ij2}, y) \in W_j \setminus U_y$, $(x_{ij1}, c) \in U_y \setminus W_j$, $(x_{ij2}, c) \in U_y \setminus W_i$. Hence the family $\{U_y, W_1, W_2, ..., W_n\}$ is independent and $sr(B) \geq n + 1$. The proof is complete. □

7. Examples

Example 7.1. Fix the cardinal τ and the infinite cardinal m. If α is an ordinal, then $|\alpha| = |\{\beta : \beta \leq \alpha \text{ and } \beta \text{ is an ordinal}\}|$. Denote $W(m) = \{\alpha : \alpha \text{ is an ordinal and } |\alpha| < m\}$. Fix a family $\{X_\alpha : \alpha \in W(m)\}$ of discrete spaces of cardinality τ. Denote $B(\tau, m) = \prod \{X_\alpha : \alpha \in W(m)\}$. If $\beta \in W(m)$ and $a_\alpha \in X_\alpha$ for all $\alpha \leq \beta$, then $H(a_\alpha : \alpha \leq \beta) = \{x = (x_\xi : \xi \in W(m)) \in B(\tau, m) : x_\alpha = a_\alpha \text{ for all } \alpha \leq \beta\}$. On $B(\tau, m)$ we consider the topology generated by the base $B = \{H(a_\alpha : \alpha \leq \beta) : a_\alpha \in X_\alpha, \alpha \leq \beta, \beta \in W(m)\}$. Since $r(B) = 1$, then the space $B(\tau, m)$ is zero-dimensional and hereditarily paracompact.

If $m = \aleph_0$, then $B(\tau, m)$ is complete metrizable. We have $|B(\tau, m)| = \tau^m$ and $d(B(\tau, m)) = \sum\{\tau^n : n < m\}$.

Example 7.2. Let τ, m be infinite cardinals and $\tau = \sum\{\tau^n : n < m\} \leq \lambda \leq \tau^m$. Fix in $B(\tau, m)$ some dense subset H of cardinality τ. Let $\{H_s : s \in S\}$, where $H \bigcap S = \emptyset$, be an infinite family of subsets of H of cardinality m such that $|H_s \bigcap H_\mu| < m$ for every pair s, μ of distinct elements of S and that $\{H_s : s \in S\}$ is maximal with respect to the last property. For every point $a = (a_\mu : \mu \in W(m)) \in B(\tau, m)$ there exists $s \in S$ such that $|H_s \bigcap H(a_\alpha : \alpha \leq \beta)| = m$ and $|H_\alpha \setminus H(a_\alpha : \alpha \leq \beta)| < m$ for every $\beta \in W(M)$. Then $|S| = \tau^m$ (for $\tau = m = \aleph_0$ see [8], Exercise 3.6.I(a)). Generate a topology on the set $Y = H \bigcup S$ by the neighbourhood system $\{B_{(y)} : y \in Y\}$, where $B_{(y)} = \{H(y, \emptyset) = \{y\}\}$ if $y \in H$ and $B_{(y)} = \{H(y, L) = \{y\} \bigcup (H_y \setminus L) : L \subseteq H, |L| < m\}$ if $y \in S$.

Every open cover of $H(y, L)$ contains a subcover of $H(y, L)$ of cardinality $< \tau$. In particular, $l(H(y, \emptyset)) \leq m$ and for $m = \aleph_0$ the sets $H(y, L)$ are compact. Fix in S some subset S_1 of cardinality λ and consider $Z = H \bigcup S_1$ as a subspace of Y. We put $X(\tau, m, \lambda) = Z \bigcup \{b\}$, where $b \in Z$. Consider that Z is an open subspace of $X(\tau, m, \lambda)$ and the neighbourhoods of the point b are the form $H(y_1, ..., y_n) = Z \setminus \bigcup\{H(y_i, L_i) : i \leq n\}$, where $n \in N$, $L_1, ..., L_n \subseteq H$, $|L_i| < m$ for every $i \leq n$ and $y_1, ..., y_n \in Z$. We have $\dim X(\tau, m, \lambda) = 0$ and $l(X(\tau, m, \lambda)) \leq m$. Moreover, $X(\tau, m, \lambda)$ is a scattered space with $is(X(\tau, m, \lambda)) = 3$ and every open cover of $X(\tau, m, \lambda)$ contains a subcover of cardinality $< m$. For $m = \aleph_0$ the space $X(\tau, m, \lambda)$ is compact. By construction, $c(X(\tau, m, \lambda)) = d(X(\tau, m, \lambda)) = \tau$ and $\psi(X(\tau, m, \lambda)) = \psi(b, X(\tau, m, \lambda)) = |X(\tau, m, \lambda)| = \lambda$.

Proposition 7.3. *Let τ, m be infinite cardinals and $\tau = \sum\{\tau^n : n < m\} < \lambda \leq \tau^m$. Then:*

1. $wr(X(\tau, m, \lambda)) = \lambda$.
2. $r_\tau(X(\tau, m, \lambda)) = 1$ *if and only if* $\lambda = \tau^+$.
3. *If* $\tau^+ < \lambda \leq \tau^m$, *then* $r_\tau(X(\tau, m, \lambda)) = \lambda$.

Proof. It is obvious that $r_\tau(X(\tau, m, \lambda)) \leq wr(X(\tau, m, \lambda)) \leq \lambda$. By Theorem 3.1, $\lambda = \psi(X(\tau, m, \lambda)) \leq c(X(\tau, m, \lambda)) + wr(X(\tau, m, \lambda)) = \tau + wr(X(\tau, m, \lambda))$. Hence $wr(X(\tau, m, \lambda)) = \lambda$.

If $\lambda = \tau^+$, then from Proposition 2.7 it follows that $r_\tau(X(\tau, m, \lambda)) = 1$. Suppose that $\lambda > \tau^+$. By Theorem 4.1, we have $\lambda = \psi(X(\tau, m, \lambda)) \leq d(X(\tau, m, \lambda)) + r_\tau(X(\tau, m, \lambda)) + \tau^+ = \tau + \tau^+ + r_\tau(X(\tau, m, \lambda))$. Hence $r_\tau(X(\tau, m, \lambda)) = \lambda$. The proof is complete. □

Corollary 7.4. *Let $m = \tau = \aleph_0 < \lambda \leq c = 2^{\aleph_0}$. Then:*

1. $X(\tau, m, \lambda)$ *is a separable scattered compact space.*
2. $wr(X(\tau, m, \lambda)) = \lambda$.
3. $r_\tau(X(\tau, m, \lambda)) = 1$ *if and only if* $\lambda = \aleph_1$.
4. $r_\tau(X(\tau, m, \lambda)) = \lambda$ *if and only if* $\aleph_2 \leq \lambda \leq c$.

Corollary 7.5. *Let $m = \aleph_0$, $\tau \geq m$, $\tau < \lambda \leq \tau^m$. Then:*

1. $X(\tau, m, \lambda)$ *is a compact scattered space.*
2. $wr(X(\tau, m, \lambda)) = \lambda$.
3. $r_\tau(X(\tau, m, \lambda)) = 1$ *if and only if* $\lambda = \tau^+$.
4. $r_\tau(X(\tau, m, \lambda)) = \lambda$ *if and only if* $\tau^+ < \lambda \leq \tau^m$.

By D_τ we denote the discrete space of cardinality τ and by A_τ we denote the Alexandroff one-point compactification of the space D_τ. It is obvious that $wr(A_\tau) = wsr(A_\tau) = 1$ for every cardinal τ.

Theorem 7.6. *Let $n \in N$, $\tau_{n+1} \geq \aleph_0$ and $\tau_1, \tau_2, ..., \tau_n$ be uncountable cardinals. Then $wr(\prod\{A_{\tau_i} : i \leq n+1\}) = wsr(\prod\{A_{\tau_i} : i \leq n+1\}) = n + 1$.*

Proof. We have $ic(A_\tau) = \aleph_0$ for every $\tau \geq \aleph_0$. Fix $1 \leq m \leq n$. Consider that $A_{\tau_i} \setminus D_{\tau_i} = \{a_i\}$. We put $X_m = \prod\{A_{\tau_i} : i \leq m\}$ and $b_m = (a_1, ..., a_m) \in X_m$. If H is a G_δ-subset of X_m and $b_m \in H$, then H contains the copy of X_m and $sr(H) = sr(X_m)$. By Theorem 7.4, $wsr(X_2) = wsr(A_{\tau_1} \times A_{\tau_2}) \geq 2$. By induction and Theorem 7.5 it follows that $wsr(X_m) \geq m$ and $wsr(X_n \times A_{\tau_{n+1}}) \geq n + 1$. This complete the proof. □

Theorem 7.7. *Let X be a locally compact scattered metalindelöf space. If $wr(X \times X) = 1$, then X is metrizable.*

Proof. Let X be a non-discrete space. Since X is locally compact, $ic(X) = \aleph_0$. By Theorem 6.4 it follows that $\psi(X) = \aleph_0$. By virtue of Telgarski's theorem [14], every scattered compact first countable space is metrizable. Hence X is locally metrizable and X has a point-countable base. In virtue of Alexandroff's and Urysohn's theorem ([1], Theorem 5.11; [10], Exercise 4.4.F and Theorem 5.3.10) the space X is metrizable. □

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