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TWO UNUSUAL ORIENTATION REVERSING HOMEOMORPHISMS OF THE ANNULUS ONTO ITSELF

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ABSTRACT. Two unusual orientation reversing homeomorphisms of the annulus onto itself are constructed. In particular, each of the two homeomorphisms gives an answer to a question posed by W. Kuperberg.

The following question was asked by W. Kuperberg: Let A be an annulus, $\{(r, \theta) : 1 \leq r \leq 3\}$, and let h be an orientation reversing homeomorphism of the annulus onto itself which does not interchange the two boundary components of the annulus. Do there always exist two disjoint subcontinua F_1 and F_2 of the annulus, each invariant under h, such that the set $F_1 \cup F_2$ separates the annulus?

In [2], Boyles constructed an orientation reversing fixed point free homeomorphism of the plane onto itself such that the orbit of every point is bounded. In this note, by modifying Boyles' construction, we provide a procedure for constructing maps defined on the annulus such that the closures of the orbits of any two points in the interior of A intersect. We use this procedure to construct the two examples below. This shows that the answer to W. Kuperberg's question is no. These examples are referred to in [3] but have not yet appeared in print.

We find it convenient to define our maps using polar coordinates (r, θ). Indeed our maps will be of the form $F(r, \theta) = (f(r), -(\theta + \beta(r, \theta))$.

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The homeomorphism in each example can be constructed by first rotating each (r,θ) to $(r,\theta+\beta(r,\theta))$, then flipping $(r,\theta+\beta(r,\theta))$ to $(r,-(\theta+\beta(r,\theta)))$, and finally contracting $(r,-(\theta+\beta(r,\theta)))$ to $(f(r),-(\theta+\beta(r,\theta)))$. The maps f and β will be defined below.

Example 1: *There is an orientation reversing homeomorphism F of then annulus A onto itself such that:*

(i) *If $r=1$ or $r=3$, then $F(r,\theta)=(r,-\theta)$,*
(ii) *If $1<r<3$ and $0\le\theta\le 2\pi$, then*

$$\bigcap_{n=1}^{\infty} \mathrm{Cl}\,(\{F^j(r,\theta) : j=n,n+1,n+2,\ldots\}) = \{(r,\theta') : r=1\}.$$

(iii) *If $1<r<3$ and $0\le\theta\le 2\pi$, then*

$$\bigcap_{n=1}^{\infty} \mathrm{Cl}\,(\{F^{-j}(r,\theta) : j=n,n+1,n+2,\ldots\}) = \{(r,\theta') : r=3\}.$$

It follows that every compact set K that is invariant under F and intersects the interior of A must contain $\{(r,\theta) : r=1\}$. Thus, there do not exist two disjoint invariant continua each of which intersects the interior of A. This assures us that the answer to W. Kuperberg's question is no.

Example 2: *There is an orientation reversing homeomorphism H of then annulus A onto itself such that:*

(i) *If $r=1$ or $r=3$, then $H(r,\theta)=(r,-\theta)$,*
(ii) *If $1<r<3$ and $0\le\theta\le 2\pi$, then $\lim_{j\to\infty} H^{2j}(r,\theta)=(1,0)$.*
(iii) *If $1<r<3$ and $0\le\theta\le 2\pi$, then $\lim_{j\to\infty} H^{-2j}(r,\theta)=(3,0)$.*

For this example it follows that every compact set K that is invariant under H and that intersects the interior of A must contain $(1,0)$.

We denote the integers and the positive integers by $\mathbb{Z}$ and $\mathbb{Z}^+$, respectively.

For each $i\in\mathbb{Z}$, let $r_i = \begin{cases} 1+\dfrac{1}{i+1} & \text{if} \quad i\ge 0 \\ 3+\dfrac{1}{i-1} & \text{if} \quad i<0. \end{cases}$

Notice that $\{r_i : i \in \mathbb{Z}\}$ is strictly decreasing, $\lim_{i\to\infty} r_i = 1$ and $\lim_{i\to-\infty} r_i = 3$.

For each $i \in \mathbb{Z}$, let $L_i : [r_{i+1}, r_i] \to [0,1]$ be the linear map

$$L_i(r) = \frac{r - r_{i+1}}{r_i - r_{i+1}}.$$

Lemma 1. *If $i \in \mathbb{Z}$ and $t \in [0,1]$, then*

(i) $L_i^{-1}(t) = t(r_i - r_{i+1}) + r_{i+1}$,
(ii) $L_i^{-1}(1) = r_i$,
(iii) $L_i^{-1}(0) = r_{i+1}$, *and*
(iv) $r_{i+1} \le L_i^{-1}(t) \le r_i$ *if* $0 \le t \le 1$.

Definition: Let $f : [1,3] \to [1,3]$ be such that

$$f(r) = \begin{cases} 1 & \text{if} \quad r = 1, \\ L_{i+1}^{-1}\left((L_i(r))^2\right) & \text{if} \quad r_{i+1} \le r \le r_i \text{ and } i \in \mathbb{Z}, \\ 3 & \text{if} \quad r = 3. \end{cases}$$

Lemma 2. *If f is the map above, then*

(i) $f(r_i) = r_{i+1}$ *for all* $i \in \mathbb{Z}$,
(ii) f^k *maps* $[r_{i+1}, r_i]$ *homeomorphically onto* $[r_{i+k+1}, r_{i+k}]$ *for all* $i \in \mathbb{Z}$ *and* $k \in \mathbb{Z}^+$, *and*
(iii) f *maps* $[1,3]$ *homeomorphically onto* $[1,3]$.

Lemma 3. *If $i \in \mathbb{Z}$ and $r_{i+1} < r < r_i$, then*

(i) $f^j(r) = L_{i+j}^{-1}\left((L_i(r))^{2^j}\right)$ for $j \in \mathbb{Z}^+$,
(ii) *there is a $j \in \mathbb{Z}^+$ such that for $k \ge j$*

$$r_{i+k+1} \le f^k(r) \le \frac{2r_{i+k+1} + r_{i+k}}{3},$$

(iii) $f^{-j}(r) = L_{i-j}^{-1}\left((L_i(r))^{-2^j}\right)$, *and*
(iv) *there is a $j \in \mathbb{Z}^+$ such that for $k \ge j$,*

$$\frac{2r_{i-k} + r_{i-k+1}}{3} \le f^{-k}(r) \le r_{i-k}.$$

Proof: (i) If $j = 1$, then by definition we have

$$f^1(r) = L_{i+1}^{-1}\left((L_i(r))^{2^1}\right).$$

If $f^k(r) = L_{i+k}^{-1}\left((L_i(r))^{2^k}\right)$, where $k \in \mathbb{Z}^+$, then $f^{k+1}(r) = f\left(L_{i+k}^{-1}\left((L_i(r))^{2^k}\right)\right) = L_{i+k+1}^{-1}\left(L_{i+k}\left(L_{i+k}^{-1}\left(\left((L_i(r))^{2^k}\right)^2\right)\right)\right) = L_{i+k+1}^{-1}\left((L_i(r))^{2^{k+1}}\right)$.

(ii) Since $0 < L_i(r) < 1$, there is a $j \in \mathbb{Z}^+$ such that $(L_i(r))^{2^j} < \dfrac{1}{3}$. If $k \geq j$, then $r_{i+k+1} \leq f^k(r) = L_{i+k+1}^{-1}\left((L_i(r))^{2^k}\right) \leq L_{i+k+1}^{-1}\left((L_i(r))^{2^j}\right) \leq L_{i+k+1}^{-1}(\frac{1}{3}) = \dfrac{1}{3}(r_{i+k} - r_{i+k+1}) + r_{i+k+1} = \dfrac{2r_{i+k+1} + r_{i+k}}{3}$.

(iii) Since $r_{i-j+1} < f^{-j}(r) < r_{i-j}, r = f^j\left(f^{-j}(r)\right) = L_i^{-1}\left(L_{i-j}\left(f(r)\right)^{2^j}\right)$. Thus, $L_i(r) = \left(L_{i-j}\left(f^{-j}(r)\right)\right)^{2^j}$. Thus, $(L_{i-j}(r))^{-2^j} = L_{i-j}\left(f^{-j}(r)\right)$.

(iv) Since $0 < L_i(r) < 1$, we may choose j so that $(L_i(r))^{-2^j} > \dfrac{2}{3}$. If $k \leq j$, then $\dfrac{2}{3} \leq L_{i-j}\left(f^{-j}(r)\right)$. Thus, $f^{-k}(r) \geq L_{i-k}\left(\dfrac{2}{3}\right) = \dfrac{2}{3}(r_{i-k} - r_{i-k+1}) + r_{i-j+1} = \dfrac{2r_{i-k} + r_{i-k+1}}{3}$. $\square$

Define:

(i) $\alpha_i = \dfrac{1}{|i| + 1}$ for $i \in \mathbb{Z}$,

(ii)
$$\beta(r,\theta) = \begin{cases} 0 & \text{if } r = 1 \text{ or } r = 3, \\ (-1)^i \alpha_i & \text{if } \dfrac{2r_i + r_{i+1}}{3} \le r \le \dfrac{2r_i + r_{i-1}}{3} \\ & \text{for } i \in \mathbb{Z}, \\ (-1)^{i-1}\left(\dfrac{\alpha_{i-1}+\alpha_i}{r_{i-1}-r_i}(3r - 2r_{i-1} - r_i) + \alpha_{i-1}\right) & \\ & \text{if } \dfrac{2r_i + r_{i-1}}{3} \le r \le \dfrac{2r_{i-1} + r_i}{3} \\ & \text{for } i \in \mathbb{Z}. \end{cases}$$

(iii) $F : A \to A$ by $F(r,\theta) = (f(r), (-(\theta + \beta(r,\theta))) \bmod 2\pi)$.

Lemma 4. *If $i, k \in \mathbb{Z}, k > 0$ and $r_i \le r \le \dfrac{2r_i + r_{i-1}}{3}$, then*

$$F^{2k}(r,\theta) = (f^{2k}(r), \theta + (-1)^i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+2k-1})).$$

Proof: If $r_i \le r \le \dfrac{2r_i + r_{i-1}}{3}$, then

$$0 \le L_{i-1}(r) \le \frac{\dfrac{2r_i + r_{i-1}}{3} - r_i}{r_{i-1} - r_i} = \frac{1}{3}.$$

Thus, $r_{i+1} = L_i^{-1}(0) \le f(r) = L_i^{-1}\Big(\big((L)_{i-1}(r)\big)^2\Big) \le L_i^{-1}\left(\dfrac{1}{9}\right) < L_i^{-1}\left(\dfrac{1}{3}\right) = \dfrac{1}{3}(r_i - r_{i+1}) + r_{i+1} = \dfrac{2r_{i+1} + r_i}{3}$.

Thus, $r_{i+1} \le f(r) \le \dfrac{2r_{i+1} + r_i}{3}$. It follows, by induction, that $r_{i+k} \le f^k(r) \le \dfrac{2r_{i+k} + r_{i+k-1}}{3}$ for all $k \in \mathbb{Z}^+$. Thus, $F^2(r,\theta) = F(F(r,\theta)) = F\big((f(r), -\theta + (-1)^{i+1}\alpha_i\big) = \big(f^2(r), -((-\theta + (-1)^{i+1}\alpha_i) + \beta(f(r), -\theta + (-1)^{i+1}\alpha_i)\big) = (f^2(r), -(-\theta + (-1)^{i+1}\alpha_i) + (-1)^{i+1}\alpha_{i+1}) = \big(f^2(r), \theta + (-1)^{i+2}(\alpha_i + \alpha_{i+1})\big) = \big(f^2(r), \theta + (-1)^i(\alpha_i + \alpha_{i+1})\big)$.

Assume, using finite induction, that $F^{2n}(r,\theta) = (f^{2n}(r), \theta + (-1)^i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+2n-1}))$ for some integer n. Then $F^{2(n+1)}(r,\theta) = F^2(f^{2n}(r), \theta + (-1)^i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+2n-1}))) = (f^{2n+2}(r), \theta + (-1)^i(\alpha_i + \cdots + \alpha_{i+2n-1}) + (-1)^i(\alpha_{i+2n} + \alpha_{i+2n+1})) = (f^{2(n+1)}(r), \theta + (-1)^i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+2n} + \alpha_{i+2n+1})) = (f^{2(n+1)}(r), \theta + (-1)^i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+2n} + \alpha_{i+2(n+1)-1}))$. □

Lemma 5. *F is a homeomorphism of A onto A.*

Proof: Suppose $F(r_1, \theta_1) = F(r_2, \theta_2)$. That is,

$$(f(r_1), \theta_1 + \beta(r_1, \theta_1)) = (f(r_2), \theta_2 + \beta(r_2, \theta_2)).$$

Since f is one to one, we have $r_1 = r_2$. Since β is a function of r alone, it follows that $\theta_1 = \theta_2$.

To see that F is an onto map, let $(r,\theta) \in A$, and notice that $F^{-1}(r,\theta) = (r, -\theta)$ if $r = 1$ or $r = 3$, and that

$$F^{-1}(r,\theta) = (f^{-1}(r), -(\theta - \beta(f^{-1}(r), \theta)),$$

otherwise. □

Lemma 6. *If $\frac{2r_i + r_{i+1}}{3} \leq r \leq r_i$, then $\frac{2r_{i-1} + r_i}{3} \leq f^{-1}(r) \leq r_{i-1}$.*

Proof: Clearly, $f^{-1}(r) \leq f^{-1}(r_i) = r_{i-1}$ and

$$L_i(r) \geq L_i\left(\frac{2r_i + r_{i+1}}{3}\right) = \frac{\frac{2r_i + r_{i+1}}{3} - r_{i+1}}{r_i - r_{i+1}} = \frac{2}{3}.$$

Since $L_i(r) = L_i\left(f\left(f^{-1}(r)\right)\right) = L_i\left(L_i^{-1}\left(\left(L_{i-1}\left(f^{-1}(r)\right)\right)^2\right)\right) = \left(L_{i-1}(f^{-1}(r))\right)^2$, we have $\frac{2}{3} < \left(\frac{2}{3}\right)^{\frac{1}{2}} \leq L_{i-1}(f^{-1}(r)) = \frac{f^{-1}(r) - r_i}{r_{i-1} - r_i} \leq 1$. Thus, $\frac{2r_{i-1} + r_i}{3} = \frac{2}{3}(r_{i-1} - r_i) + r_i \leq f^{-1}(r) \leq r_{i-1}$. It follows by induction that $\frac{2r_{i-k} + r_{i+k-1}}{3} \leq f^{-k}(r) \leq r_{i-k}$ for $k \in \mathbb{Z}^+$. □

Lemma 7. *If* $\dfrac{2r_i + r_{i+1}}{3} \leq r \leq r_i$, *then* $F^{-2k}(r,\theta) = \left(f^{-2k}(r), \theta + (-1)^i(\alpha_{i-1} + \alpha_{i-2} + \cdots + \alpha_{i-2k})\right)$.

Proof: $F^2(f^{-2}(r), \theta + (-1)^i(\alpha_{i-1} + \alpha_{i-2})) = F(F(f^{-2}(r), \theta + (-1)^i(\alpha_{i-1} + \alpha_{i-2}))) = F(f^{-1}(r), -(\theta + (-1)^i(\alpha_{i-1} + \alpha_{i-2}) - \beta(f^{-2}(r), \theta + (-1)^i(\alpha_{i-1} + \alpha_{i-2}))) = F(f^{-1}(r), -(\theta + (-1)^i (\alpha_{i-1} + \alpha_{i-2}) + (-1)^{i-2}\alpha_{i-2})) = F(f^{-1}(r), -(\theta - (-1)^i(\alpha_{i-1}) = F(f^{-1}(r), -(\theta+(-1)^{i-1}\alpha_{i-1}) = (r, -(-\theta+(-1)^{i-1}\alpha_{i-1})-\beta(f^{-1}(r), -\theta + (-1)^{i-1}\alpha_{i-1}) = (r, \theta + (-1)^i\alpha_{i-1} + (-1)^{i-1}\alpha_{i-1}) = (r, \theta)$.

The lemma now follows using finite induction. □

Properties of Example 1: If $r_i \leq r \leq \dfrac{2r_i + r_{i-1}}{3}$ for some i then, according to Lemma 4, we have

$$F^{2k}(r,\theta) = \begin{cases} (f^{2k}(r), \theta + \alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+2k-1}) & \text{if } i \text{ is even} \\ (f^{2k}(r), \theta - (\alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+2k-1})) & \text{if } i \text{ is odd} \end{cases}$$

That is, the sequence of forward even iterates of the $F^j(r,\theta)$ tends toward the circle $r = 1$ and rotates in the counterclockwise or clockwise direction depending on whether i is even or i is odd. Since $\sum_{k=1}^{\infty} \alpha_k = \infty$ and $\lim_{k\to\infty} \alpha_k = 0$, it follows that $\bigcap_{n=1}^{\infty} \mathrm{Cl}\left(\{F^j(r,\theta) : j = n, n+1, n+2, \ldots\}\right) = \{(r,\theta') : r = 1\}$. The above behavior of the forward iterates does not depend on the restriction $r_{i+k} \leq r \leq \dfrac{2r_i + r_{i-1}}{3}$ since, according to Lemma 3 (ii), $\dfrac{2r_{i+1+k} + r_{i+k}}{3} \leq f^k(r) \leq r_{i+k}$ when $r_{i+1} \leq r < r_i$ and k are large. Likewise, $F^{-2k}(r,\theta) = (f^{-2k}(r), \theta + (-1)^i(\alpha_{i-1} + \alpha_{i-2} + \cdots + \alpha_{i-2k}))$ so that the backward iterates tend toward $\{(r,\theta) : r = 3\}$ and rotate slowly around $\{(r,\theta) : r = 3\}$.

Example 2 is constructed exactly like Example 1 except that for $0 \leq \theta < \pi$, the number α_i is replaced by the functions

$$\delta_i(\theta) = \min\left(\frac{\theta}{2}, \alpha_i, \pi - \frac{\theta}{2}\right).$$

If $1 < r < 3$ and we write $H^k((r,\theta)) = (f^k(r), \theta_k)$, then $\delta_i(\theta) + \delta_{i+1}(\theta_1) + \cdots + \delta_{i+2k-1}(\theta_{i+2k-2}) \le \frac{\theta}{2} + \frac{\theta}{4} + \cdots + \frac{\theta}{2^k} < \theta$. It follows that the even iterates can approach, but not cross, the line $\theta = 0$. Since $\sum_{k=1}^{\infty} \alpha_k = \infty$, the iterates cannot stop short of the line $\theta = 0$. Thus, $\lim_{k\to\infty} H^{2k}(r,\theta) = (1,0)$ if $1 < r < 3$. A similar argument shows that $\lim_{k\to\infty} H^{-2k}(r,\theta) = (3,0)$.

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