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A CHARACTERIZATION OF A CLASS OF PIECEWISE LINEAR MODELS WITH A MAXIMUM NUMBER OF FIXED POINTS

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Abstract. We study the dynamical behavior of a class of piecewise linear maps of $\mathbf{R}^n$, designated Brain-State-in-a-Box neural network models. It is shown that such networks have either at most $3^n - 2^n + 1$ fixed points or a continuum set of fixed points. Conditions on the connecting weights that guarantee a maximum number of isolated fixed points are established and the stability type of each fixed point determined.

1. Introduction

The Brain-State-in-a-Box model (BSB) introduced in [1] (see also [2] and [12]) is an artificial neural model inspired by the anatomy and physiology of the brain. BSB models are oversimplifications of reality that provide excellent theoretical framework to simulate a large spectrum of psychological behaviors, such as categorical perception [1], adaptation [6], visual word perception [8], and multistable perception [14]. Each model represents the discrete activity of a group of n interconnected neurons The network's activity is a vector in $\mathbf{R}^n$ whose update is done by the action of an $n \times n$ matrix, designated connecting matrix, followed by the action of a piecewise linear function, designated transfer function.

The qualitative orbital behavior of BSB models has been investigated by several researchers using different techniques from Matrix Theory [10] and [13] to Gradient Descent and Lyapunov functions

[3] and [9]. In 1994, Lillo, et al [15] introduced a generalization of the BSB model (GBSB) and a learning rule that successfully retrieves prototype patterns stored as asymptotically stable fixed points of a network with a non-symmetric connecting matrix. Such a network is used to perform as an associative memory for which simulations have shown that the number of spurious states is as small or smaller than in any other model. Moreover, non-symmetric interconnection is a desirable feature that facilitates certain implementation issues [7], [16], [17], and [19].

Given a mathematical neural model that simulates some brain function, the location of fixed points and their stability type is of great interest since it decides their potential use as fundamental memories (see [18]). In this paper we consider a generalization of the BSB model whose transfer function keeps the network's activity level in the hypercube $[0,\, 1]^n$. We establish conditions on the connecting matrix that guarantee a maximum number of isolated fixed points (Theorem 3.2). We also present a characterization of those fixed points that are stable, conditionally stable, and unstable.

2. Basic Definitions and Notation

We consider a fully connected neural network with n neurons. A network's state is an assignment of real numbers in the unit interval to the neurons composing the net; each such number measures the average firing rate of a given neuron in a certain time interval. A state vector, $\mathbf{x} = (x_i)_{i=1}^n$, is regarded as a point in the hypercube $\mathcal{H} = [0, 1]^n$. Information traveling from neuron i to neuron j, x_i changes linearly via a multiplicative factor, called connecting weight and represented by ω_{ji}. We denote by $\mathbf{W}$ the matrix of all connecting weights $[\,\omega_{ij}]_{i,\,j=1,\,\ldots,\,n}$. The neuron j changes its incoming signal, $\sum_{i=1}^n \omega_{ji} x_i$, via the piecewise linear transfer function:

$$\sigma(x) = \begin{cases} 0 & if \quad x \leq 0 \\ x & if \quad 0 < x < 1 \\ 1 & if \quad x \geq 1. \end{cases}$$

The dynamics of the net is considered to be the discrete time evolution of an arbitrary input, given by the vector map T, satisfying $\pi_j\left(T\left(\mathbf{x}\right)\right) = \sigma\left(\sum_{i=1}^n \omega_{ji} x_i\right)$, for $j = 1, \ldots n$, where π_j represents the standard projection onto the j-th component. Such an artificial neural network is designated of saturated linear type. The transfer function simulates the response of a neuron outside some pre-assigned threshold value. In fact, no activity is spontaneously generated; the neuron saturates after a certain value of the signal's intensity and leaves it unchanged otherwise. For simplicity of notation we define $\sigma(\mathbf{x})$ to be $(\sigma(x_i))_{i=1,\ldots,n}$.

Definition 2.1. A point $\mathbf{x}$ in $\mathcal{H}$ is a fixed state vector of the network or simply a fixed point of T if $T(\mathbf{x}) = \mathbf{x}$.

Definition 2.2. Given two disjoint subsets of $\{1, 2, \cdots, n\}$, $\mathbf{I}$ and $\mathbf{J}$, the face of type $(\mathbf{I}, \mathbf{J})$ is the subset $\Delta_{(\mathbf{I}, \mathbf{J})}$ (or just Δ for simplicity of notation) of $\mathcal{H}$ such that: (1) $\pi_i(\Delta) = 0$ or 1 if $i \in \mathbf{I}$ or $\in \mathbf{J}$, respectively, and (2) $\pi_i(\Delta)$ is equal to the open interval $(0, 1)$ for $i \notin \mathbf{I} \cup \mathbf{J}$.

The dimension of Δ is the number of elements $i \in \{1, 2, \cdots, n\} \setminus \mathbf{I} \cup \mathbf{J}$. The set of all k-faces (or faces of dimension k) is denoted by $\mathcal{F}^k$, and $\mathcal{F}$ represents the set of all possible faces in $\mathcal{H}$. In particular, we notice that the cardinality of $\mathcal{F}^0$ is 2^n, since 0-faces are the vertices of $\mathcal{H}$. Moreover, 1-faces are the edges of $\mathcal{H}$, and the only n-face is the open hypercube. A face, $\Delta_{(\mathbf{I}, \mathbf{J})}$, is said to be non-adjacent to $\mathbf{O} = (0)_{i=1,\ldots,n}$ if $\mathbf{J} \neq \emptyset$; otherwise, it is said to be adjacent to $\mathbf{O}$. We denote the set of all k-faces adjacent to $\mathbf{O}$ by $\mathbf{Adj}_k$ and its complement by $\mathbf{NAdj}_k$. It follows from the definitions provided that a vertex, $\mathbf{x}$, of $\mathcal{H}$ is a fixed point under T if and only if $\pi_i(\mathbf{W}\mathbf{x}) \begin{cases} \geq 1 & \text{if} \quad \pi_i(\mathbf{x}) = 1 \\ \leq 0 & \quad \text{otherwise.} \end{cases}$
In fact, if $\mathbf{x}$ is a vertex of $\mathcal{H}$ and $\mathbf{J} = \{\, i : \pi_i(\mathbf{x}) = 1 \,\}$ then we have that:

$$\pi_i(T(\mathbf{x})) =$$

$$\sigma\left(\textstyle\sum_{j \in \mathbf{J}} \omega_{ij}\right) = \begin{cases} 1 & \text{if} \quad i \in \mathbf{J} \iff \pi_i(\mathbf{W}\mathbf{x}) = \sum_{j \in \mathbf{J}} \omega_{ij} \geq 1 \\ 0 & \text{if} \quad i \notin \mathbf{J} \iff \pi_i(\mathbf{W}\mathbf{x}) = \sum_{j \in \mathbf{J}} \omega_{ij} \leq 0. \end{cases}$$

A nontrivial region consisting of fixed points is highly sensitive to small perturbations which creates creates serious computational difficulties since long term iterations may accumulate stepwise errors that grow exponentially fast. Computer generated orbits might be misleading by presenting a behavior not encountered in the real system (cf. [5]).

Remark 2.1. If $\Delta \in \mathbf{Adj}_k$, $\mathbf{x} \in \Delta$, and $T(\mathbf{x}) = \mathbf{x}$ then T has infinitely many fixed points.

Let $(\mathbf{I}, \mathbf{J}(= \emptyset))$ be the type of the face Δ, since $(x) = \{x_i\}_{i=1,\dots,n}$ is a fixed point of T then we have that

$$\sum_j \omega_{ij} x_j \begin{cases} \leq 0 & \text{if} \quad i \in \mathbf{I} \\ = x_i & \quad \text{otherwise.} \end{cases}$$

The segment connecting $\mathbf{O}$ to $\mathbf{x}$ is pointwise fixed. If $\mathbf{y} = \{\lambda x_i\}_i$, with $0 < \lambda < 1$, then

$$\pi_i\left(T(\mathbf{y})\right) = \sigma\left(\textstyle\sum_j \omega_{ij} \lambda x_j\right) =$$

$$\sigma\left(\lambda \textstyle\sum_j \omega_{ij} x_j\right) = \begin{cases} \leq 0 & \iff \quad i \in \mathbf{I} \\ \sigma\left(\lambda x_i\right) = \lambda x_i & \iff \quad i \notin \mathbf{I}. \end{cases}$$

Remark 2.2. If $\mathbf{x}$ and $\mathbf{y}$ are two distinct fixed points in Δ, a k-face non-adjacent to $\mathbf{O}$, then T has infinitely many fixed points.

If $(\mathbf{I}, \mathbf{J}(\neq \emptyset))$ is the type of Δ then the segment connecting $\mathbf{x}$ to $\mathbf{y}$ is pointwise fixed. In fact, for $0 < \lambda < 1$, we have that $T\left(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\right) = \sigma\left(\mathbf{W}\left(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\right)\right)$ and

$$\sigma\left(\lambda\mathbf{W}\mathbf{x} + (1-\lambda)\mathbf{W}\mathbf{y}\right) = \begin{cases} \lambda x_i + (1-\lambda) y_i & \text{if} \quad i \notin \mathbf{I} \cup \mathbf{J} \\ 0 \;\text{ or }\; 1 & \text{if} \quad i \in \mathbf{I} \,\text{or}\, \mathbf{J}, \end{cases}$$

respectively.

The statements of these two remarks are summarized in the following Proposition.

Proposition 2.1. *If T has finitely many fixed points then every fixed point of T is a vertex of $\mathcal{H}$ or is in the interior of a face non-adjacent to $\mathbf{O}$. Moreover, the interior of a face non-adjacent to $\mathbf{O}$ contains at most one fixed point.*

3. Networks with a Maximum Number of Isolated Fixed Points

In this section we assume that T has finitely many fixed points. Proposition 2.1 allows us to determine a sharp upper bound of the number of fixed points (cf. Theorem 3.1 and Example 1). We also determine conditions on the connecting weights that are equivalent to the attainability of the upper bound established in Theorem 3.1 (cf. Theorem 3.2). It follows that the set of stable state vectors grows exponentially fast with the number of neurons composing the network.

Theorem 3.1. *If T has finitely many fixed points then it has at most* $3^n - 2^n + 1$.

Proof: We denote the number of fixed points of T by $\#\mathrm{Fix}(T)$. T has at most one fixed point in the interior of each non-adjacent (to $\mathbf{O}$) k-face and $\#\mathbf{NAdj}_k = \begin{pmatrix} n \\ k \end{pmatrix}(2^{n-k} - 1)$. Therefore, we have that,

$$\#\mathrm{Fix}(T) \leq \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix}(2^{n-k} - 1) =$$

$$\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} 2^{n-k} - \sum_{k=1}^{n} \begin{pmatrix} n \\ k \end{pmatrix} = 3^n - 2^n + 1.$$

□

The following result is a straightforward corollary of Theorem 3.1 and Proposition 2.1.

Corollary 3.1. *T has exactly $3^n - 2^n + 1$ fixed points if and only if all the vertices of $\mathcal{H}$ are fixed and each non-adjacent k-face has exactly one fixed point in its interior.*

We provide an example of a 2-dimensional network with exactly six fixed points. Thus, the given upper bound is sharp. The 2-dimensional network with connecting matrix $\mathbf{W} = \begin{bmatrix} 4 & -1 \\ -1 & 5 \end{bmatrix}$ has exactly six fixed points:

$$\{(0,\, 0),\, (1,\, 0),\, (0,\, 1),\, (1,\, 1),\, (\tfrac{1}{3},\, 1),\, (1,\, \tfrac{1}{4})\}.$$

The next theorem establishes conditions on the network connecting matrix under which this upper bound is actually attained.

We represent by $\widetilde{\mathbf{W}} = [\tilde{\omega}_{ij}]$ the matrix $\mathbf{W} - \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Let $\mathbf{N} = \{1, 2, \ldots, n\}$, $p \in \mathbf{N}$, and $\mathbf{A}$, $\mathbf{A}_0$, and $\mathbf{A}_1$ are three subsets of $\mathbf{N}$. We define the operator $\mathcal{L}^p_{\mathbf{A}}$, on the set of all $n \times n$ matrices, which associates to each matrix $\mathbf{M} = [m_{ij}]$ an $n \times n$ matrix $\mathcal{L}^p_{\mathbf{A}}(\mathbf{M}) = [a_{ij}]_{i,j=1,\ldots,n}$, where, for every i,

$$a_{ij} = \begin{cases} m_{ij} & \text{if} \quad j \neq p \\ \sum_{k\in\mathbf{A}} m_{ik} & \text{if} \quad j = p. \end{cases}$$

If r and s represent the cardinalities of $\mathbf{A}_0$ and $\mathbf{A}_1$, respectively, we define the operator $\mathcal{L}_{\{\mathbf{A}_0,\,\mathbf{A}_1\}}$ that associates to each $n \times n$ matrix $\mathbf{M}$, an $(n-r)\times(n-s)$ matrix obtained from $\mathbf{M}$ by deleting those rows specified in $\mathbf{A}_0$ and those columns specified in $\mathbf{A}_1$. When $\mathbf{A}_0 = \mathbf{A}_1$ we simply denote this operator by $\mathcal{L}_{\mathbf{A}_0}$. Given an $n \times n$ matrix $\mathbf{M}$, $\det(\mathbf{M})$ represents the determinant of $\mathbf{M}$. We denote by $\circ$ the usual map composition.

Theorem 3.2. *A saturated linear network with n neurons has exactly* $3^n - 2^n + 1$ *fixed points if and only if* $det(\widetilde{\mathbf{W}}) \geq 0$, *and for every pair of disjoint subsets of* $\mathbf{N}$, $\{\mathbf{A}_0,\, \mathbf{A}_1\}$ *with* $\mathbf{A}_1 \neq \emptyset$, *and* $\{i\} \subset \mathbf{A}_0 \cup \mathbf{A}_1$,

$$det\left(\mathcal{L}_{\mathbf{A}_0\cup\mathbf{A}_1\setminus\{i\}} \circ \mathcal{L}^i_{\mathbf{A}_1}\right)(\widetilde{\mathbf{W}}) \begin{cases} > 0 & if \quad i \in \mathbf{A}_1 \\ < 0 & if \quad i \in \mathbf{A}_0. \end{cases}$$

The next lemma is a consequence of the well-known formula that computes the determinant of a matrix along the entries of a given row (cf. [11, ch. 8]). This lemma will be used in the proof of the theorem.

Lemma 3.1. *If* $\mathbf{B}_0$ *and* $\mathbf{B}_1$ *are two disjoint subsets of* $\mathbf{N}$ *and* $k \in \mathbf{B}_0 \cup \mathbf{B}_1$ *then*

$$det\left(\mathcal{L}_{\mathbf{B}_0\cup\mathbf{B}_1\setminus\{k\}}\circ\mathcal{L}^k_{\mathbf{B}_1}\right)\left(\widetilde{\mathbf{W}}\right) = \sum_{j\in\mathbf{B}_1} \widetilde{\omega}_{kj}\, det\left(\widetilde{\mathbf{W}}_c\right) - \sum_{t=1}^{p} \widetilde{\omega}_{ki_t}\, det\left(\widetilde{\mathbf{W}}^t_\Delta\right).$$

Proof: If $\mathbf{B}_2 = \{i_1,\, i_2,\, \ldots,\, i_p\}$, the complement of $\mathbf{B}_0 \cup \mathbf{B}_1$, and $l = 1, \ldots,$ or $p+1$, we denote by $\mathbf{M}^l$ the $(p+1)-$matrix $\left(\mathcal{L}_{\mathbf{B}_0\cup\mathbf{B}_1\setminus\{k\}}\circ\mathcal{L}^k_{\mathbf{B}_1}\right)\left(\widetilde{\mathbf{W}}\right)$, where l is the smallest integer such that $i_l > k$. This matrix is obtained from $\mathcal{L}_{\mathbf{B}_0\cup\mathbf{B}_1}\ (\widetilde{\mathbf{W}})$ by including the row vector

$$\left(\tilde{\omega}_{ki_1},\, \tilde{\omega}_{ki_2},\, \ldots,\, \tilde{\omega}_{ki_{l-1}},\, \sum_{j\in B_1} \tilde{\omega}_{kj},\, \tilde{\omega}_{ki_l},\, \ldots,\, \tilde{\omega}_{ki_p}\right)$$

at the l-position and the column vector

$$\left(\sum_{j\in B_1}\tilde{\omega}_{i_1 j},\ \sum_{j\in B_1}\tilde{\omega}_{i_{l-1} j},\ \sum_{j\in B_1}\tilde{\omega}_{kj},\ \sum_{j\in B_1}\tilde{\omega}_{i_l j},\ \ldots,\ \sum_{j\in B_1}\tilde{\omega}_{i_p j}\right)^t$$

at the l-position. Without loss of generality, we may assume $k = 1$, therefore $l = 1$. In this case we have that

$$\det(\mathbf{M}^1) = \textstyle\sum_{t=1}^{p+1}(-1)^{1+t} m_{1t} \det\left(\mathcal{L}_{\{\{1\},\{t\}\}}(\mathbf{M}^1)\right),$$

where m_{1t} represents the $(1,\, t)-$entry of $\mathbf{M}^1$. The matrix $[\tilde{\omega}_{i_l i_s}]_{l,s=1,\ldots,p}$ is denoted by $\widetilde{\mathbf{W}}_c$ and $\widetilde{\mathbf{W}}^t_\Delta$ represents the matrix $\widetilde{\mathbf{W}}_c$ with the $t^{th}-$column replaced by the vector

$$(\textstyle\sum_{j\in B_1}\tilde{\omega}_{i_1 j},\ \sum_{j\in B_1}\tilde{\omega}_{i_2 j},\ \ldots,\ \sum_{j\in B_1}\tilde{\omega}_{i_p j}).$$

Since $\det\left(\mathcal{L}_{\{1\}}(\mathbf{M}^1)\right) = \det\left(\widetilde{\mathbf{W}}_c\right)$ and $\det\left(\mathcal{L}_{\{\{1\},\{t\}\}}(\mathbf{M}^1)\right) = (-1)^{t-2}\det\left(\widetilde{\mathbf{W}}^{t-1}_\Delta\right)$ we have that

$$\det(\mathbf{M}^1) = m_{11}\det\left(\widetilde{\mathbf{W}}_c\right) - \textstyle\sum_{t=2}^{p+1}(-1)^{2t-2} m_{1t}\det\left(\widetilde{\mathbf{W}}^{t-1}_\Delta\right) =$$

$$\textstyle\sum_{j\in B_1}\tilde{\omega}_{kj}\det\left(\widetilde{\mathbf{W}}_c\right) - \sum_{t=2}^{p+1}\tilde{\omega}_{k i_{t-1}}\det\left(\widetilde{\mathbf{W}}^{t-1}_\Delta\right).$$

Therefore,

$$\det(\mathbf{M}^1) = \sum_{j\in B_1}\tilde{\omega}_{kj}\det\left(\widetilde{\mathbf{W}}_c\right) - \sum_{t=1}^{p}\tilde{\omega}_{k i_t}\det\left(\widetilde{\mathbf{W}}^t_\Delta\right).$$

This completes the proof of the Lemma. □

Proof of Theorem 3.2: We show that the condition stated in the Theorem implies the existence of $3^n - 2^n + 1$ fixed points. It is sufficient to prove that every vertex of $\mathcal{H}$ is fixed, and every face non-adjacent to $\mathbf{O}$ has a fixed point in its interior (cf. Proposition 2.1).

Let $\mathbf{B}_0$ and $\mathbf{B}_1(\neq \emptyset)$ be two subsets of $\mathbf{N}$. We denote by $\mathbf{B}_2 = \mathbf{N}\setminus(\mathbf{B}_0\cup\mathbf{B}_1)$ and by Δ the face of type $(\mathbf{B}_0\setminus\mathbf{B}_1,\, \mathbf{B}_1)$ such that $\mathbf{x}\in\Delta$ if and only if

$$\pi_i(x) = \begin{cases} 1 & \text{if} \quad i\in\mathbf{B}_1 \text{ and} \\ 0 & \text{if} \quad i\in\mathbf{B}_0\setminus\mathbf{B}_1. \end{cases}$$

Case 1. $\mathbf{B}_2 = \emptyset$.

In this case Δ is a vertex of $\mathcal{H}$ and $\pi_i(T(\mathbf{x})) = \sum_{j\in\mathbf{B}_1}\omega_{ij} =$
$det\big(\mathcal{L}_{\mathbf{N}\setminus\{i\}}\circ\mathcal{L}^i_{\mathbf{B}_1}\big)(\mathbf{W}) \begin{cases} >1 & \text{if} \quad i\in\mathbf{B}_1 \\ <0 & \text{if} \quad i\in\mathbf{B}_0\setminus\mathbf{B}_1. \end{cases}$

Consequently, $\mathbf{x}$ is a fixed point of T.

Case 2. $\mathbf{B}_2 \neq \emptyset$.

Let $\mathbf{B}_2 = \{i_1, i_2, \ldots, i_p\}$ and, for simplicity of notation, assume that $\mathbf{B}_0 \cap \mathbf{B}_1 = \emptyset$. If T has a fixed point, $\mathbf{x} = \{x_1, \ldots, x_n\} \in \text{int}(\Delta)$, then it satisfies

$$\sum_{j\in\mathbf{B}_1}\omega_{kj} + \sum_{t=1}^{p}\omega_{ki_t}x_{i_t} \begin{cases} = x_k & \text{if} \quad k\in\mathbf{B}_2 & (1) \\ \geq 1 & \text{if} \quad k\in\mathbf{B}_1 & (2) \\ \leq 0 & \text{if} \quad k\in\mathbf{B}_0. & (3) \end{cases}$$

System (1) is equivalent to

$$\sum_{t=1}^{p}\widetilde{\omega}_{ki_t}x_{i_t} = -\sum_{j\in\mathbf{B}_1}\widetilde{\omega}_{kj} \quad (\,k\in\mathbf{B}_2\,),$$

and it has a unique solution if the determinant of the coefficient matrix, $\widetilde{\mathbf{W}}_c$, is not equal to 0. Since $\det(\widetilde{\mathbf{W}}_c) = \det\big(\mathcal{L}_{\mathbf{B}_0\cup\mathbf{B}_1}\big)\big(\widetilde{\mathbf{W}}\big) = \det\big(\mathcal{L}_{\mathbf{B}_0\cup\mathbf{B}_1\setminus\{i_t\}}\circ\mathcal{L}^{i_t}_{\{i_t\}}\big)(\widetilde{\mathbf{W}})$, the condition stated in the theorem (with $\mathbf{A}_0 = \mathbf{B}_0\cup\mathbf{B}_1$ and $\mathbf{A}_1 = \{i_t\}$) implies that $\det(\widetilde{\mathbf{W}}_c) > 0$. Therefore, system (1) has a unique solution given by

$$x_{i_t} = -\frac{\det\left(\widetilde{\mathbf{W}}^t_\Delta\right)}{\det\left(\widetilde{\mathbf{W}}_c\right)},$$

with

$$\det\left(\widetilde{\mathbf{W}}^t_\Delta\right) \;=\; \det\big(\mathcal{L}_{\mathbf{B}_0\cup\mathbf{B}_1}\circ\mathcal{L}^{i_t}_{\mathbf{B}_1}\big)\left(\widetilde{\mathbf{W}})\right).$$

Since $i_t \notin \mathbf{B}_1$ we have that $\det\big(\mathcal{L}_{\mathbf{B}_0\cup\mathbf{B}_1}\circ\mathcal{L}^{i_t}_{\mathbf{B}_1}\big)\left(\widetilde{\mathbf{W}}\right) < 0$ which implies $x_{i_t} > 0$.

On the other hand, the matrices $\widetilde{\mathbf{W}}^t_\Delta$ and $\widetilde{\mathbf{W}}_c$ are identical except at the t-column. We define a new matrix $\widetilde{\mathbf{W}}^t_*$ obtained from $\widetilde{\mathbf{W}}_c$ by adding its t-column with the t-column of $\widetilde{\mathbf{W}}^t_\Delta$. We have that

$$\det\left(\widetilde{\mathbf{W}}^t_\Delta\right) + \det\left(\widetilde{\mathbf{W}}_c\right) = \det\left(\widetilde{\mathbf{W}}^t_*\right).$$

This shows that

$$\det\left(\widetilde{\mathbf{W}}_*^t\right) = \det\bigl(\mathcal{L}_{\mathbf{B}_0\cup\mathbf{B}_1}\circ\mathcal{L}^{i_t}_{\mathbf{B}_1\cup\{i_t\}}\bigr)\left(\widetilde{\mathbf{W}}\right) > 0,$$

which implies $x_{i_t} < 1$. Moreover, inequalities (2) and (3) follow from the Lemma 3.1.

Conversely, if T has $3^n - 2^n + 1$ fixed points we first prove that for every pair of disjoint sets $\mathbf{A}_0$ and $\mathbf{A}_1$ such that $\mathbf{A}_1 \neq \emptyset$ and $\mathbf{A}_0 \cup \mathbf{A}_1 = \mathbf{N}$ we have

$$\det\bigl(\mathcal{L}_{\mathbf{N}\setminus\{i\}}\circ\mathcal{L}^{i}_{\mathbf{A}_1}\bigr)\left(\widetilde{\mathbf{W}}\right)\begin{cases} > 0 & \text{if} \quad i \in \mathbf{A}_1 \\ < 0 & \text{if} \quad i \in \mathbf{A}_0. \end{cases}$$

If the point $\mathbf{x} \in \mathcal{H}$ (such that $\pi_i(x) = 1$ if $i \in \mathbf{A}_1$ and $\pi_i(x) = 0$ if $i \in \mathbf{A}_0$) is a fixed point then

$$\det\bigl(\mathcal{L}_{\mathbf{N}\setminus\{i\}}\circ\mathcal{L}^{i}_{\mathbf{A}_1}\bigr)\left(\widetilde{\mathbf{W}}\right)\begin{cases} \geq 0 & \text{if} \quad i \in \mathbf{A}_1 \\ \leq 0 & \text{if} \quad i \in \mathbf{A}_0. \end{cases}$$

We need to prove that these inequalities are strict. Let Δ be 1-face satisfying $\pi_j(\text{int}(\Delta)) = 1$ if and only if $j \in \mathbf{A}_1 \setminus \{i\}$. Since T has a unique fixed point $\mathbf{x}$ in $\text{int}(\Delta)$, x_i satisfies the following equation

$$\omega_{ii}x_i + \sum_{s\in\mathbf{A}_1\setminus\{i\}} \omega_{is} = x_i.$$

Since T has finitely many fixed points then $\omega_{ii} > 1$ and $\sum_{s\in\mathbf{A}_1} \omega_{is} < 0$, if $i \notin \mathbf{A}_1$. If $\mathbf{A}_1 = \{i\}$ then $n = 1$ (degenerate case) or Δ is adjacent to $\mathbf{O}$ which contradicts Remark 2.1. Moreover, if $i \in \mathbf{A}_1$, since $x_i < 1$, we have that $\sum_{s\in\mathbf{A}_1} \omega_{is} > 1$. This proves that the inequalities obtained above are strict.

Now, we proceed by induction. Let us assume that for every $\mathbf{B}_0$ and $\mathbf{B}_1$, disjoint subsets of $\mathbf{N}$ such that $\mathbf{B}_1 \neq \emptyset$ and $\#(\mathbf{N} \setminus \mathbf{B}_0 \cup \mathbf{B}_1) \leq k - 1$, we have

$$\det\left(\mathcal{L}_{\mathbf{B}_0\cup\mathbf{B}_1\setminus\{i\}}\circ\mathcal{L}^{i}_{\mathbf{B}_1}\right)\left(\widetilde{\mathbf{W}}\right)\begin{cases} > 0 & \text{if} \quad i \in \mathbf{B}_1 \\ < 0 & \text{if} \quad i \in \mathbf{B}_0. \end{cases}$$

We want to show that given $\mathbf{A}_0$ and $\mathbf{A}_1$, disjoint subsets of $\mathbf{N}$, satisfying $\mathbf{A}_1 \neq \emptyset$ and $\mathbf{A}_2 = (\mathbf{N} \setminus \mathbf{A}_0 \cup \mathbf{A}_1) = \{j_1, \ldots, j_k\}$ with $k < n - 1$, we have

$$\det\left(\mathcal{L}_{\mathbf{A}_0\cup\mathbf{A}_1\setminus\{i\}}\circ\mathcal{L}^{i}_{\mathbf{A}_1}\right)\left(\widetilde{\mathbf{W}}\right)\begin{cases} > 0 & \text{if} \quad i \in \mathbf{A}_1 \\ < 0 & \text{if} \quad i \in \mathbf{A}_0. \end{cases}$$

Let Δ be a k-face such that $\pi_i(\mathrm{int}(\Delta)) = 1$ if and only if $i \in \mathbf{A}_1$. T has a unique fixed point, $\mathbf{x} \in \mathrm{int}(\Delta)$, which satisfies the relations

$$\sum_{j\in\mathbf{A}_1} \omega_{ij} + \sum_{t=1}^{p} \omega_{ij_t} x_{j_t} \begin{cases} = x_i & \text{if} \quad i \in \mathbf{A}_2 & \text{(a)} \\ \geq 1 & \text{if} \quad i \in \mathbf{A}_1 & \text{(b)} \\ \leq 0 & \text{if} \quad i \in \mathbf{A}_0. & \text{(c)} \end{cases}$$

System (a) is equivalent to $\sum_{t=1}^{k} \widetilde{\omega}_{ij_t} x_{j_t} = -\sum_{j\in\mathbf{A}_1} \widetilde{\omega}_{ij}$ (for $i \in \mathbf{A}_2$) and it has a unique solution if and only if

$$\det\,(\widetilde{\mathbf{W}}_c) = \det\,\left(\mathcal{L}_{\mathbf{A}_0\cup\mathbf{A}_1\setminus\{j_t\}} \circ \mathcal{L}^{j_t}_{\{j_t\}}\right)\left(\widetilde{\mathbf{W}}\right) \neq 0.$$

Let $\mathbf{B}_0 = \mathbf{A}_0 \cup \mathbf{A}_1,$ and $\mathbf{B}_1 = \{j_t\}$, (therefore, $\#\,(\mathbf{N} \setminus (\mathbf{B}_0 \cup \mathbf{B}_1))$ $= k - 1$) then the hypothesis implies that $\det(\widetilde{\mathbf{W}}_c) =$ $\det\left(\mathcal{L}_{\mathbf{A}_0\cup\mathbf{A}_1\setminus\{j_t\}} \circ \mathcal{L}^{j_t}_{\{j_t\}}\right)\left(\widetilde{\mathbf{W}}\right) > 0$. The solution of system (a) is given by

$$x_{j_t} = -\frac{\det(\widetilde{\mathbf{W}}^t_\Delta)}{\det(\widetilde{\mathbf{W}}_c)}, \;\; t = 1, \ldots, k.$$

Since $x_{j_t} > 0$ we have that $\det\,(\widetilde{\mathbf{W}}^t_\Delta) = \det\left(\mathcal{L}_{\mathbf{A}_0\cup\mathbf{A}_1} \circ \mathcal{L}^{j_t}_{\mathbf{A}_1}\right)\left(\widetilde{\mathbf{W}}\right)$ < 0. Lemma 3.1 allows us to write inequalities (b) and (c) as follows

$$\det\,\left(\mathcal{L}_{\mathbf{A}_0\cup\mathbf{A}_1\setminus\{i\}} \circ \mathcal{L}^{i}_{\mathbf{A}_1}\right)\left(\widetilde{\mathbf{W}}\right) =$$

$$\textstyle\sum_{j\in\mathbf{A}_1} \widetilde{\omega}_{ij}\ \det(\widetilde{\mathbf{W}}_c) - \sum_{s=1}^{k} \widetilde{\omega}_{ij_s} \det(\widetilde{\mathbf{W}}^t_\Delta) \begin{cases} \geq 0 & \text{if} \quad i \in \mathbf{A}_1 \\ \leq 0 & \text{if} \quad i \in \mathbf{A}_0. \end{cases}$$

Once again, it remains to show that these two inequalities are strict. We consider $i \in \mathbf{A}_0 \cup \mathbf{A}_1$ and define a $(k+1)$-face, Δ of type $(\mathbf{A}_0 \setminus \{i\},\, \mathbf{A}_1 \setminus \{i\})$, such that $\pi_k(\mathrm{int}(\Delta)) = 1$ if $k \in \mathbf{A}_1 \setminus \{i\}$ and $\pi_k(\mathrm{int}(\Delta)) = 0$ if $k \in \mathbf{A}_0 \setminus \{i\}$. Similar arguments show that

$$0 \,<\, x_i = -\frac{\det(\widetilde{\mathbf{W}}^i_\Delta)}{\det(\widetilde{\mathbf{W}}_c)} \,<\, 1.$$

The induction hypothesis implies that
$\det\left(\mathcal{L}_{\mathbf{A}_0\cup\mathbf{A}_1\setminus\{i\}} \circ \mathcal{L}^{i}_{\mathbf{A}_1}\right)\left(\widetilde{\mathbf{W}}\right) \;=\; \det(\widetilde{\mathbf{W}}^i_\Delta) < 0$ if $i \in \mathbf{A}_0$ and
$\det\left(\mathcal{L}_{\mathbf{A}_0\cup\mathbf{A}_1\setminus\{i\}} \circ \mathcal{L}^{i}_{\mathbf{A}_1}\right)\left(\widetilde{\mathbf{W}}\right) \;= \det(\widetilde{\mathbf{W}}^i_\Delta) > 0$ if $i \in \mathbf{A}_1$.

It remains to prove that $\det(\widetilde{\mathbf{W}}) \geq 0$. If Δ is the $(n-1)-$face of type $(\emptyset,\; \{n\})$ then the only fixed point of T in the $\mathrm{int}(\Delta)$ satisfies the system

$$\sum_{j=1}^{n-1} \widetilde{\omega}_{ij} x_j = -\widetilde{\omega}_{in}, \;\; i = 1, \cdots, n-1$$

and the inequality

$$\sum_{j=1}^{n-1} \widetilde{\omega}_{nj} x_j + \widetilde{\omega}_{nn} \geq 0.$$

This implies that det $(\widetilde{\mathbf{W}}) \geq 0$.

4. Stability Properties

Information stored in stable fixed points can be retrieved from corrupted inputs. It is important to determine which fixed points are stable since they provide good storage places. We start by reviewing the definition of stable and unstable fixed point used in [15] and introduce the definition of conditionally stable fixed point. We denote by $d(\mathbf{a}, \mathbf{b})$ the standard euclidean distance in $\mathbf{R}^n$ between the vectors $\mathbf{a}$ and $\mathbf{b}$. This metric induces a topology in $\mathcal{H}$, a neighborhood in $\mathcal{H}$, is the intersection of a neighborhood in $\mathbf{R}^n$ with $\mathcal{H}$.

Definition 4.1. A fixed point $\mathbf{x}$ is called *stable* if and only if there exists an $\epsilon > 0$ so that for every $\mathbf{y} \in \mathcal{H}$, $\epsilon-$close to $\mathbf{x}$ (i.e., $d(\mathbf{x}, \mathbf{y}) < \epsilon$), $T(\mathbf{y}) = \mathbf{x}$.

A fixed point $\mathbf{x}$ is *unstable* if and only if there exists an $\epsilon > 0$ so that for every $\mathbf{y} \in \mathcal{H}$, $\epsilon-$close to $\mathbf{x}$, $\liminf_{n\to+\infty} d(T^n(\mathbf{y}), \mathbf{x}) > 0$.

A fixed point $\mathbf{x}$ of T is called *conditionally stable* if and only if it is neither stable nor unstable and there exists an $\epsilon > 0$, so that for every $\mathbf{y_0}$ and $\mathbf{y_1} \in \mathcal{H}$, $\epsilon-$close to $\mathbf{x}$, $T(\mathbf{y_0}) = \mathbf{x}$ and $\liminf_{n\to+\infty} d(T^n(\mathbf{y_1}), \mathbf{x}) > 0$.

Lemma 4.1. *If T has finitely many and a maximum number of fixed points, then:*
1. $\mathbf{O}$ *is unstable.*
2. Every vertex of $\mathcal{H}$, different from $\mathbf{O}$*, is a stable fixed point.*
3. If $\mathbf{x}$ *is a fixed point of T in a k-face, then* $\mathbf{x}$ *is conditionally stable.*

Proof:

(1) It is enough to notice that given a state vector $\mathbf{x} \neq \mathbf{O}$ such that $\pi_i(\mathbf{x}) = \delta > 0$ and $\pi_j(\mathbf{x}) = 0$ for $j \neq i$ then $\pi_i(T(\mathbf{x})) = \omega_{ii}\delta > \delta$ (since $\omega_{ii} > 1$) and $\pi_j(T(\mathbf{x})) \leq 0$ for $j \neq i$. This implies that $d(T^n(\mathbf{x}), \mathbf{O}) \geq d(\mathbf{x}, \mathbf{O})$, for all $n \geq 1$.

(2) This statement follows from Theorem 3.2, and the continuity of the map $\phi \;:\; \mathbf{x} \;\rightarrow \sum_j \omega_{ij} x_j$.

(3) Let $\epsilon > 0$ and $\mathbf{y}$ be a state vector such that $\pi_i(\mathbf{y}) = \pi_i(\mathbf{x}) + \epsilon$, for some i, and $\pi_k(\mathbf{y}) = \pi_k(\mathbf{x})$, for $k \neq i$. It follows that $\pi_k(T(\mathbf{y})) = \pi_k(\mathbf{x}) + \omega_{ki}\epsilon$. Therefore, $d(T(\mathbf{y}), \mathbf{x}) > \epsilon > 0$. Now, we denote by Δ the k-face containing $\mathbf{x}$ in its interior and let $(\mathbf{A}_0, \mathbf{A}_1)$ be its type. Since $\Delta \in \mathbf{NAdj}_k$, the set $\mathbf{A}_1$ is nonempty (cf. Definition 2.2). Theorem 3.2 implies that for every $\mathbf{y} \in \Delta$ we have that

$$\sum_{j \in \mathbf{A}_0 \cup \mathbf{A}_1} \omega_{ij} \pi_j(\mathbf{y}) + \sum_{j \in \mathbf{A}_1} \omega_{ij} \begin{cases} > 1 & \text{if} \quad i \in \mathbf{A}_1 \\ < 0 & \text{if} \quad i \in \mathbf{A}_0. \end{cases}$$

The continuity of T and compactness of Δ guarantee the existence of a neighborhood of Δ in $\mathcal{H}$ given by the cartesian product, $\prod_{i=1}^n \mathcal{N}_i$, whose components satisfy the following:

$$\pi_i(\mathcal{N}_i) = \begin{cases} \pi_i(\Delta) & \text{if} \quad i \notin \mathbf{A}_0 \cup \mathbf{A}_1 \\ [0,\, \epsilon] & \text{if} \quad i \in \mathbf{A}_0 \\ [1-\epsilon,\, 1] & \text{if} \quad i \in \mathbf{A}_1 \end{cases}$$

Moreover, given $a \in [0, \epsilon]$ and $b \in [1-\epsilon, 1]$, the set $\prod_{i=1}^n \mathcal{N}_i^*$ where $\mathcal{N}_i^* = \mathcal{N}_i$ if $i \in \mathbf{A}$, $\mathcal{N}_i^* = a$ if $i \notin \mathbf{A}_0 \cup \mathbf{A}_1$, and $\mathcal{N}_i^* = b$ if $i \in \mathbf{A}_1$ has image under T equal to Δ. This implies the existence of a path in $\prod_{i=1}^n \mathcal{N}_i$ whose image under T is equal to $\mathbf{x}$.

The following Proposition follows from the previous Lemma.

Proposition 4.1. *If T has a maximum number of fixed points then every vertex of $\mathcal{H}$ different from $\mathbf{O}$ is stable, $\mathbf{O}$ is unstable, and the remaining fixed points are conditionally stable.*

We notice that a k-face with a fixed point in its interior is expanded under the action of W; therefore, we conclude that the only stable fixed points are the 0-faces different from $\mathbf{O}$.

5. Conclusions

In this paper, we have studied the dynamical behavior of a generalized BSB neural model with activation levels restricted to the hypercube $[0, 1]^n$. We characterize a class of such networks that exhibit a maximum number of isolated fixed points in terms of determinants of sub-matrices of connecting weights. It is proved that there exist one unstable fixed point (the origin), $2^n - 1$ stable fixed points and $3^n - 2^{n+1} + 1$ conditionally stable fixed points, if T has a maximum number of isolated fixed points. Stable fixed points provide good storage places for easily recoverable information, and stored patterns can be retrieved from corrupted inputs. Conditionally stable fixed points can be interpreted as "conditional memories" or "creative memories" where pattern recovery depends on the initial input noise. The information stored in a conditionally stable fixed point is retrieved only if the input belongs to the stable "pseudo-manifold" of the fixed point. The unstable fixed points are "accidental memories" where pattern recovery is purely accidental and highly improbable.

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