A SEMILINEAR MODEL FOR EXPONENTIAL DYNAMICS AND TOPOLOGY

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Abstract. We present a model over the plane that recreates the same dynamics involved for the complex exponential family. This model is based on a one parameter family of continuous, semilinear maps. Under certain assumptions over the parameter value, we show the continuum obtained from the semilinear map resembles the one obtained for $E_\lambda(z)$.

1. Introduction

The complex exponential family $E_\lambda(z) = \lambda e^z$ exhibits both rich topology and interesting dynamics. It is known that if $\lambda$ is real and $\lambda > 1/e$, then $E_\lambda$ admits an invariant set in the strip $0 \leq \text{Im} z \leq \pi$ that is an indecomposable continuum [3]. This is a closed connected set which cannot be decomposed into two (not necessarily distinct) closed, connected sets. Such sets have a complicated topological structure [1], [5], [6], [7].

In contrast to this rich topology, the dynamics on this invariant set are quite tame. There is a unique repelling fixed point in the set. All other orbits either eventually land on the real line and then tend to $\infty$, or else they accumulate on both the orbit of 0 and a point at $\infty$.

Similar invariant indecomposable continua have been found in a variety of complex exponentials with $\lambda$ complex [8]. There is also an uncountable collection of different, non-invariant indecomposable continua.
continua in the Julia sets of these maps [4]. Unfortunately, very little is known about the actual topology of these invariant sets. Also, how this topology depends on the parameter $\lambda$ is an open question.

In an effort to simplify some of these questions, we propose in this paper a simpler family of maps that shares many of the topological and dynamical properties of the complex exponential family. Our map is a piecewise semilinear map of the strip $0 \leq \text{Im } z \leq 1$. This map mimics the behavior of $E_\lambda$, $\lambda > 1/e$ in that there is an invariant indecomposable continuum in this strip on which our map behaves dynamically like $E_\lambda$ (see Figure 1). Because of the semilinearity, our map is significantly easier to work with in many respects. We illustrate this by computing a type of kneading invariant for this family of maps. This has allowed the second author to show that each member of this family is not topologically conjugate to any other member, despite the fact that their gross dynamical properties are the same (see [8]). Another advantage of our model is the fact that we can construct an uncountable number of curves homeomorphic to the real line that belong to the continuum. These curves are in fact composants. We conjecture that every point in the continuum that is not accessible from the exterior of the strip lies in such a curve.

2. Definition of the map

Let $S$ denote the strip $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}$. Let $L_1 \subset S$ be the open half strip

$$L_1 = \{(x, y) \in S \mid x > 0, \ 1/3 < y < 2/3\}.$$ 

Let $\lambda > 0$ (later we will further restrict $\lambda$). We define a family of homeomorphisms $h_\lambda : S - L_1 \to S - \{(\lambda, 0)\}$ as follows. We first decompose $S - L_1$ into three subregions:

$$\begin{align*}
H_1 &= \{(x, y) \mid x \geq 0, \ 0 \leq y \leq 1/3\} \\
H_2 &= \{(x, y) \mid x \geq 0, \ 2/3 \leq y \leq 1\} \\
H_3 &= \{(x, y) \mid x \leq 0\}.
\end{align*}$$
We further subdivide $H_3$ into 3 substrips of equal height:

$A_1 = \{(x, y) | x \leq 0, \ 0 \leq y \leq 1/3\}$

$A_2 = \{(x, y) | x \leq 0, \ 1/3 \leq y \leq 2/3\}$

$A_3 = \{(x, y) | x \leq 0, \ 2/3 \leq y \leq 1\}$.

Finally, we define

$$h_\lambda(x, y) = \begin{cases} 
(\lambda + e^x, 3y) & \text{if } (x, y) \in H_1 \\
(\lambda - e^x, 3(1 - y)) & \text{if } (x, y) \in H_2 \\
e^x(1, 3y) + (\lambda, 0) & \text{if } (x, y) \in A_1 \\
e^x(3 - 6y, 1) + (\lambda, 0) & \text{if } (x, y) \in A_2 \\
e^x(-1, 3(1 - y)) + (\lambda, 0) & \text{if } (x, y) \in A_3.
\end{cases}$$

One checks easily the following facts:

1. $h_\lambda$ maps $H_1$ onto the half strip $\{(x, y) \in S | x \geq \lambda + 1\}$.
2. $h_\lambda$ maps $H_2$ onto the half strip $\{(x, y) \in S | x \leq \lambda - 1\}$.
3. $h_\lambda$ is an expansion on $H_1 \cup H_2$ (strict expansion if $x \geq \delta > 0$).
Figure 2. Domain and range for $h_\lambda$

(4) $h_\lambda$ maps $H_3$ onto the rectangle $\{(x, y) \in S \mid \lambda - 1 \leq x \leq \lambda + 1\}$, missing only the point $(\lambda, 0)$.

(5) Note that $h_\lambda$ is not a contraction on $H_3$. However, if $x$ is sufficiently negative, then $h_\lambda$ is a strong contraction. By a slight abuse of language we will call the image $h_\lambda(H_3) = \{(x, y) \in S \mid \lambda - 1 \leq x \leq \lambda + 1\}$ the contracting rectangle.

(6) $h_\lambda$ maps $S - L_1$ homeomorphically onto $S - \{(\lambda, 0)\}$.

(7) On the real axis, $h_\lambda$ is given by the function $x \to e^x + \lambda$.
This function is conjugate to $x \to e^\lambda \cdot e^x$ via the conjugacy $x \to x - \lambda$.

(8) $h_\lambda$ maps $y = 1$ to the half line $(-\infty, \lambda)$ on the real axis, and maps the boundary of $L_1$ onto the line $y = 1$.

In Figure 2, we display the domain and range of $h_\lambda$. Note that the image of $H_3$ consists of three triangular regions: on the right, $T_r$, the image of $A_1$; in the center, $T_c$, the image of $A_2$; and on the left, $T_l$, the image of $A_3$. This is one aspect of the linearity of our map: $E_\lambda$ would map this region to a semicircular region.

The map $h_\lambda$ is similar to a complex exponential map of the form $E_\mu(z) = \mu e^z$, $\mu > 1/e$, on the strip $0 \leq y \leq \pi$. $E_\mu$ maps this strip onto the upper half plane (minus the origin) with a region analogous to our $L_1$ mapped above $y = \pi$. Thus, we think of $L_1$
as the set of points in $S$ whose orbit leaves $S$ after one iteration, although we do not define $h_\lambda$ on $L_1$.

For later use, note that the image of a vertical line $x = \nu$ in $H_3$ is a linear horseshoe curve. This curve is the portion of the boundary of a rectangle formed by the three straight lines:

1. $x = e^\nu + \lambda$, $0 < y \leq e^\nu$
2. $y = e^\nu$, $\lambda - e^\nu \leq x \leq \lambda + e^\nu$
3. $x = e^\nu - \lambda$, $0 < y \leq e^\nu$.

We call a region bounded by two such linear horseshoe curves a linear horseshoe region.

3. The indecomposable continuum

For each $j \geq 2$, let $L_j = h_\lambda^{-1}(L_{j-1})$. Each $L_j$ is an open simply connected subset of $S$ that contains the half-strip $x > 0$, $1/3^j < y < 2/3^j$, among (many) other points. Note also that the $L_j$ are disjoint, since $h_\lambda(L_{j+1}) = L_j$. We think of $L_j$ as the set of points that “escape from $S$” after $j$ iterations of $h_\lambda$.

Let $\partial L_j$ denote the boundary of $L_j$. If $(x, y) \in \partial L_j$, then $h_\lambda^j(x, y)$ lies on the line $y = 1$; $h_\lambda^{j+1}(x, y)$ lies on the real axis to the left of $\lambda$; and $h_\lambda^{j+2}(x, y)$ lies to the right of $\lambda$. Any point on $\partial L_j$ thus has orbit that eventually tends to $\infty$ along the real axis.

Let $\Lambda_\lambda$ denote the closure of $\bigcup_j \partial L_j$. Our main result in this paper is:

**Theorem 3.1.** $\Lambda_\lambda$ is an indecomposable continuum.

To prove this, we will show that there exists a simply connected domain $U$ whose boundary is $\Lambda_\lambda$ and we will exhibit a prime end of $U$ whose impression contains $\Lambda_\lambda$. Hence, by a result of Rutt [9], $\Lambda_\lambda$ is either indecomposable or the union of two indecomposable continua \(^1\). The proof of Theorem 3.1 will rule out the latter case.

Denote by $\gamma$ the union of the boundaries of the $L_j$. Now obviously, this choice of $\gamma$ is neither continuous nor compact. However, we may compactify the set as in [3] by first compressing the strip $S$ to a bounded horizontal strip and then by identifying points on the “backward” orbit of $(\lambda, 0)$. That is, we identify the point corresponding to $(-\infty, 0)$ with $(-\infty, 1), (\infty, 1)$ with $(\infty, 2/3), (\infty, 1/3)$

\(^1\)We thank J. C. Mayer for showing us this argument using prime ends.
with \((\infty, 2/9)\), and so forth. Then we must show that the union of the preimages of \(y = 0\) accumulates everywhere on itself.

**Proposition 3.2.** The curve \(\gamma\) accumulates everywhere on itself.

**Proof:** Note first that the boundaries of the \(L_j\) accumulate everywhere on \(y = 0\). Indeed, the boundary of \(L_j\) contains a horizontal line segment of the form \(x = t, y = 1/3^j\) with \(t \geq 0\). Hence, the boundaries accumulate on \(x \geq 0\). Consider any vertical line \(x = \nu \leq 0, 0 < y < \tau\), where \(0 < \tau < 1/3\). Then \(h_\lambda\) maps this line to a vertical line of the form \((e^\nu + \lambda, 3y e^\nu)\) which crosses infinitely many of the horizontal lines above. Hence, the boundaries accumulate everywhere on \(y = 0\). Now apply \(h_\lambda^{-k}\). It follows that the boundaries of the \(L_j\) accumulate everywhere on each \(\partial L_k\). In fact, we have shown more: Let \((x, y) \in \partial L_k\) and let \(\mathcal{N}\) be a neighborhood of \((x, y)\). Then there exists \(k_0\) such that, if \(k > k_0\), then \(\partial L_k\) meets \(\mathcal{N}\). \(\square\)

It follows that the closure of \(\gamma\) is \(\Lambda_\lambda\). There is a natural embedding of the curve \(\gamma\) into the Riemann sphere that places its unique endpoint at infinity. The last Proposition implies that the points in \(\gamma\) are the only points accessible from the “exterior” of \(\Lambda_\lambda\). Denote by \(U\) the exterior region of the curve. Clearly, \(U\) is an open simply connected region of the plane whose boundary is \(\gamma\). Define by \(\{A_k\}\) a chain of crosscuts such that \(A_k\) lies in the interior of \(U\) and the endpoints of each \(A_k\) are the point at infinity and the \(k\)th iterate of the backward orbit of \((\lambda, 0)\). It is easy to see that \(\{A_k\}\) is a fundamental chain having \(N_k\) as the component of \(U - A_k\) bounded by \(A_k\) and \(\gamma\) minus the arc \([h^{-k}(\lambda, 0), \infty]\).

Let \(\eta\) be the prime end associated to this fundamental chain. As \(\gamma\) accumulates everywhere on itself, the impression of \(\eta\) given by

\[
I(\eta) = \bigcap_k N_k,
\]

must contain \(\gamma\). But \(I(\eta)\) is by definition a compact set; hence, it contains the closure of \(\gamma\).

By Rutt’s result, \(\overline{\gamma}\) is either indecomposable or the union of two indecomposable continua. Assume \(\overline{\gamma} = A \cup B\), where \(A\) and \(B\) are indecomposable. Without loss of generality, assume that the point at infinity belongs to \(A\) and there is a point \(p \in \gamma\) for which the arc \([p, \infty)\) of \(\gamma\) is also contained in \(A\). If no such point exists, \(B\) must
contain the arc \([p, \infty)\). But since \(B\) is a closed set in the topology inherited from the Riemann sphere, we must have \(\infty \in B\). In this case, we choose \(B\).

If \(\gamma \cap B \neq \emptyset\), then for any \(q \in \gamma \cap B\) there exists an open neighborhood \(\mathcal{W}\) of \(q\), such that \(\mathcal{W} \cap A = \emptyset\). By Proposition 3.2, there exists \(k_0 > 0\) such that, if \(k > k_0\), \(L_k\) meets \(\mathcal{W}\). Using the interior of \(L_k\) and \(\mathcal{W}\), we can construct an open annulus that bounds the arc \([p, \infty]\) away from infinitely many points in \(\gamma\). Since \(A\) is connected, then \(A\) is completely contained inside the open annulus.

But \(A\) is indecomposable and contains a ray of the form \([x, \infty]\), \(x \in \mathbb{R}\). Then there is a point in \(z \in A\) that lies above the real line and below \(\partial L_k\) (otherwise, \(A\) would be decomposable).

Then, there exists \(n > k > k_0\) for which \(L_n\) lies in between \(z\) and \(\mathbb{R}^+\), and moreover, \(L_n\) enters the neighborhood \(\mathcal{W}\). We can construct a second annular region that will bound the point \(z\) away from the arc \([p, \infty]\). But this contradicts the fact that \(A\) is connected.

Hence, \(\gamma \cap B = \emptyset\) and \(\gamma \subset A\). Since \(A\) is closed, it follows that \(B = \emptyset\) and \(\gamma\) is indecomposable. This ends the proof of Theorem 3.1. \(\square\)

**Proposition 3.3.** The curve \(\gamma\) is a composant of \(\Lambda_\lambda\).

**Proof:** Assume otherwise. Denote by \(\kappa\) the composant of the point at infinity. Let \(z \in \kappa - \gamma\) and \(H\) be a proper subcontinuum that contains both \(z\) and \(\infty\). Clearly, \(\gamma \cap H \neq \emptyset\). Without loss of generality, assume there exists a point \(x\) on the real line such that the arc \([x, \infty]\) is the maximal arc that contains the point at infinity and belongs to \(\gamma \cap H\). Let \(A = H - [x, \infty]\). Since \(H\) is connected, \(A\) must accumulate on \(x\). But as the regions \(L_k\) accumulate on the real line, there exists \(N > 0\) for which \(A\) also accumulates on \(\partial L_k\), for all \(k > N\). Since \(H\) is closed, it follows that

\[
\bigcup_{k > N} \partial L_k \subset H.
\]

By Proposition 3.2, \(\bigcup_{k > N} \partial L_k = \gamma\), which contradicts the fact that \(H\) is a proper subcontinuum. \(\square\)
4. Dynamics of $h_\lambda$

For the remainder of this paper, we will consider only $\lambda$-values larger than $\lambda_0$, where $\lambda_0$ is given by the following proposition.

**Proposition 4.1.** There exists a unique $\lambda_0 > 1$ that solves the equation

$$\lambda - \exp(\lambda - 1) = \ln(2/3).$$

The proof is straightforward calculus.

If $\lambda > \lambda_0 > 1$, it follows that the contracting rectangle lies to the right of $x = 0$ in $S$. For convenience, we introduce the real valued functions $f_\lambda(x) = \lambda + e^x$ and $g_\lambda(x) = \lambda - e^x$. Note that $f_\lambda$ and $g_\lambda$ are the real parts of $h_\lambda$ on $H_1$ and $H_2$, respectively. In this section we will prove:

**Theorem 4.2.** The map $h_\lambda$ has a unique repelling fixed point $p_\lambda$ in $S$. All other orbits have $\alpha$-limit set given by \{p_\lambda\}. The $\omega$-limit set is either

1. the point at $\infty$ given by $x = \infty, y = 0$, in which case the orbit eventually lands on the real axis, or
2. the orbit of $\lambda, 0$ together with points at $\infty$ to the left and right in $S$, in which case the orbit eventually cycles through $H_1, H_2$ and $H_3$, accumulating on the orbit of $\lambda, 0$.

Before proving this theorem, consider the rectangle $R = \{(x, y) \in S | 0 \leq x \leq \lambda + 1\}$. Note that $R$ contains the contracting rectangle since $\lambda > \lambda_0 > 1$. There are certain points in $R$ whose orbit leaves $S$. For example, if $(x, y) \in R$ with $1/3 < y < 2/3$ and $x > 0$, then $(x, y) \in L_1$. Similarly, if $1/3^k < y < 2/3^k$ and $x \geq 0$, then $(x, y) \in L_k$. Let $W$ denote $R$ with all the open strips $1/3^k < y < 2/3^k$ removed. Then $W$ is a union of closed rectangles $R_k$, $k = 0, 1, 2, \ldots$, where $R_k = \{(x, y) \in R | 2/3^{k+1} \leq y \leq 1/3^k\}$. Note that $R_0 \subset H_2$ but $R_1 \subset H_1$ for $j \geq 1$.

We define the first return map $\phi_\lambda : W \to R$, by $\phi_\lambda(x, y) = h^k_\lambda(x, y)$ where $h^k_\lambda(x, y) \in W$, but $h^i_\lambda(x, y) \not\in W$ for $i = 1, \ldots, k - 1$. Note that the first iterate under $h_\lambda$ of the left-hand boundary of $R_k$, $k \geq 1$, is the right-hand boundary of $R_{k-1}$. For technical reasons we will not consider these points as the first return points; rather, we will consider the next return as the first return points.
Proposition 4.3. For any \( k \geq 1 \), \( \phi_\lambda(R_k) \) is a linear horseshoe region that is located strictly below \( R_k \) in \( W \).

Proof: If \( (x, y) \in R_k, k \geq 1 \), then \( h_i^k(x, y) \in H_1 \) for \( i = 0, \ldots, k-1 \), and \( h_k^k(x, y) \in H_2 \). We claim that \( h_{k+1}^k(x, y) \in H_3 \). Indeed, if \( k = 1 \), \( h_2^1(R_1) \) is the rectangle bounded on the right by \( x = g_\lambda(f_\lambda(0)) = \lambda - \exp(\lambda + 1) \) which is negative for \( \lambda > 0 \). On the left, this rectangle is bounded by \( x = g_\lambda \circ f_\lambda^k(0) \). It follows similarly that \( h_{k+1}^k(R_k) \) is also a rectangle in \( H_3 \) whose right boundary is given by \( x = g_\lambda \circ f_\lambda^k(0) \). It follows that the image of \( h_{k+1}^k(R_k) \) is a linear horseshoe region that lies in the contracting rectangle, hence in \( R \).

The maximal \( y \)-coordinate in \( h_{k+2}^k(R_k) \) is given by the \( y \)-coordinate of the image of the right hand boundary of \( h_{k+1}^k(R_k) \cap A_2 \). This \( y \)-coordinate is \( \exp(g_\lambda \circ f_\lambda^k(0)) \). Thus, we need to show that

\[
\exp(g_\lambda \circ f_\lambda^k(0)) < \frac{2}{3^{k+1}}.
\]

Notice first that the map \( g_\lambda \circ f_\lambda^k(0) = \lambda - \exp(f_\lambda^k(0)) \) is a decreasing function of \( \lambda \) when \( \lambda > 0 \). One checks easily that \( g_\lambda \circ f_\lambda^k(0) < -\exp^{k+1}(0) \), which in turn implies that \( \exp(g_\lambda \circ f_\lambda^k(0)) < 1/\exp^{k+2}(0) \). Since \( 3^{k+1} < 2 \exp^{k+2}(0) \) holds for all \( k \), the required inequality holds for \( \lambda = 0 \) and hence for all \( \lambda > 0 \). □

It follows from the above Proposition that any point in \( R_k, k \geq 1 \), whose forward orbit remains for all time in \( S \), must repeatedly visit \( W \), and each time this orbit returns to \( W \) it does so in an \( R_k \) with strictly larger \( k \)-value. This is not true in \( R_0 \), since the image of \( R_0 \) is the rectangle \( \lambda - \exp(\lambda + 1) \leq x \leq \lambda + 1 \) which contains \( R_0 \). Indeed we have:

Proposition 4.4. There is a repelling fixed point \( p_\lambda \) in \( R_0 \).

Proof: The real function \( g_\lambda(x) = \lambda - e^x \) has a fixed point \( x_\lambda \) in the interval \( 0 \leq x \leq \lambda + 1 \) since \( g_\lambda \) is decreasing. Indeed, \( g_\lambda(0) = \lambda - 1 > 0 \), and \( g_\lambda(\lambda + 1) = \lambda - \exp(\lambda + 1) < 0 \) when \( \lambda > 1 \). Similarly, \( y \to 3(1-y) \) has a fixed point at \( y = 3/4 \). Hence, \( p_\lambda = (x_\lambda, 3/4) \) is fixed for \( h_\lambda \) and

\[
Dh_\lambda(x_\lambda, 3/4) = \begin{pmatrix}
-\exp(x_\lambda) & 0 \\
0 & -3
\end{pmatrix},
\]

so \( p_\lambda \) is repelling. □
Proposition 4.5. If \((x, y) \in R_0\) and \((x, y) \neq p_\lambda\), then the orbit of \((x, y)\) must eventually leave \(R_0\) and enter either \(L_1\) or \(H_1\).

Proof: We divide \(R_0\) into two rectangles:
\[
U_0 = \{(x, y) \in R_0 | x \geq \lambda - 1\}
\]
\[
U_1 = \{(x, y) \in R_0 | x \leq \lambda - 1\};
\]
i.e., \(U_0\) lies in the contracting rectangle, \(U_1\) does not.

The image of \(U_0\) is the rectangle
\[
\lambda - \exp(\lambda + 1) \leq x \leq \lambda - \exp(\lambda - 1) < 0
\]
contained in \(H_3\). We claim that, since \(\lambda > \lambda_0\), the image of this rectangle, i.e., \(h_\lambda^2(U_0)\), lies below \(y = 2/3\) in the contracting rectangle. To see this, note that the vertical line \(x = \lambda - \exp(\lambda - 1)\) is mapped to a linear horseshoe curve of height \(\exp(\lambda - \exp(\lambda - 1))\).

Hence, \(h_\lambda^2(U_0)\) lies below \(y = 2/3\) provided
\[
\exp(\lambda - \exp(\lambda - 1)) < 2/3.
\]
But this is precisely the condition that determines \(\lambda_0\). Hence, \(h_\lambda^2(U_0)\) meets only \(L_1\) and \(H_1\), not \(H_2\), and so we are done in this case.

Now suppose \((x, y) \in U_1\). If \(h_\lambda(x, y) \in L_1 \cup H_1\), then we are done.

If \(h_\lambda(x, y) \in H_3\), then \(h_\lambda^2(x, y)\) lies in the contracting rectangle, and hence, in one of \(U_0, L_1,\) or \(H_1\). In all three cases we are done.

Finally, if \(h_\lambda(x, y)\) remains in \(H_2\), then there is a positive integer \(n\) such that \(h_\lambda^n(x, y) \notin U_1\) (for otherwise \((x, y)\) is the fixed point). Then \(h_\lambda^n(x, y)\) lies in one of \(H_1, H_3,\) or \(U_0\) and we are again done. \(\square\)

5. Dynamics of \(\phi_\lambda\)

In this section we describe the set of points in \(\Lambda_\lambda\) whose orbits under the return map \(\phi_\lambda\) have the same “itinerary.” The itinerary of \((x, y)\) is a sequence of integers \((s_0 s_1 s_2 \ldots)\) with \(s_j \geq 1\) such that \(\phi_\lambda^k(x, y) \in R_{s_k}\) for each \(k\). Note that we do not define the itinerary of points in \(R_0\).

Recall that \(\phi_\lambda\) maps each \(R_k\) for \(k \geq 1\) to a linear horseshoe region inside the contracting rectangle that lies strictly below \(R_k\).

In particular, \(\phi_\lambda(R_k)\) meets infinitely many \(R_j\) with \(j > k\) in a pair of rectangles. \(\phi_\lambda(R_k)\) may intersect other \(R_j\) with smaller indices, but this intersection will not be a pair of rectangles (see Figure 3).
In this case, $\phi_\lambda(R_k)$ meets $R_j$ in the two triangular regions $T_\ell$ and $T_r$ lying below the diagonals in the contracting rectangle. We say that $\phi_\lambda(R_k)$ cuts completely across $R_j$ in this case. Note that, if $\phi_\lambda(R_k)$ meets $R_j$ in $T_\ell$ or $T_r$, then $\phi_\lambda$ maps vertical line segments in $\phi_\lambda^{-1}(R_j) \cap R_k$ to vertical line segments in $R_j$. Indeed, the only region where vertical lines are not preserved by $\phi_\lambda$ is $A_2$, and such an orbit does not visit $A_2$ before returning to $W$.

For any $k \geq 1$, let $\ell(k)$ denote the smallest integer such that $\phi_\lambda(R_k)$ cuts completely across $R_{\ell(k)}$. Clearly, $\ell(k) > k$, and, in fact, $\ell(k) \gg k$ for $k$ large.

Let $\nu$ be a continuous curve lying in some $R_k$. We say that $\nu$ is a horizontal curve if $\nu$ meets each vertical line $x = \nu$ with $0 \leq \nu \leq \lambda + 1$ exactly once. A horizontal strip is a region in $R_k$ between two nonintersecting horizontal curves.

Now let $t = (t_1, t_2, t_3, \ldots)$ be a sequence of positive integers that satisfy $t_{k+1} \geq \ell(t_k)$ for $k \geq 1$. Such a sequence is called admissible for $\phi_\lambda$. We have

Theorem 5.1. Suppose $t$ is an admissible sequence for $\phi_\lambda$. Then

$$\{(x, y) \in R_{t_1} \mid \text{the itinerary of } (x, y) \text{ is } (t_1, t_2, t_3, \ldots)\}$$
is a Cantor set of horizontal curves in $\Lambda_\lambda$.

**Proof:** Let $V_{t_1 \ldots t_k}$ denote the set of points whose itinerary begins with $t_1 \ldots t_k$. We have that $V_{t_1 t_2} = R_{t_1} \cap \phi_\lambda^{-1}(R_{t_2})$ is a pair of horizontal strips in $R_{t_1}$. Inductively, $V_{t_1 \ldots t_{k+1}}$ consists of $2^k$ horizontal strips in $R_{t_1}$, and $V_{t_1 \ldots t_{k+1}} \subset V_{t_1 \ldots t_k}$. It follows that the nested intersection of the $V_{t_1 \ldots t_k}$ is a Cantor set of horizontal components in $R_{t_1}$. Since $\phi_\lambda$ maps and contracts vertical vectors to vertical vectors, it follows that each of these components is actually a horizontal curve. Since each of these horizontal curves is an accumulation of horizontal curves that lie in $\partial L_n$ for some $n$, it follows that these curves lie in $\Lambda_\lambda$. □

We now turn to the construction of a second curve $\mu$ that accumulates everywhere on itself and on $\gamma$. This curve is constructed by “connecting” various pieces of the Cantor set of curves that correspond to a given admissible sequence. The admissible sequence $(s_0 s_1 s_2 \ldots)$ involved will have the property that the digits $s_j$ tend to $\infty$ extremely rapidly.

For concreteness, and to begin the construction, we choose $s_0 = 1$. So our sequence will be $(1 s_1 s_2 \ldots)$. We will specify the $s_j$ inductively. Now $\phi_\lambda(R_1) = h_\lambda^3(R_1)$ is a linear horseshoe region that cuts completely across $R_k$ for any $k \geq \ell(1)$. Choose any $s_1 \geq \ell(1)$. Then, for any choice of admissible sequence that begins $1 s_1$, there is a Cantor set of horizontal curves in $R_1$ corresponding to this itinerary. For any such choice, we may select one such horizontal curve, say $\tau_1$. The curve $h_\lambda^3(\tau_1)$ is a horizontal segment lying in $R_{s_1}$ inside one of the horizontal curves whose itinerary is $(s_1 s_2 s_3 \ldots)$. Call this horizontal curve $\tau_{s_1}$. Now the preimage of $\tau_{s_1}$ under $h_\lambda^3$ is a curve that crosses $R_1$ in two horizontal curves, one of which is $\tau_1$. Let $\tau_{1 s_1}$ denote the preimage of $\tau_{s_1}$ under $h_\lambda^3$. Note that $\tau_{1 s_1}$ extends beyond the boundaries of $R_1$ and is mapped in one-to-one fashion onto $\tau_{s_1}$ by $h_\lambda^3$ (see Figure 4).

We now continue this procedure inductively. For any $s_2 > \ell(s_1)$, the image of $R_{s_1}$ under $\phi_\lambda$ is again a linear horseshoe region that cuts completely across $R_{s_2}$, and there is a Cantor set of horizontal curves in this strip that corresponds to any admissible itinerary $(s_2 s_3 s_4 \ldots)$. Note that $\phi_\lambda = h_\lambda^{s_1+2}$ in $R_{s_1}$. As above, $\phi_\lambda(\tau_{s_1})$ lies in one of these curves, say $\tau_{s_2}$. Now pull back this horizontal curve by $h_\lambda^{-(s_1+2)}$. As above, we get a new curve $\tau_{s_1 s_2}$ that cuts completely
Figure 4. The curve $\tau_{1s_1}$ is the extension of $\tau_1$ obtained by pulling back the curve $\tau_{s_1}$ under $\phi^{-1}$.

across $R_{s_1}$ twice and extends $\tau_{s_1}$. If we now pull $\tau_{s_1s_2}$ back by the original map, $h^{-3}_\lambda$, we obtain a new curve which we call $\tau_{1s_1s_2}$. This curve crosses the strip $R_1$ in at least four horizontal curves and extends $\tau_{1s_1}$.

We now put additional restrictions on $s_2$ to control the behavior of $\tau_{1s_1s_2}$. Toward that end, we break up the right hand portion of the strip $S$ into countably many rectangles $Q_j, j = 0, 1, \ldots$ where

$$Q_j = \{(x, y) \in S \mid h^j_\lambda(0) \leq x \leq h^{j+1}_\lambda(0)\}.$$ 

Note that $Q_0$ contains all of the rectangles $R_i$, while $Q_j$ contains the images of $R_1$ under $h^j_\lambda$, provided that $i \geq j$. In particular, note that $h^3_\lambda(R_{s_1+3})$ is contained in $Q_3$ far to the right of $R_{s_1}$.

We now choose $s_2$ so that not only $\phi_\lambda(R_{s_1})$ cuts completely across $R_{s_2}$, but also $\phi_\lambda(R_{s_1+3})$ does as well. Equivalently, the four linear horseshoe regions $\phi_\lambda(R_{s_1+i})$ for $i = 0, 1, 2, 3$ cut completely across $R_{s_2}$. It follows that $\tau_{s_1s_2}$ is a curve that cuts twice completely across not only $R_{s_1}$, but also across the horizontal extensions of this rectangle in $Q_j$ for $j = 1, 2, 3$. Now consider the pull back of this curve to $\tau_{1s_1s_2}$. This curve meets $R_1$ as above, but it must also cut completely across $R_{s_1+3}$ (see Figure 5). That is, the extended curve $\tau_{1s_1s_2}$ meets both $R_1$ and the much lower $R_{s_1+3}$. 
Figure 5. Two steps in the construction of $\mu$.

By choosing $s_2$ larger so that many more linear horseshoe regions of the form $\phi_\lambda(R_{s_1+k})$ cross $R_{s_2}$, we may guarantee that the extended curve crosses not only $R_{s_1+3}$, but also the extensions of this rectangle to the right into $Q_1, Q_2, \ldots Q_n$.

Continuing in such fashion, we may choose the $s_j$ so that the curves $\tau_{1s_1s_2\ldots s_n}$ accumulate everywhere on the positive real axis as $n \to \infty$. If we choose $\mu$ to be the countable union of such extensions, we find a curve that accumulates everywhere on the positive reals. It is not hard to see that, in fact, $\mu$ accumulates on all of $\gamma$ and, hence, on itself. Since $\mu$ is dense in $\Lambda_\lambda$, a similar argument as in Proposition 3.3 shows that $\mu$ is a composant of $\Lambda_\lambda$.

References


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