ON THE COMPARISON OF HEREDITY OF GENERALIZED METRIC PROPERTIES TO MAPPING SPACES AND HYPERSPACES

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Abstract. We compare the heredity of some generalized metric properties to mapping spaces with compact open topology and hyperspaces of compact subsets with finite topology. Especially, we treat here spaces with a $G_δ$-diagonal, $G_δ^*$-diagonal, regular $G_δ$-diagonal, paracompact M-space, M-space, topologically complete spaces and Moore spaces.

1. Introduction

Throughout this paper, all spaces are assumed to be regular $T_1$ unless the contrary is stated explicitly. For a space $X$, we denote the topology of $X$ by $\tau(X)$.

In this paper, we compare the heredity of generalized metric properties of a space $Z$ to the hyperspace $\mathcal{K}(Z)$ of non-empty compact subsets of $Z$ and to the mapping space $C(X, Z)$ with the compact domain. That is, we study two topological operations under classes of topological spaces in terms of generalized metric properties as follows: Let $\mathcal{P}$ be a class of spaces with some generalized metric property;

(I) if $Z \in \mathcal{P}$, then does $\mathcal{K}(Z) \in \mathcal{P}$?
(II) if $X$ is a compact space and $Z \in \mathcal{P}$, then does $C(X, Z) \in \mathcal{P}$?

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Here, $\mathcal{K}(Z)$ is the space of non-empty compact subsets of $Z$ with a finite topology in the sense of [4], called frequently the “Vietoris topology” also, which has a base

$$\{(U_1, \ldots, U_k) | U_1, \ldots, U_k \in \tau(X) \text{ and } k = 1, 2, \ldots\},$$

where $\langle U_1, \ldots, U_k \rangle =$

$$\left\{ K \in \mathcal{K}(Z) \left| K \subset \bigcup_i U_i \text{ and } K \cap U_i \neq \emptyset \text{ for each } i \right. \right\},$$

and $C(X, Z)$ has the compact-open topology which has a base

$$\{W(K_1, \ldots, K_n; U_1, \ldots, U_n) | K_1, \ldots, K_n \in \mathcal{K}(X), U_1, \ldots, U_n \in \tau(Z), n = 1, 2, \ldots\},$$

where $W(K_1, \ldots, K_n; U_1, \ldots, U_n) =$

$$\{f \in C(X, Z) | f(K_i) \subset U_i \text{ for each } i \}.$$

For brevity, we write $\langle \mathcal{U} \rangle$ in place of $\langle U_1, \ldots, U_k \rangle$ when $\mathcal{U} = \{U_1, \ldots, U_k\}$.

Both $\mathcal{K}(Z)$ and $C(X, Z)$ are regular $T_1$ for a regular $T_1$-space $Z$ ([4, Theorem 4.9.10] and [2, Theorem 3.4.13], respectively).

As for undefined terms in generalized metric spaces treated here, refer to [3].

2. The case of M-spaces

Čoban showed in [1, Proposition 2] that if $f$ is a perfect mapping of a completely regular space $X$ onto a completely regular space $Y$, then the mapping $f^* : \mathcal{K}(X) \to \mathcal{K}(Y)$ defined by

$$f^*(K) = f(K), \ K \in \mathcal{K}(X),$$

is also perfect. Using this, he showed that paracompact M-spaces are closed under the operation $I$. But we can show the perfectness of $f^*$ for more general spaces, the proof of which is more complicated than his. We note that his proof indeed proceeds via the Stone-Čech compactifications $\beta X$ and $\beta Y$ and we do not use them.

**Proposition 2.1.** Let $X, Y$ be $T_2$-spaces and let $f$ be a perfect mapping of $X$ onto $Y$. Then $f^* : \mathcal{K}(X) \to \mathcal{K}(Y)$ is also perfect.
Proof: $f^*$ is continuous [4, Theorem 5.10.1] and obviously onto. Since for each $K \in \mathcal{K}(Y)$, $\mathcal{K}(f^{-1}(K))$ is compact, $(f^*)^{-1}(K)$ is compact. Thus it remains to show that $f^*$ is closed. To this end, let $\hat{O}$ be any open subset of $\mathcal{K}(X)$ such that $(f^*)^{-1}(K) \subset \hat{O}$. Take finite collections $\mathcal{U}_i$, $i = 1, \cdots, k$ of open subsets of $X$ such that

\[(f^*)^{-1}(K) \subset \bigcup_i \langle \mathcal{U}_i \rangle \subset \hat{O},\]

\[\langle \mathcal{U}_i \rangle \cap (f^*)^{-1}(K) \neq \emptyset, \quad i = 1, \cdots, k.\]

Let $\{\mathcal{U}(\delta)|\delta \in \Delta\}$ be the totality of finite covers of $f^{-1}(K)$ by members of $\bigcup_i \mathcal{U}_i$. Obviously $\Delta$ is finite. Let $\delta \in \Delta$ and let $\mathcal{V}(1), \cdots, \mathcal{V}(s(\delta))$ be the totality of subcollections of $\mathcal{U}(\delta)$ such that $(Y \setminus f(X \setminus \bigcup \mathcal{V}(i))) \cap K \neq \emptyset$ for each $i$. Let

\[\hat{V}(\delta) = \left\{ Y \setminus f \left( X \setminus \bigcup \mathcal{V}(i) \right) \bigg| i = 1, \cdots, s(\delta) \right\},\]

which is an open neighborhood of $K$ in $\mathcal{K}(Y)$. Thus, if we define

\[\hat{V} = \bigcap \{\hat{V}(\delta)|\delta \in \Delta\},\]

then $\hat{V}$ is an open neighborhood of $K$ in $\mathcal{K}(Y)$. To show the closedness of $f^*$, it suffices to show that $(f^*)^{-1}(\hat{V}) \subset \bigcup_i \langle \mathcal{U}_i \rangle$. On the contrary, assume that there exists

\[L \in (f^*)^{-1}(\hat{V}) \setminus \bigcup_i \langle \mathcal{U}_i \rangle.\]

By (1), there exists $i(0)$ such that $f^{-1}(K) \in \langle \mathcal{U}_{i(0)} \rangle$. Let $\mathcal{U}_{i(0)} = \mathcal{U}(\delta_0)$, $\delta_0 \in \Delta$. If we set

\[\mathcal{U}'_{i(0)} = \{U \in \mathcal{U}_{i(0)}|U \cap L = \emptyset\},\]

then $\mathcal{U}'_{i(0)} \neq \emptyset$ because $f(L) \in \hat{V}(\delta_0)$ by (2) and $L \notin \langle \mathcal{U}_{i(0)} \rangle$. We show the validity of

\[f \left( f^{-1}(K) \setminus \bigcup \mathcal{U}'_{i(0)} \right) = K.\]

Otherwise, there exists

\[p \in K \setminus f \left( f^{-1}(K) \setminus \bigcup \mathcal{U}'_{i(0)} \right),\]
which implies $f^{-1}(p) \subset \bigcup U_i'(0)$. But this is a contradiction because $f(L) \cap (Y \setminus f(X \setminus \bigcup U_i'(0))) \neq \emptyset$ by (3) and by the construction of $V(\delta)$. Again, by (1) there exists $i(1)$ such that
$$f^{-1}(K) \setminus \bigcup U_i'(0) \in \langle U_i(1) \rangle.$$  
Note that $i(0)$ and $i(1)$ are distinct. Choose $\delta_1 \in \Delta$ such that $U_i'(0) \cup U_i'(1) = U(\delta_1)$. Set
$$U_i'(1) = \{ U \in U_i(1) | L \cap U = \emptyset \},$$
then $U_i'(1) \neq \emptyset$, for otherwise, $L \in \langle U_i(1) \rangle$ follows and this is a contradiction to (1). By the same argument as (4), we have
$$f \left( f^{-1}(K) \setminus \bigcup (U_i'(0) \cup U_i'(1)) \right) = K.$$  
Then there exists $i(2)$ with $i(2) \neq i(0), i(1)$ such that
$$f^{-1}(K) \setminus \bigcup (U_i'(0) \cup U_i'(1)) \in \langle U_i(2) \rangle.$$  
Repeating this process as many times as possible, we should come to a contradiction because $\bigcup \langle U_i \rangle$ covers $(f^*)^{-1}(K)$. □

Quasi-perfect mappings do not have this property, even if both $X$ and $Y$ are completely regular. To see it, it suffices to recall that the existence of a countably compact completely regular space $X$ such that $K(X)$ is not M [6, Example 2] and that M-spaces are characterized as a quasi-perfect preimage of a metric space.

The next two examples show that paracompact M-spaces and M-spaces are not closed under the operation $\Pi$.

**Example 2.2.** There exists compact spaces $X$ and $Y$ such that $C(X, Y)$ is not paracompact M.

**Construction:** Let $X$ be the unit interval $[0, 1]$ with the usual topology. It is well known that $C(X, X)$ is not countably compact. Let $Y$ be the long segment due to Michael, i.e., $Y$ consists of $\omega_1$ together with $\alpha < \omega_1$ and $0 \leq t < 1$ and has the interval topology [2, Problem 3.12.19]. We show that $C(X, Y)$ is not paracompact M. Here, we identify $C(X, Y)$ with the subspace
$$\{ (x, f(x)) | x \in X \} \mid f \in C(X, Y)$$
of $K(X \times Y)$, [2, Problem 3.12.27(j)]. Assume that $C(X, Y)$ is paracompact M. Let $(\hat{U}_n)$ be an M-sequence for $C(X, Y)$. Since
$C(X,Y)$ is paracompact, we can assume that each $\hat{U}_n$ is locally finite in $C(X,Y)$. Let $\hat{F}_n$ be a locally finite closed cover of $C(X,Y)$ shrinking $\hat{U}_n$. For each $n$, there exists an open neighborhood $\hat{O}(n)$ of

$f = \{(x,\omega_1) | x \in X\}$

such that

$$\hat{O}(n) \cap \left( \bigcup \{ \hat{F} \in \hat{F}_n | f \not\in \hat{F} \} \right) = \emptyset,$$

where $\alpha_n < \omega_1$ and $U_i \in \tau(X)$ for $i = 1, \cdots, k_n$. Let $\alpha = \sup_n \alpha_n$ and let

$$\hat{G}(n) = \langle \{ U_i \times [(\alpha,0),\omega_1] | i = 1, \cdots, k_n \} \rangle \cap C(X,Y), \ n \in \omega.$$

Then $\hat{G}(n) \subset S(f,\hat{F}_n) \subset S(f,\hat{U}_n)$. Since $(\hat{U}_n)$ is an M-sequence for $C(X,Y)$, $\bigcap_n \hat{G}(n)$ is compact. But this is a contradiction. For, we can easily notice that $\bigcap_n \hat{G}(n)$ contains the closed subspace homeomorphic to $C(X,X)$ which is not countably compact.

**Example 2.3.** There exists a compact space $X$ and a countably compact space $Y$ such that $C(X,Y)$ is not a w∆-space.

**Construction:** Let $Z_1$, $Z_2$ be countably compact spaces such that $Z_1 \times Z_2$ is not countably compact [2, Example 3.10.19]. We separate the non-limit ordinals in $\omega_1$ to $L_1$ and $L_2$ as

$L_1 = \{ \alpha + 2n + 1 | n \in \omega \text{ and } \alpha \text{ is a limit ordinal} \}$,

$L_2 = \{ \alpha + 2n + 2 | n \in \omega \text{ and } \alpha \text{ is a limit ordinal} \}$.

For each $\alpha \in \omega_1 + 1$, let

$$\alpha^* = \{ \alpha \} \times Z_i \quad \text{if } \alpha \in L_i \ (i = 1,2)$$

$$\alpha^* = \{ \alpha \} \quad \text{otherwise}$$

and let $Y = \bigcup \{ \alpha^* | \alpha \in \omega_1 + 1 \}$. We topologize $Y$ as follows: For each $\alpha \in L_i (i = 1,2)$ we make $\alpha^*$ a clopen subspace homeomorphic to $Z_i$. For each limit ordinal $\alpha$ in $\omega_1 + 1$, let $\{ (\beta^*, \alpha^*) | \beta < \alpha \}$ be a neighborhood base of $\alpha$ in $Y$, where

$$\langle \beta^*, \alpha^* \rangle = \bigcup \{ \gamma^* | \beta < \gamma \leq \alpha \}.$$  

Then obviously $Y$ is countably compact. Let $X = \omega_1 + 1$ with order topology. Now, we show that $C(X,Y)$ is not a w∆-space. Assume that there exists a w∆-sequence $(\hat{U}_n)$ for $C(X,Y)$. As in
the previous example, we identify $C(X, Y)$ with the subspace of $K(X \times Y)$. Let

$$f = \{(\alpha, \omega_1) | \alpha \in \omega_1 + 1\}.$$

For each $n$, there exists a closed neighborhood $\hat{O}(n)$ of $f$ in $C(X, Y)$ such that

$$\hat{O}(n) = \langle \{ \overline{U_i} \times [\alpha^*_n, \omega^*_1] | i = 1, \ldots, k_n \} \rangle \cap C(X, Y) \subset S(f, \hat{U}_n),$$

where $\alpha_n$ is a limit ordinal and $\{U_i | i = 1, \ldots, k_n\}$ is an open cover of $\omega_1 + 1$. ($[\alpha^*_n, \omega^*_1]$ is defined similarly to the above.) Let $\alpha = \sup_n \alpha_n$. Then

$$C = \bigcap_n \langle \{ \overline{U_i} \times [\alpha^*_n, \omega^*_1] | i = 1, \ldots, k_n \} \rangle \cap C(X, Y)$$

is a countably compact closed subset of $C(X, Y)$. Since as easily seen, $Z_1 \times Z_2$ is embedded in $C$ as a closed subset, $C$ cannot be countably compact, a contradiction.

3. The case of topologically complete spaces

A space $X$ is called topologically complete if there exists a uniformity $\mu$ compatible with $\tau(X)$ such that $(X, \mu)$ is complete. Such spaces coincide with inverse limits of metric spaces [8]. Using this, Zenor proved that topologically complete spaces are closed under the operation I [9]. Similarly, we show that they are closed under the operation II when the domain $X$ is a $k$-space. For it, we prepare the following notation. For a space $X$, let $K(X) = \{K_\alpha | \alpha \in A\}$ and we introduce the order $\leq$ in $A$ as follows: For $\alpha, \beta \in A$, $\alpha \leq \beta$ if and only if $K_\alpha \subset K_\beta$. For a directed set $B$, we introduce the order $\leq$ in $A \times B$ as follows: For $\lambda_1 = (\alpha_1, \beta_1), \lambda_2 = (\alpha_2, \beta_2) \in A \times B$, $\lambda_1 \leq \lambda_2$ if and only if $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$. For $\alpha_1, \alpha_2 \in A$ with $\alpha_1 \leq \alpha_2$, let $i_{\alpha_1, \alpha_2}$ be the inclusion mapping of $K_{\alpha_1}$ into $K_{\alpha_2}$.

**Proposition 3.1.** Let $X$ be a $k$-space and $Y = \lim \{Y_\beta, \pi_{\beta_1, \beta_2} | \beta_1, \beta_2 \in B, \beta_1 > \beta_2\}$. Then $C(X, Y)$ is the inverse limit of $\{C(K_{\alpha_1}, Y_\beta) | \alpha, \beta \in A \times B\}$.

**Proof:** As the bonding mappings, we define

$$\Pi_{\lambda_1, \lambda_2} : C(K_{\alpha_1}, Y_{\beta_1}) \to C(K_{\alpha_2}, Y_{\beta_2}),$$

for $\lambda_1(\alpha_1, \beta_1), \lambda_2(\alpha_2, \beta_2) \in A \times B$ and $\lambda_2 < \lambda_1$ as follows:

$$\Pi_{\lambda_1, \lambda_2}(f) = \pi_{\beta_1, \beta_2} \circ f \circ i_{\alpha_2, \alpha_1}, \ f \in C(K_{\alpha_1}, Y_{\beta_1}).$$
Since it is easy to check for $\lambda_i = (\alpha_i, \beta_i) \in A \times B$ ($i = 1, 2, 3$) with $\lambda_3 < \lambda_2 < \lambda_1$

$$\Pi_{\lambda_2 \lambda_3} \circ \Pi_{\lambda_1 \lambda_2}(f) = \Pi_{\lambda_2 \lambda_3} \circ (\pi_{\beta_1 \beta_2} \circ f \circ i_{\alpha_2 \alpha_1})$$

$$= \pi_{\beta_2 \beta_3} \circ (\pi_{\beta_1 \beta_2} \circ f \circ i_{\alpha_2 \alpha_1}) \circ i_{\alpha_3 \alpha_2}$$

$$= \pi_{\beta_1 \beta_3} \circ f \circ i_{\alpha_3 \alpha_1}$$

$$= \Pi_{\lambda_1 \lambda_3}(f),$$

$\{C(K_{\alpha}, Y_{\beta}), \Pi_{\lambda_1 \lambda_2}|_{\lambda_1, \lambda_2 \in A \times B, \lambda_2 < \lambda_1}\}$ forms an inverse limit system. Let

$$P = \lim_{\leftarrow} C(K_{\alpha}, Y_{\beta})$$

be its inverse limit. Then we show that $C(X, Y) \cong P$. To this end, we define a homeomorphism $\Phi$ of $C(X, Y)$ onto $P$ as follows:

Let $\pi_{\beta}: Y \to Y_{\beta}$ be the projection for each $\beta \in B$. Let $f$ be an arbitrary element of $C(X, Y)$. Then we let

$$\Phi(f) = (f_{\lambda})_{\lambda \in A \times B},$$

where for each $\lambda = (\alpha, \beta) \in A \times B$,

$$f_{\lambda} = (\pi_{\beta} \circ f)|_{K_{\alpha}}.$$ 

Then clearly $(f_{\lambda}) \in P$. To show that $\Phi$ is one-to-one, let $f \neq g, f, g \in C(X, Y)$. There exist $x \in X$ and $\beta \in B$ such that $(\pi_{\beta} \circ f)(x) \neq (\pi_{\beta} \circ g)(x)$. Pick $\alpha \in A$ with $K_{\alpha} = \{x\}$. For this $\lambda = (\alpha, \beta)$, clearly $f_{\lambda} \neq g_{\lambda}$, which means $\Phi(f) \neq \Phi(g)$. To see that $\Phi$ is onto, let $(f_{\lambda}) \in P$. Let $x$ be an arbitrary point of $X$ and pick $\alpha(x) \in A$ with $K_{\alpha(x)} = \{x\}$. For each $\beta \in B$, $f_{(\alpha(x), \beta)}(x) \in Y_{\beta}$. Then we define a correspondence $f: X \to Y$ such that

$$f(x) = (f_{(\alpha(x), \beta)}(x))_{\beta \in B}.$$ 

Note that $f|_{K_{\alpha}} = (f_{(\alpha, \beta)})_{\beta \in B}$ for each $\alpha \in A$. Since $X$ is a k-space, $f \in C(X, Y)$ and obviously $\Phi(f) = (f_{\lambda})$. To see the continuity, let $\lambda = (\alpha, \beta) \in A \times B$. Let $C_{\alpha}$ be a compact subset of $K_{\alpha}$ and $U_{\beta}$ an open subset of $Y_{\beta}$. Then we have the equality

$$\Phi^{-1}(\Pi_{\lambda}^{-1}(W(C_{\alpha}, U_{\beta}))) = W\left(C_{\alpha}; \pi_{\beta}^{-1}(U_{\beta})\right),$$

where $\Pi_{\lambda}: P \to C(K_{\alpha}, Y_{\beta})$ is the projection, which proves the continuity. Conversely, to see the continuity of $\Phi^{-1}$, let $\alpha \in A$ and
let $U_\beta$ be an open subset of $Y_\beta$. Then it is easy to see

$$\Phi \left( W \left( K_\alpha ; \pi_\beta^{-1}(U_\beta) \right) \right) = \Pi_{(\alpha,\beta)}^{-1}(W(K_\alpha;U_\beta)),$$

which proves the continuity. Hence $\Phi$ is a homeomorphism. □

**Theorem 3.2.** Let $X$ be a $k$-space. If $Y$ is a topologically complete space, then so is $C(X,Y)$.

**Proof:** Let $Y = \lim\leftarrow Y_\alpha$, where $Y_\alpha$’s are metric spaces. By the above proposition, $C(X,Y) = \lim\leftarrow C(K_\alpha,Y_\beta)$. Since $C(K_\alpha,Y_\beta)$ is metrizable [2, Exercise 4.2.H], $C(X,Y)$ is topologically complete. □

4. The case of $G_\delta$-diagonals.

From the definitions [3, Definitions 2.1, 2.10], the following implication holds true:

Regular $G_\delta$-diagonal $\rightarrow$ $G_\delta^*$-diagonal $\rightarrow$ $G_\delta$-diagonal.

It is shown in [6, Example 3] that spaces with a $G_\delta$-diagonal are not closed under the operation $I$, but this example is not a $T_2$-space. So, we give here another example.

**Example 4.1.** There exists a $T_2$-space $Z$ with a $G_\delta$-diagonal such that $K(Z)$ has no $G_\delta$-diagonal.

**Construction:** Let $X = X(1) \cup X(2)$, where

$$X(1) = \left\{ (x,y) \in \mathbb{R}^2 \left| y = 0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \right. \right\},$$

$$X(2) = \mathbb{R} \times \{ -1 \}.$$

Topologize $X$ as follows: Each $\mathbb{R} \times \{ \frac{1}{n} \}$ is an open Michael line. For each $p = (x,0)$ or $(x,-1)$ has a neighborhood base $\{ N(p;\varepsilon) | \varepsilon > 0 \}$, where $N(p;\varepsilon)$’s are defined by;

(1) if $p = (x,0)$ and $x \in \mathbb{Q}$ (the rationals), then $N(p;\varepsilon) =$

$$\{ p \} \cup \left\{ (x',y') \in X(1) \left| 0 \leq y' \leq \frac{1}{\sqrt{2}}|x' - x| \text{ and } 0 \not= |x' - x| < \varepsilon \right. \right\};$$

(2) if $p = (x,-1)$ and $x \in \mathbb{Q}$, then $N(p;\varepsilon) =$

$$\{ p \} \cup \left\{ (x',y') \in X(1) \left| \frac{1}{\sqrt{2}}|x' - x| < y' < \varepsilon \right. \right\};$$
(3) if \( p = (x, 0) \) and \( x \in \mathbb{R} \setminus \mathbb{Q} \), then \( N(p; \varepsilon) = \{p\} \cup \{(x', y') \in X(1) \mid 0 \leq y' \leq |x' - x| \text{ and } 0 \neq |x' - x| < \varepsilon\} ; \)

(4) if \( p = (x, -1) \) and \( x \in \mathbb{R} \setminus \mathbb{Q} \), then \( N(p; \varepsilon) = \{p\} \cup \{(x', y') \mid |x' - x| < y' < \varepsilon\} . \)

Then \( X \) has a \( G_\delta \)-diagonal. In fact, since \( X(1) \) is submetrizable, there exists a \( G_\delta \)-diagonal sequence \( \{U(n) \mid n = 1, 2, \cdots \} \) for \( X(1) \).

For each \( n \), let \( V(n) = U(n) \cup \left\{ N\left(p; \frac{1}{n}\right) \mid p \in X(2) \right\} . \)

Then it is easy to see that \( (V(n)) \) is a \( G_\delta \)-diagonal sequence for \( X \).

Let \( Y \) be the Michael line and let \( Z = X \cup_f Y \) be the adjunction space of \( X \) and \( Y \) with respect to \( f : X(2) \rightarrow Y \) such that \( f((x, -1)) = x, \ x \in \mathbb{R} . \)

By [5, Lemma] \( Z \) has a \( G_\sigma \)-diagonal. We show that \( K(Z) \) has no \( G_\delta \)-diagonal. For a contradiction, assume that \( K(Z) \) has a \( G_\delta \)-diagonal sequence \( (\hat{U}(n)) \). For each \( s \in \mathbb{R} \), let

\[
K(s) = \left\{ (s, y) \mid y = -1, 1, \frac{1}{2}, \frac{1}{3}, \cdots \right\},
\]

\[
L(s) = K(s) \cup \{(s, 0)\}.
\]

Then \( K(s), L(s) \in K(Z) \) and \( K(s) \neq L(s) \). Since \( (\hat{U}(n)) \) is a \( G_\delta \)-diagonal sequence, there exists \( n(s) \) such that \( L(s) \notin S(K(s), \hat{U}(n(s))) \). By the second category theorem, there exists \( n \) such that

\[
A = \text{Int}_\mathbb{R} (\text{Cl}_\mathbb{R}(\{ s \in \mathbb{R} \mid n(s) = n \})) \neq \emptyset,
\]

where \( \text{Int}_\mathbb{R}, \text{Cl}_\mathbb{R} \) means the operator in the usual sense. Take \( r \in A \cap \mathbb{Q} \). Since \( U(n) \) covers \( K(Z) \), there exists \( \hat{U} \in \hat{U}(n) \) such that \( L(r) \in \hat{U} \). Then it is easily observed that there exists \( s \in \mathbb{R} \) with \( n(s) = n \) such that both \( K(s) \) and \( L(s) \) belong to \( \hat{U} \), a contradiction.

Note that this example is not regular. As for \( G^*_\delta \)-diagonals, we do not know the following:

**Question 4.2.** If a space \( X \) has a \( G^*_\delta \)-diagonal, then does \( K(X) \) have a \( G^*_\delta \)-diagonal?
As a partial answer, we can easily show the following: (1) If a space $X$ has a sequence $(U(n))$ of open covers of $X$ such that $K = \bigcap_n S(K, U(n))$ for each $K \in \mathcal{K}(X)$, then $\mathcal{K}(X)$ has a $G^*_\delta$-diagonal. (2) If a space $X$ has a sequence $(U(n))$ of open covers of $X$ such that $K = \bigcap_n S(K, U(n))$ for each $K \in \mathcal{K}(X)$, then $\mathcal{K}(X)$ has a $G_\delta$-diagonal.

We give the positive results to I and II.

**Theorem 4.3.** If a space $X$ has a regular $G_\delta$-diagonal, then so does $\mathcal{K}(X)$.

**Proof:** Recall the characterization of a space $X$ having a regular $G_\delta$-diagonal [10, Theorem 1] that there exists a sequence $(U(n))$ of open covers of $X$ such that if $x, y \in X$ with $x \neq y$, then there exist $n \in \omega$ and neighborhoods $U, V$ of $x, y$, respectively, such that $U \cap S(V, U(n)) = \emptyset$. Suppose that $X$ has such a sequence $(U(n))$ with $U(n + 1) < U(n)$, $n \in \omega$. For each $n$, let

$$\hat{U}(n) = \{\langle U_0 \rangle | U_0 \subset U(n) \text{ is finite}\}.$$  

Then $\hat{U}(n)$ is an open cover of $\mathcal{K}(X)$. We show that $(\hat{U}(n))$ has the required property. Let $K, L \in \mathcal{K}(X)$ and $K \neq L$. Suppose that there exists a point $p \in K \setminus L$. Then there exist neighborhoods $U, V$ of $p, L$ in $X$, respectively, and $n \in \omega$ such that $U \cap S(V, U(n)) = \emptyset$. Then it is easy to see that

$$\langle X, U \rangle \cap S(\langle V \rangle, \hat{U}(n)) = \emptyset.$$  

Since $\langle X, U \rangle, \langle V \rangle$ are neighborhoods of $K, L$ in $\mathcal{K}(X)$, respectively, we can say that $(\hat{U}(n))$ has the required property. \[\square\]

**Theorem 4.4.** Let $X$ be such a space that has a $\sigma$-compact dense subset. If $Y$ has a $G_\delta$-diagonal, $G^*_\delta$-diagonal, regular $G_\delta$-diagonal, then so does $C(X, Y)$, respectively.

**Proof:** We show only the case of a regular $G_\delta$-diagonal, and the others are similar. By the assumption on $X$, there is no loss if we assume that $X = \bigcup_n X_n$, where $X_n \in \mathcal{K}(X)$, $n \in \omega$. By the characterization stated above, there exists a sequence $(U(n))$ of open covers satisfying the same condition as there. For $m, n \in \omega$, let $\Delta(m, n)$ be the totality of pairs $\delta = (\mathcal{K}(\delta), U(\delta))$ of subsets $\mathcal{K}(\delta), U(\delta)$ of $\mathcal{K}(X), U(n)$, respectively, such that $\mathcal{K}(\delta) = \{K_1, \ldots, K_t\}$ is a finite cover of $X_m$ and $U(\delta) = \{U_1, \ldots, U_t\}$. For each $\delta \in \Delta(m, n)$,
let
\[ W(\delta) = W(K_1, \ldots, K_t; U_1, \ldots, U_t) \]
and \( W(m, n) = \{ W(\delta) | \delta \in \Delta(m, n) \} \). Since each \( X_m \) is compact, it is easy to see that \( W(m, n) \) covers \( C(X, Y) \). To see that \( (W(m, n)) \) has the required property, let \( f \neq g, f, g \in C(X, Y) \). Then there exist \( m \in \omega \) and \( x_0 \) such that \( x_0 \in X_m \) and \( f(x_0) \neq g(x_0) \). There exist neighborhoods \( O \) and \( O' \) of \( f(x_0) \) and \( g(x_0) \) in \( Y \), respectively, and \( n \), such that \( S(O, U(n)) \cap O' = \emptyset \). It is easily checked that
\[ S(W(\{x_0\}; O), W(m, n)) \cap W(\{x_0\}; O') = \emptyset. \]
Hence, \( C(X, Y) \) has a regular \( G_\delta \)-diagonal. □

5. **The case of Moore spaces and the conclusion**

It is known that Moore spaces are closed under the operation I, [5]. Moreover, it is shown in [7] that Moore spaces with a regular \( G_\delta \)-diagonal are closed under the operation I and II. But we do not know whether Moore spaces are closed under the operation II.

**Question 5.1.** Let \( X \) be a compact space and let \( Y \) be a Moore space. Then is \( C(X, Y) \) Moore?

As the conclusion, the results in this are shown by the following figure, where +, - means the operation I and II holds true or not, respectively.

<table>
<thead>
<tr>
<th>spaces</th>
<th>operation I</th>
<th>operation II</th>
</tr>
</thead>
<tbody>
<tr>
<td>M-space</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>paracompact M-space</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>topologically complete space</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( G_\delta )-diagonal</td>
<td>- (( T_2 )-space)</td>
<td>+</td>
</tr>
<tr>
<td>( G_\delta^* )-diagonal</td>
<td>?</td>
<td>+</td>
</tr>
<tr>
<td>regular</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( G_\delta )-diagonal</td>
<td>+</td>
<td>?</td>
</tr>
<tr>
<td>Moore space</td>
<td>+</td>
<td>?</td>
</tr>
<tr>
<td>Moore space with a regular</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( G_\delta )-diagonal</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>
References


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