THE HEREDITARILY METALINDÊOF PROPERTY
OF $k$–SPACES WITH $\sigma$–HCP CLOSED

$k$–NETWORKS

LIANG–XUE PENG

Abstract. In this paper, we prove that a regular $k$–space with a $\sigma$–HCP
$k$–network is a hereditarily metalindêof space. Thus, the question that appeared in [5] and [6] is answered.

1. Introduction

If $X$ is a space, a family $\mathcal{P}$ of closed subsets of $X$ is a $k$–network
for $X$, if for every compact subset $K \subset X$ and an open neighborhood $U$ of $K$, there is a finite $\mathcal{P}^* \subset \mathcal{P}$, such that $K \subset \cup \mathcal{P}^* \subset U$. A space $X$ is called an $\aleph$–space if $X$ has a $\sigma$–locally finite $k$–network (cf. [2] and [7]). A network for a space $X$ is a collection $\mathcal{F}$ of subsets of $X$ such that whenever $x \in U$ with $U$ open, there exists $F \in \mathcal{F}$ with $x \in F \subset U$. A space $X$ is a $\sigma$–space if $X$ has a $\sigma$–discrete network (cf. [1]). A space is called a sequential space if a subset $F \subset X$ is closed in $X$ if and only if $F$ contains all limit points of sequences from $F$ (cf. [3]). We know that $k$–spaces and sequential spaces are equivalent for $\sigma$–spaces (cf. [1]). A space $X$ is a metalindêof space if every open cover $U$ of $X$ has a point–countable open refinement (cf. [1]).

The properties of $\aleph$–spaces have been studied by many topologists. In [2], Foged proved that a sequential space or $k$–space with a $\sigma$–locally finite $k$–network ($k$–and–$\aleph$ space) is a hereditarily metalindêof space, and a normal $k$–and–$\aleph$ space is a paracompact

Key words and phrases. sequential space, $k$–network, metalindêof, $k$–space.
space. From [4], we know that a space with a \( \sigma \)-hereditarily closure preserving (\( \sigma \)-HCP) \( k \)-network need not be an \( \aleph \)-space. So it is necessary to study the properties of spaces with \( \sigma \)-HCP \( k \)-networks. By a space we mean a regular topological space.

In [5], Liu raised the following question: Are \( k \)-spaces with \( \sigma \)-HCP \( k \)-networks hereditarily metalindelöf? And the same question was also raised by Liu and Tanaka in [6]. Is every \( k \)-space with a \( \sigma \)-HCP \( k \)-network a metalindelöf space? In this paper, we prove that a \( k \)-space with a \( \sigma \)-HCP \( k \)-network is a hereditarily metalindelöf space. Thus, the question appearing in [5] and [6] is answered.

**Lemma 1** [2]. A \( k \)-and-\( \aleph \) space is a hereditarily metalindelöf space.

**Lemma 2** [3]. The following are equivalent for a regular space.

(a) \( X \) is a \( \sigma \)-space.

(b) \( X \) has a \( \sigma \)-locally finite network.

(c) \( X \) has a \( \sigma \)-discrete network.

(d) \( X \) has a \( \sigma \)-closure preserving network.

By Lemma 2, we know that a space with a \( \sigma \)-HCP \( k \)-network is a \( \sigma \)-space.

**Theorem 1.** A \( k \)-space with a \( \sigma \)-HCP \( k \)-network is a hereditarily metalindelöf space.

**Proof:** A space \( X \) is hereditarily metalindelöf iff every collection \( \mathcal{U} \) of open sets has a point–countable open refinement. Since \( X \) is a \( \sigma \)-space, it is easy to see that any \( \mathcal{U} \) has a \( \sigma \)-closed discrete (in \( X \)) refinement. Thus, it is enough to show that every closed discrete family \( \mathcal{F} \) of subsets of \( X \) may be expanded to a point–countable open family.

\( X \) is a regular space, so we may assume that \( X \) has a \( \sigma \)-HCP closed \( k \)-network. Let \( \mathcal{P} = \bigcup \{ \mathcal{P}_n : n \in \omega \} \) be a \( \sigma \)-HCP closed \( k \)-network of \( X \) and \( \mathcal{P}_n \subset \mathcal{P}_{n+1} \) for each \( n \in \omega \). Let \( \mathcal{F} = \{ F_\alpha : \alpha \in \Lambda \} \) be a discrete closed family of \( X \). Let \( \mathcal{P}(\emptyset) = \mathcal{F} \). For \( n \in \omega \), let \( P_\alpha^*(n) = \bigcup \{ P : P \in \mathcal{P}_n, P \cap F_\alpha \neq \emptyset, P \cap F_\beta = \emptyset \) for all \( \beta \in \Lambda \setminus \{ \alpha \} \}, and let \( P_\alpha(n) = P_\alpha^*(n) \setminus \cup \{ P_\beta^*(n) : \beta \in \Lambda \setminus \{ \alpha \} \} \). Then \( \mathcal{P}(n) = \{ P_\alpha(n) : \alpha \in \Lambda \} \) is a pairwise disjoint family in \( X \), and for any \( \Lambda' \subset \Lambda \), \( \cup \{ P_\beta^*(n) : \beta \in \Lambda' \} \) is a closed subset of \( X \).
For any finite sequence \( \delta \) of \( \omega \), suppose \( \mathcal{P}(\delta) \) has been defined. That is, \( \mathcal{P}(\delta) = \{ P_\alpha(\delta) : \alpha \in \Lambda \} \) is a pairwise disjoint family, where \( P_\alpha(\delta) = P^*_\alpha(\delta) \cup \{ P^*_\beta(\delta) : \beta \in \Lambda \setminus \{ \alpha \} \} \), and \( \cup \{ P^*_\beta(\delta) : r \in \Lambda' \} \) is closed for any \( \Lambda' \subseteq \Lambda \). Now, for \( n \in \omega \), we construct \( \mathcal{P}(\delta_n) \). Let \( P^*_\alpha(\delta_n) = \cup \{ P : P \in \mathcal{P}_n, P \cap P_\alpha(\delta) \neq \emptyset \} \), \( P \cap \{ P^*_\beta(\delta) : \beta \neq \alpha \} = \emptyset \). Then \( \cup \{ P^*_\beta(\delta_n) : r \in \Lambda' \} \) is closed for any \( \Lambda' \subseteq \Lambda \). Let \( P_\alpha(\delta_n) = P^*_\alpha(\delta_n) \cup \{ P^*_\beta(\delta_n) : \beta \neq \alpha \} \). Then \( \mathcal{P}(\delta_n) = \{ P_\alpha(\delta_n) : \alpha \in \Lambda \} \) is a pairwise disjoint family of \( X \).

Let \( U_\alpha = \cup \{ P_\alpha(\delta) : \delta \) is a finite sequence in \( \omega \} \). We will show that \( U_\alpha = \{ U_\alpha : \alpha \in \Lambda \} \) is a point–countable open family of \( X \) and \( F_\alpha \subset U_\alpha \) for \( \alpha \in \Lambda \).

For any \( x \in F_\alpha \), there is an open neighborhood \( V_x \) of \( X \), such that \( V_x \cap F_{\beta} \) is \( \emptyset \) for \( \beta \in \Lambda \setminus \{ \alpha \} \). Then there is \( n \in \omega \), and \( P \in \mathcal{P}_n \), such that \( x \in P \subset V_x \). Since \( \cup \{ P_{\beta}(n) : \beta \in \Lambda \setminus \{ \alpha \} \} \cap F_\alpha = \emptyset \), we have \( x \in P_\alpha(n) \). Thus, \( F_\alpha \subset U_\alpha \).

To prove \( U_\alpha \) is open, we need only to prove that it is sequential open in \( X \). Suppose a sequence \( Z \) converges to \( x \), and \( x \in U_\alpha \). Then there is a finite sequence \( \delta \) in \( \omega \), such that \( x \in P_\alpha(\delta) = P^*_\alpha(\delta) \cup \{ P^*_\beta(\delta) : \beta \in \Lambda \setminus \{ \alpha \} \} \). And we know that \( \cup \{ P^*_\beta(\delta) : \beta \in \Lambda \setminus \{ \alpha \} \} = M \) is a closed subset of \( X \). So there is an open neighborhood \( V_x \) of \( x \), such that \( V_x \cap M = \emptyset \). Then there are some \( n \in \omega \) and \( \mathcal{P}^* \subset \mathcal{P}_n \) such that \( Z \) is eventually in \( \cup \mathcal{P}^* \subset V_x \), and \( P^* \subset V_x \) contains \( x \). So \( \cup \mathcal{P}^* \subset P^*_\alpha(\delta_n) \). For any \( \beta \in \Lambda \setminus \{ \alpha \} \), \( P^*_\beta(\delta_n) \cap P^*_\alpha(\delta) = \emptyset \), so \( x \in X \setminus \cup \{ P^*_\beta(\delta_n) : \beta \in \Lambda \setminus \{ \alpha \} \} \). Thus, \( Z \) is eventually in \( P_\alpha(\delta_n) = P^*_\alpha(\delta_n) \cup \{ P^*_\beta(\delta_n) : \beta \in \Lambda \setminus \{ \alpha \} \} \). So \( U_\alpha \) is an open set of \( X \).

Suppose \( \{ U_\alpha : \alpha \in \Lambda \} \) is not point–countable. Then there is \( x \in X \) such that \( |\{ \alpha : x \in U_\alpha \}| > \omega \). Thus, there is a finite sequence \( \delta \) in \( \omega \), satisfying \( x \in P_\alpha(\delta) \), where \( \alpha \in \{ \beta : x \in U_\beta \} \). Thus, \( \{ P_\alpha(\delta) : \alpha \in \Lambda \} \) is not a pairwise disjoint family. Contradiction.

Now we have proved that \( \{ U_\alpha : \alpha \in \Lambda \} \) is a point–countable open family of \( X \) and \( F_\alpha \subset U_\alpha \) for \( \alpha \in \Lambda \). Thus, \( X \) is a hereditarily metalindelöf space.

From the proof of Theorem 1 or Theorem 6 of [8], we have the following corollary.

**Corollary 1.** If \( X \) is a \( k \)-space and has a \( \sigma \)-HCP weak base, then \( X \) is a metalindelöf space.
References


Department of Applied Mathematics, Beijing Polytechnic University, Beijing 100022 CHINA

E-mail address: lxuepeng@263.net.cn