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GRAPHS OF NMS FLOWS ON S^3 WITH UNKNOTTED SADDLE PERIODIC ORBITS

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Abstract. In this paper we build dual graphs for the Non-Singular Morse-Smale systems on S^3 characterized by I, II and III Wada operations which have no heteroclinic trajectories connecting two saddles orbits.

Moreover, we show that these dual graphs are in one to one correspondence with these kind of NMS flows on S^3.

1. Introduction

Morse-Smale flows form a open subset of C^∞-vector fields and they are the structurally stable flows on 2-dimensional manifolds. In paper [4], M. M. Peixoto studies Morse-Smale flows on compact orientable two-dimensional manifolds. To every flow of such type, he associates a graph endowed with finite combinatorial structure and proves that these graphs (called Peixoto graphs) are in one to one correspondence with classes of topological equivalence of Morse-Smale flows.

X. Wang [6] obtains a complete classification of the C*-algebras of all Morse-Smale flows on closed 2-manifolds in terms of dual graphs, showing that the coloured dual graphs are exactly the complete combinatorial invariant of the topological conjugacy classes of the Morse-Smale flows on closed 2-manifolds.

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For the three dimensional case, Morse-Smale flows are not dense but they define an open set in the set of C^1-vector fields; in most cases they can be studied from Non-Singular Morse-Smale flows, hereafter called NMS flows.

For the case of NMS flows on the 3-sphere a topological characterization of the set of the periodic orbits has been made by M. Wada [5] in terms of knots and links, using a generator, the Hopf link, and six operations. The Hopf link, denoted by h, consists of two linked trivial knots corresponding to one attractive and one repulsive periodic orbit; the six operations are basically split sums and cabling ($l \cdot l'$ denote the split sum of l and l'). Nevertheless, different flows can be characterized by the same link. The flow is completely characterized by its round handle decomposition and we use it to obtain the dual graphs.

Our aim is to obtain the dual graphs associated to the NMS systems on S^3. In this paper we obtain dual graphs associated to NMS systems on S^3 with unknotted saddle periodic orbits and no heteroclinic trajectories connecting two saddle orbits. In Section 3 we obtain dual graphs corresponding to Wada operations applied once on Hopf links from the global picture of the corresponding flow. We also define the NMS gluing of graphs and prove that NMS flows on S^3 coming from I, II and III Wada operations and with no heteroclinic trajectories connecting two saddles orbits, are obtained from the NMS gluing of dual graphs corresponding to each operation (Th. 2). Finally, we show that these kind of NMS flows can be reproduced from the associated dual graphs. So, these dual graphs are in one to one correspondence with these kind of NMS flows on S^3(Cor. 1).

2. Topological description of NMS flows on S^3

NMS flows are characterized by the following simple qualitative features:

1. There are a finite number of periodic orbits, all of hyperbolic generic type.
2. The α and the $\omega-$limit set of every trajectory is a closed orbit.
3. If γ is a closed orbit then denote by $W^s(\gamma)$ and $W^u(\gamma)$ respectively the stable and unstable manifolds of γ. Then, $W^u(\gamma)$ and $W^s(\gamma')$ are transversally intersecting submanifolds for all γ and γ'.

Closed orbits constitute the nonwandering set $\Omega = \Omega(X)$ of a NMS flow X and they can only be sources, sinks or saddle orbits. A source (sink) is a periodic orbit which is the $\alpha\,(\omega)$-limit set of all nearby trajectories. Singularities are not considered in NMS flows. A closed orbit of a NMS flow on a 3-manifold is attracting, saddle or repelling if the dimension of its unstable manifold is equal to one, two or three, respectively.

Definition 1. A pair $(M, \partial_- M)$ of a manifold M and a compact submanifold of M, $\partial_- M$, where the flow is inwards, or by abuse of notation, a manifold M is called

a *round 0-handle* if $(M, \partial_- M) \cong \left(D^2 \times S^1, \emptyset\right)$,

a *round 1-handle* if it consists of disk bundles over S^1, $B_s \bigoplus_{S^1} B_u$

a *round 2-handle* if $(M, \partial_- M) \cong \left(D^2 \times S^1, \partial D^2 \times S^1\right)$.

So, the round handles, that are diffeomorphic to tori, correspond to a 0-handle when there is a repulsive periodic orbit in the core, to a 2-handle if there is an attractive periodic orbit in the core and to a 1-handle if the orbit is a saddle; 0, 1 and 2 are called the indices of the periodic orbits, respectively.

Fattened round 1-handles have been introduced by Morgan [3]. A fattened round 1-handle is a 3-manifold Y_2 obtained from a manifold Y_1 by attaching a round 1-handle by means of one or two attaching circles $c_1 = \varphi\left(\{-1\} \times \{0\} \times S^1\right)$ and $c_2 = \varphi\left(\{+1\} \times \{0\} \times S^1\right)$ depending on whether the round 1-handle is twisted or not. So, while round 1-handles are always disk bundles over S^1, fattened round 1-handles may be different types of manifolds since they take into account the way the round 1-handles have been attached.

The simplest case of a NMS flow on S^3 corresponds to a flow which has only two periodic orbits: one repulsive and one attractive. These periodic orbits can be represented as the core of two complementary tori and the flow lines go from the sink to the source. These two periodic orbits form the $(0, 2)$-Hopf link. For the other cases, the link of periodic orbits are obtained using Wada operations with the Hopf link as a generator [5].

Theorem 1 (Wada)**.** *"Every indexed link which consists of all the closed orbits of a Non-Singular Morse-Smale flow on S^3 is obtained from Hopf links by applying the following six operations.*

Conversely, every indexed link obtained from Hopf links by applying the operations is the set of all the closed orbits of some Non-Singular Morse-Smale flow on S^3".

OPERATIONS: For given indexed links l_1 and l_2, the six operations are defined as follows. Let $l_1 \cdot l_2$ denote the split sum of the links l_1 and l_2 and $N(k, M)$ a regular neighbourhood of k in M.

I$(l_1, l_2) = l_1 \cdot l_2 \cdot u$, *where u is an unknot with index 1.*

II$(l_1, l_2) = l_1 \cdot (l_2 - k_2) \cdot u$, *where k_2 is a component of l_2 of index 0 or 2.*

III$(l_1, l_2) = (l_1 - k_1) \cdot (l_2 - k_2) \cdot u$, *where k_1 is a component of l_1 of index 0 and k_2 is a component of l_2 of index 2.*

IV$(l_1, l_2) = (l_1 \# l_2) \cup m$. *The connected sum $(l_1 \# l_2)$ is obtained by composing a component k_1 of l_1 and a component k_2 of l_2, each of which has index 0 or 2. The index of the composed component $k_1 \# k_2$ is equal to either $i(k_1)$ or $i(k_2)$. Finally, m is a meridian of $k_1 \# k_2$ with $i = 1$.*

V(l_1) : *Choose a component k_1 of l_1 of index 0 or 2, and replace $N(k_1, S^3)$ by $D^2 \times S^1$ with three indexed circles in it; $\{0\} \times S^1$, k_2 and k_3. Here, k_2 and k_3 are parallel (p, q)-cables on $\partial N(\{0\} \times S^1, D^2 \times S^1)$, where p is the number of longitudinal turns and q the number of the transverse ones. The indices of $\{0\} \times S^1$ and k_2 are either 0 or 2, and one of them is equal to $i(k_1)$. The index of k_3 is 1.*

VI(l_1)*: Choose a component k_1 of l_1 of index 0 or 2. Replace $N(k_1, S^3)$ by $D^2 \times S^1$ with two indexed circles in it; $\{0\} \times S^1$ and the (2,q)-cable k_2 of $\{0\} \times S^1$. The index of $\{0\} \times S^1$ is 1, and $i(k_2)=i(k_1)$.*

These operations are obtained from the round handle decomposition of S^3, introduced by Asimov [1] and Morgan [3]. We analyze this round handle decomposition to obtain the phase portrait of NMS flows on S^3. We draw the corresponding stable and unstable manifolds of the saddle orbit that appears in each Wada operation, called u orbit and represented by an unknot (see figures 1, 2, 3, 5 and 4 represented in S^3; in figures 4 and 5, S^3 is obtained by identifying upper and lower boundaries of the two solid cones, so a periodic orbit can also be represented as a vertical line).

From the pictures of the flow, we also obtain the canonical regions for the different cases, i.e., the connected components left when periodic orbits and invariant manifolds of saddle orbits are removed.

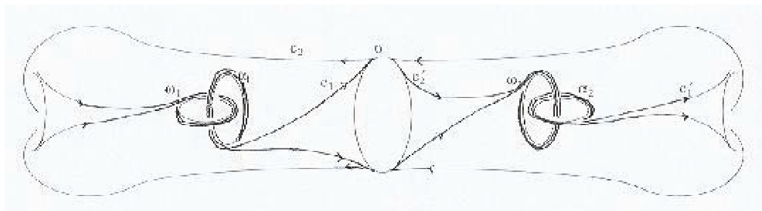

FIGURE 1. Flow associated to operation $I(h, h)$

We observe that different topological regions appear depending on the attachment of the round 1-handle.

2.1. Operations *I*, *II* and *III*

The link $I(h, h) = h \cdot h \cdot u$ is obtained when the round 1-handle is attached to two tori by means of two inessential circles. The resulting link is formed by two repulsive periodic orbits, α_1 and α_2, a saddle orbit σ and two sinks ω_1 and ω_2. The saddle orbit is in the round 1-handle, so its stable and unstable manifolds have to be connected to the repulsive and attractive orbits in a non-essential way. These manifolds, represented in figure 1, delimit the four "cylindrical" canonical regions that appear in this case (recalling that this flow is represented in S^3).

These "cylinders" divide S^3 delimiting attracting (repulsing) basins of ω_1 and ω_2 (α_1 and α_2), so the complete flow and the phase portrait are obtained.

Let us consider the link $II\,(h, h) = h{\cdot}d{\cdot}u$ formed by two repulsive periodic orbits, α_1 and α_2, a saddle orbit σ and a sink ω (d denotes a trivial unknot corresponding to a repulsive or attractive periodic orbit). In this case the round 1-handle is attached to one torus by means of an inessential circle and to the other one by an essential circle. We observe that the Hopf link h must be necessarily in the torus where the round 1-handle is attached in a non-essential way. Following a similar reasoning to the previous one, we can see in figure 2 the corresponding stable and unstable manifolds of the orbits. It can be observed that invariant manifolds of σ also delimit the three canonical regions of this case.

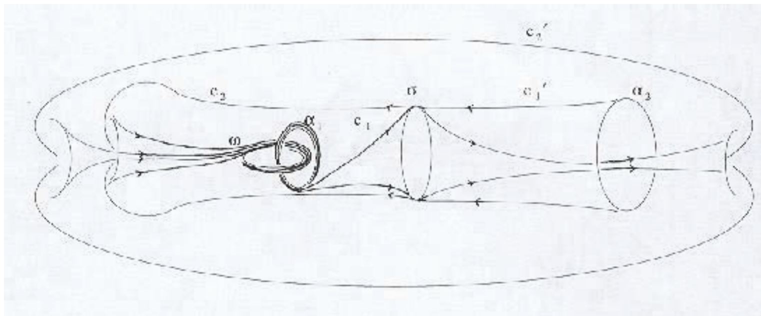

FIGURE 2. Flow associated to operation $II(h,h)$

The link $II\,(h,h)$ formed by one repulsive, one saddle and two attractive periodic orbits is obtained when the round 1-handle is attached to one torus by means of two inessential circles. The corresponding phase portrait is symmetric to the previous one.

Finally, the link $III(h,h) = d \cdot d \cdot u$ formed by one repulsive periodic orbit α, a saddle σ and a sink ω corresponds to the case of the attachment of a round 1-handle to one torus by means of one essential and one inessential circle. The phase portrait is depicted in figure 3. In this case, there are only two canonical regions.

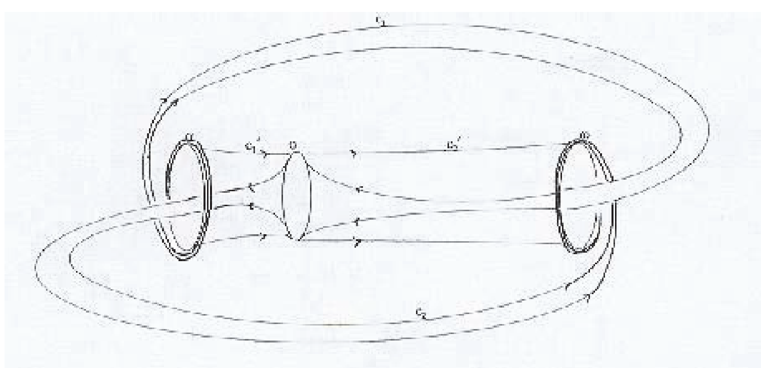

FIGURE 3. Flow associated to operation $III(h,h)$

Let us observe that saddle periodic orbits of the flows coming from these three operations are unknotted.

2.2. Operations *IV* and *V*

The link $IV(h,h)$ appears when the round 1-handle is essentially attached to two tori. This link is formed by two repulsive periodic orbits, α_1 and α_2, a saddle periodic orbit σ and one sink ω linking them (see fig. 4).

The link $V(h)$ is obtained when the round 1-handle is attached to one torus by means of two essential circles. When operation V implies $(1,0)$-cables the corresponding link is formed by one repulsive periodic orbit, α, a saddle σ and two sinks ω_1 and ω_2, one of the sinks linking the others. The phase portrait is shown in figure 5.

In both cases, invariant manifolds of the saddle orbit define two canonical regions. We observe that this type of attachment provides tori and Reeb leaves for stable and unstable manifolds of σ.

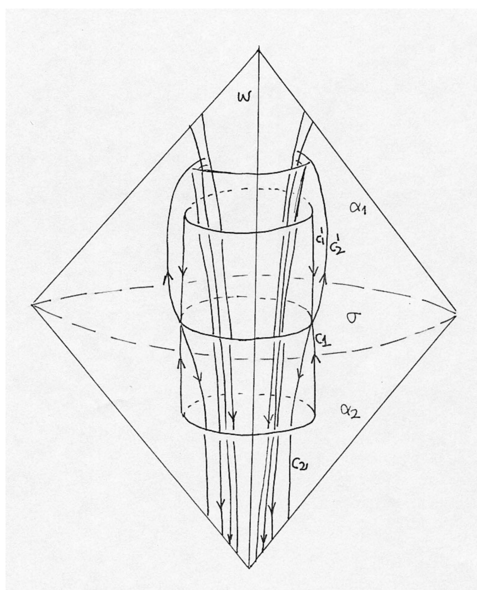

FIGURE 4. Flow associated to operation $IV(h,h)$

To obtain non trivial cables in operation V, it is necessary to attach a round 1-handle by means of attaching circles corresponding to toroidal knots. When they are essential circles on the torus, the saddle orbit inside the fattened round 1-handle and the periodic orbit in the core of the solid torus left, become two parallel $(p, q)-$cables of the orbit in the core of the torus. In this case, the canonical regions obtained are the same but instead of having a cylinder going from the saddle orbit to the other ones, a Seifert surface is obtained for stable and unstable manifolds of σ.

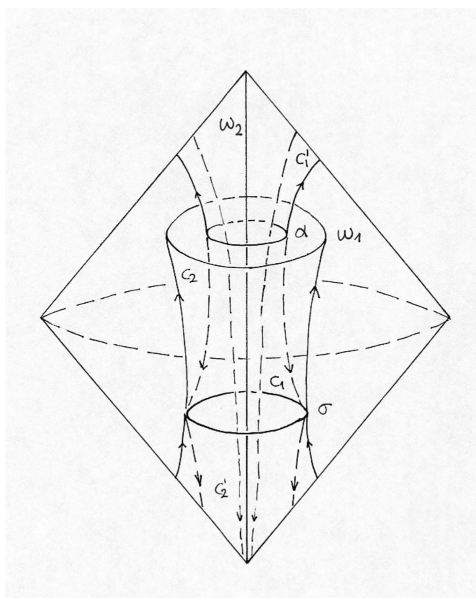

FIGURE 5. Flow associated to operation $V(h)$

2.3. Operation $\boldsymbol{VI}$

The link $VI(h)$, formed by one repulsive periodic orbit α, one saddle σ and one sink ω, is obtained from the Hopf link when the round 1-handle is twisted. The stable manifold of saddle orbit σ forms a Möebius band with the saddle orbit in its center and the repulsive periodic orbit, that forms a $(2, q)$-cable, follows the border of this band; as there exist only one stable and one unstable manifolds of σ, only one canonical region appears (see fig. 6).

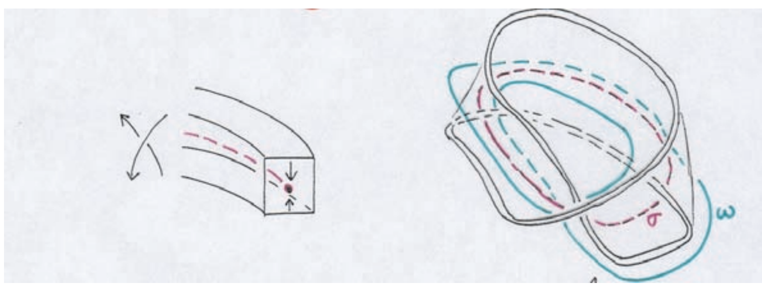

FIGURE 6. Twisted round 1-handle

3. DUAL GRAPHS OF NMS FLOWS ON S^3

We build dual graphs for Non-Singular Morse-Smale flows generalizing dual graphs definition for Morse flows on closed, orientable 2-manifolds (see [6]).

Definition 2. Let $(S^3, \mathcal{F})$ be a Non-Singular Morse-Smale flow on S^3. The *dual graph* $\hat{G}(\mathcal{F}) = (V, E)$ of the flow $\mathcal{F}$ is a graph defined as follows:

(1) The vertices are in one to one correspondence with the canonical regions;
(2) The geometric edges connecting two vertices are in one to one correspondence with the separatrices in the common boundary of the closures of two canonical regions.

Let us build dual graphs for the basic flows, those coming from the six Wada's operations applied once on Hopf links. From the flows depicted above we can observe the canonical regions and the separatrices between them.

Note that, in contrast of what happens in 2-dimensional manifolds where canonical regions are 2-balls[1], in this case we obtain different topological types of canonical regions corresponding to the different ways of attaching the round 1-handle.

Let us call *basic graphs* the dual graphs of these kind of flows corresponding to each Wada operation applied once on Hopf links (see fig. 7).

[1]The case of the polar flow, with an attractive and a repulsive fixed points and only one canonical region, S^2, corresponds to the Hopf link, with an attractive and a repulsive periodic orbits and S^3 as canonical region.

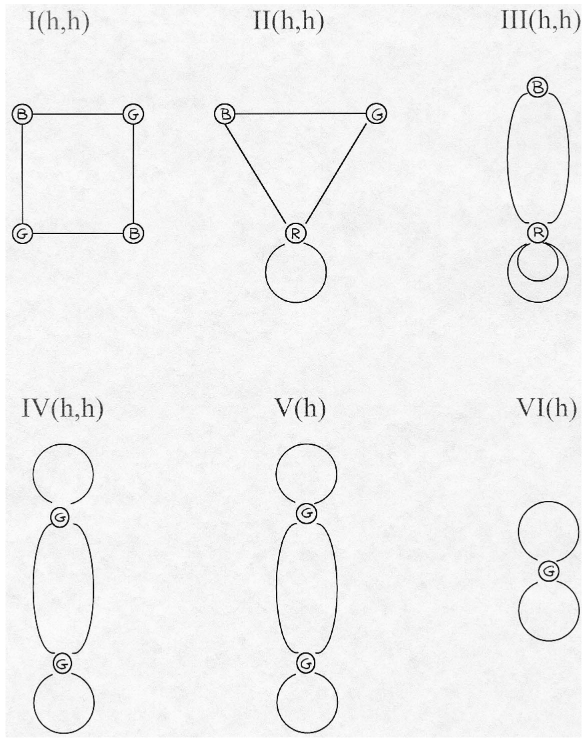

FIGURE 7. Basic dual graphs

Different types of vertices are obtained in these graphs. Some of them correspond to canonical regions that do not surround any periodic orbit. Other vertices correspond to canonical regions surrounding unknotted attractive or repulsive periodic orbits, so, new essential attachments can not be applied on this type of orbits, and other vertices correspond to canonical regions surrounding linked periodic orbits, i.e. Hopf links, therefore, esential and inessential attachments can be made on them. So, different kinds of vertices can be distinguished in these graphs.

Definition 3. *Blue vertices* correspond to canonical regions that no enclose any periodic orbit; *red vertices* correspond to canonical regions surrounding orbits admitting only inessential attachments; and *green vertices* correspond to canonical regions surrounding orbits admitting any kind of attachment.

Wada's theorem assures that the link of periodic orbits of a NMS flow can be obtained from Hopf links applying Wada operations. Each of these operations corresponds to a different attachment of one round 1-handle on one or two tori. When only one Wada operation has been applied on Hopf links, that means that only one attachment has been made. We obtain the dual graph from the corresponding picture of the flow where the canonical regions and the separatrices can be easily observed. We show that it is possible to obtain dual graphs without the picture of the flow for the case of flows characterized by Wada operations involving inessential attachments, that is, operations I, II and III, and with no heteroclinic trajectories connecting two saddles orbits. (See Th.2 and Cor. 1).

Definition 4. We say that two graphs are *glued* when one vertex of one graph is identified with one vertex of the other graph.

Definition 5. We say that two graphs are *NMS glued* when the identified vertices correspond to the canonical region where attachments of one round 1-handle has been made in such a way that the resulting flow is a NMS flow with no heteroclinic trajectories connecting two saddles orbits.

In consequence, NMS gluing of graphs is only valid on those vertices that admit the corresponding type of attachment and the following property is obtained:

Proposition 1. *When two graphs are NMS glued, the identified vertices satisfy one of the following statements:*

(1) *Blue vertices can not be identified with any other.*
(2) *Two red vertices can not be identified.*
(3) *A green vertex is obtained if and only if two green vertices are identified.*
(4) *A red vertex is obtained if and only if one green and one red vertices are identified.*

Proof.

(1) Blue vertices correspond to canonical regions enclosing no periodic orbit, so new attachment are not admissible, that means that no identification involving blue vertices is possible.

(2) Red vertices correspond to canonical regions surrounding unknotted attractive or repulsive periodic orbits. These orbits are involved in the essential-inessential attachement of the round 1-handle, so they are not in the core of the fattened round 1-handle.

In order to obtain the complete flow as there are no heteroclinic trajectories connecting two saddles orbits, essential circles must not be transformed in trivial circles. So, as this transformation is not possible, it is necessary to have a repulsive orbit (at least) in the core of the torus or fattened round handle where the round 1-handle is attached.

Then, these orbits do not admit a new essential attachment, that means that identification of two red vertices is not possible.

(3) Green vertices correspond to canonical regions surrounding linked periodic orbits, that means that all the involved attachments of the round 1-handle have been inessential. Then, to obtain again a green vertex it necessary to attach round 1-handles by means of inessential circles, i.e., to identify two green vertices.

Conversely, a vertex coming from the identification of two green vertices corresponds to a canonical region enclosing orbits where only inessential attachments have been made, so these orbits admits any kind of new attachments; therefore this vertex is green.

(4) Red vertices correspond to canonical regions enclosing unknotted orbits. As we have said before in the second point, a new round 1-handle can be attached to these orbits only by means of inessential circles, so the other vertex has to be green. Moreover, the result of this identification is a red vertex. □

On the other hand, it is shown in [2] that operations I, II and III commute. From the following proposition we can conclude that the NMS gluing of the corresponding graphs also commute.

Proposition 2. *When two round 1-handles are attached to the same torus involving inessential circles, the NMS flow obtained does not depend on the order of attachment of each round 1-handle.*

Proof. The manifold obtained after applying operations I, II or III is equivalent to the connected sum of tori (see [5]). When two round 1-handles are attached to one torus by means of inessential circles corresponding to two of these operations, the result is a torus with two disjoint 3-balls inside and the saddle orbit corresponding to each attachment is inside each ball. So, the result does not depend on the order of attachment of the round 1-handle.

Similarly, the result of attaching two round 1-handles by means of one essential and one or two inessential circles to the same torus is also a torus with two disjoint 3-balls in it, although in this case there are not any periodic orbit in the core of this torus. So, the result is also independent of the order of attachment. □

Moreover, it can be observed that when a round 1-handle is inessentially attached to a torus with one periodic orbit in its core, it does not break the invariant manifolds corresponding to the boundary of the canonical region where it is attached.

Theorem 2. *Dual graphs for NMS flows characterized by I, II and III Wada operations and with no heteroclinic trajectories connecting two saddles orbits, are obtained from the NMS gluing of dual graphs corresponding to each operation.*

Proof. For the case of a flow characterized by two operations of this type it is easy to obtain the flow and observe that the associated dual graph corresponds to the NMS gluing of the two basic graphs.

As we only consider NMS flows coming from I, II and III Wada operations the resulting manifold is a connected sum of tori. The complete flow on S^3 is obtained when toroidal holes are filled with attractive or repulsive orbits.

Let us suppose a graph that comes from n NMS gluings of n basic graphs. This graph represents a NMS flow coming from I, II and III Wada operations and with no heteroclinic trajectories connecting two saddles orbits, where each of its vertices corresponds to a canonical region of this flow. As we have shown above, each canonical region enclose periodic orbits that admit certain types of attachment and each attachment corresponds to a different NMS gluing of the vertices of the graph (see proposition 1).

A new NMS flow is obtained attaching a new round 1-handle, involving, at least, one inessential circle to one torus. For attaching a new round 1-handle to a torus we have to trough out the complementary orbit of this torus, obtaining then a torus with the n disjoint 3-balls coming from the n previous attachments (see [5]). This new attachment corresponds to a connected sum of tori and a new disjoint 3-ball appears inside the torus. So, the separatrices of stable and unstable manifolds of the previous saddle orbits are not modified by the new ones. That means that the edges of the different graphs remain and we only identify the vertices corresponding to the involved canonical regions.

This is equivalent to a NMS gluing of the corresponding basic graph and the previous one. □

Moreover,

Corollary 1. *NMS flows coming from I, II and III Wada operations and with no heteroclinic trajectories connecting two saddles orbits, can be reproduced from these dual graphs.*

Proof. The graph of these NMS flows comes from the NMS gluing of basic graphs. As each of these operations has a different associated graph, from proposition 1 we know how to separate this graph into basic graphs and we also know the type of attachment of round 1-handles made. So, the NMS flow can be reproduced. □

Duals graphs associated to NMS flows coming from IV, V and VI Wada operations are more complicated because essential attachments imply sometimes the breaking of the boundary of the different canonical regions and the corresponding dual graphs can not be obtained identifying vertices.

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