APPLICATIONS OF INDEPENDENT FAMILIES

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Abstract. We study maximal independent families (hereafter: mifs) and their applications to topological questions. We prove that if there exists either an \((\omega, \omega_1)\)-mif of size \(2^{\omega_1}\) with open density \(\omega\), or an \((\omega, \omega_1)\)-mif of size \(\leq 2^{\omega_1}\), then there exists an \(\omega\)-resolvable, not maximally resolvable, Tychonoff space.

1. Introduction

Following [7], a family \(I \subseteq \mathcal{P}(X)\) is called a \((\theta, \kappa)\)-independent family on \(X\) if for every two disjoint subfamilies \(\{A_\alpha : \alpha < \theta_1\}\) and \(\{B_\beta : \beta < \theta_2\}\) where \(\theta_1, \theta_2 < \theta\), the intersection \(\bigcap\{A_\alpha : \alpha < \theta_1\} \cap (\bigcap\{X \setminus B_\beta : \beta < \theta_2\})\) has size \(\kappa\). Such a family is called separated if for every two points \(s, t \in \kappa\) there exists \(I \in \mathcal{I}\) such that \(s \in I\) and \(t \notin I\).

Let \(<X, T>\) be a topological space. It is crowded if it has no isolated points. \(X\) is called \(\kappa\)-resolvable if there exists a partition of \(X\) into \(\kappa\)-many dense subsets. If \(X\) has no two disjoint dense subsets, then \(X\) is called irresolvable [9]. Following notation in [6], if every nonempty open subspace (respectively, crowded subset) is irresolvable, then \(X\) is open-hereditarily (respectively, hereditarily) irresolvable, in short \(OHI\) (respectively, \(HI\)). The concept of hereditarily irresolvable spaces was first defined as SI spaces in [9]. The dispersion character \(\Delta(X)\) is defined as \(\min\{|U| : U \neq \emptyset, U \in T\}\). \(X\) is maximally resolvable

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if $X$ is $\Delta(X)$-resolvable. The density of $X$ is denoted by $d(X)$. The \textit{open density} of $X$ is denoted by $od(X)$ and is defined as $\text{min}\{d(U) : \emptyset \neq U \in T\}$.

Independent families have been studied in [8], [11], [14]. In [8], it is proved that on every infinite cardinal $\kappa$, there exist an $(\omega, \kappa)$-independent family of size $2^\kappa$. In [14], K. Kunen proved that the existence of a maximal $(\omega_1, \kappa)$-independent family on some cardinal $\kappa > \omega$ is equiconsistent with the existence of a measurable cardinal. A maximal $(\omega_1, \kappa)$-independent family on $\kappa$ induces a zero-dimensional Baire irresolvable topology on $\kappa$. (See also [15]). On the other hand, Kunen and Tall showed in [16] that it is consistent that there exists no Baire irresolvable spaces.

Independent families have also been used to construct irresolvable spaces. In [6], E.K. van Douwen used maximal independent families to construct $n$-resolvable, not $n + 1$-resolvable, Tychonoff spaces. In [7], a different construction is given by F. Eckertson. In the same paper, F. Eckertson studied the relation of two concepts of maximality: a maximal $(\omega, \kappa)$-independent family and a maximal independent (maximal $(\omega, \omega)$-independent) family. In [4], it is proved that for every $\kappa \geq \omega$: (1) every $(\omega, \kappa)$-independent family on $\kappa$ with open density $\kappa$ is contained in some maximal $(\omega, \kappa)$-independent family which is also maximally independent; (2) there exists a maximal $(\omega, \kappa)$-independent family on $\kappa$ of size $2^\kappa$ that is also maximally independent.

Independent families relate to the following question, which is asked first in [2] and then in [3]: “Is there an $\omega$-resolvable Tychonoff space which is not maximally resolvable?” This question was discussed in several places: [7], [3]. Eckertson in [7] showed that, assuming the existence of “a crowded, HI, strong $P_\kappa$ space”, there exists such a space. In [10], it was proved that, assuming Luzin’s Hypothesis, $2^\omega = 2^{\omega_1}$, there exists such a space of size $\omega_1$. It is still unknown to us whether such a space exists in ZFC.

In this paper, we study maximal independent families of various sizes. We show, in section 2 and section 3, that: (1) For any infinite cardinal $\kappa$, if there is a maximal $(\omega, \kappa)$-independent family (briefly, an $(\omega, \kappa)$-mif) of size $2^\kappa$ with open density $< \kappa$, then there exists an $\omega$-resolvable Tychonoff space of size $\kappa$ which is not maximally resolvable; and (2) If there exists an $(\omega, \omega_1)$-mif on $\omega_1$ of size $\tau$ such that $\log(\tau) < \omega_1$, then there also exists an $\omega$-resolvable Tychonoff space which is not maximally resolvable.
2. \((\omega, \kappa)\)\text{-mif with open density} < \kappa

The following result from [7] will be useful to us.

**Theorem 2.1.** Let \(i_\kappa\) be the smallest cardinal \(\tau\) such that there exists a maximal \((\omega, \kappa)\)-independent family of size \(\tau\) on \(\kappa\). Then the following hold:

1. If \(\kappa = \log i_\kappa\), then every \((\omega, \kappa)\)-mif on \(\kappa\) is also maximally independent.
2. If \(\log i_\kappa < \kappa\), then there exists an \((\omega, \kappa)\)-mif on \(\kappa\) which is not maximally independent.

Each \((\omega, \kappa)\)-independent family \(I\) on a set \(X\) induces on \(X\) the topology with the subbase

\[
\{A_1 \cap ... \cap A_n \cap (X \setminus B_1) \cap ... \cap (X \setminus B_m) : n, m < \omega, A_i, B_j \in I\}.
\]

We use the same symbol \(I\) to denote that topology. The following results were proved in [4].

**Theorem 2.2.**

1. For any cardinal \(\lambda\) such that \(\log(2^\kappa) \leq \lambda \leq \kappa\), there exists a dense hereditarily irresolvable subset of \(\{0, 1\}^{2^\kappa}\) of size and open density \(\lambda\).
2. Let \(I\) be a separated \((\omega, \kappa)\)-independent family of size \(\tau\) on \(\kappa\). The topology induced by \(I\) is homeomorphic to a dense subset of the Cantor cube \(\{0, 1\}^\tau\).

Very little is known, even if ZFC is augmented with additional axioms, about the (possible) cardinalities of maximal \((\omega, \kappa)\)-independent families on a cardinal \(\kappa \geq \omega\). For example, the following question is not known to us.

**Problem 2.3.** Is it true in ZFC that every \((\omega, \kappa)\)-mif on \(\kappa\) is of size \(2^\kappa\)?

However, the question whether every \((\omega, \kappa)\)-mif of size \(2^\kappa\) is also maximally independent has the following answer.

**Theorem 2.4.** Let \(\tau, \kappa\) be two infinite cardinals such that \(\tau < \kappa\). The following are equivalent:

1. \(2^\tau = 2^\kappa\).
2. There exists an \((\omega, \kappa)\)-mif \(I\) of size \(2^\kappa\) on \(\kappa\) which is not maximally independent and for which \(od(I) = \tau\).
Proof. (1) $\rightarrow$ (2). By Theorem 2.2, there exists in the Cantor cube $\{0,1\}^{2^\kappa}$ an HI dense subset $D_1$ of size and open density $\kappa$, and an HI dense subset $D_2$ of size and open density $\tau$. Since each irresolvable dense subset of size and open density $\lambda$ in $\{0,1\}^{2^\kappa}$ induces a separating $(\omega,\lambda)$-mif on $\lambda$, we have a separating $(\omega,\kappa)$-mif $I$ on the set $D_1$ of size $2^\kappa$ such that $\text{od}(I) = \kappa$, and a separating $(\omega,\tau)$-mif $J$ on the set $D_2$ of size $2^\kappa$ such that $\text{od}(J) = \tau$.

List $I_1$ as $\{I_\alpha : \alpha < 2^\kappa\}$. Choose an element $J \in I_2$, and list $I_2 \setminus \{J\}$ as $\{J_\alpha : \alpha < 2^\kappa\}$. We define a new independent family on $D_1 \cup D_2$ by $K = \{I_\alpha \cup J_\alpha : \alpha < 2^\kappa\}$.

Obviously, $K$ is an $(\omega,\kappa)$-independent family of size $2^\kappa$ on the set $D_1 \cup D_2$. It is easy to check that $K$ is in fact an $(\omega,\kappa)$-mif.

Certainly, the family $\{J\} \cup K$ is an independent family on $\kappa \cup \tau$, which implies that the family $K$ is not maximally independent and $\text{od}(K) \leq \tau$. Since $\text{od}(I) = \kappa$ and $\text{od}(J) = \tau$, it follows that $\text{od}(K) = \tau$.

(2) $\rightarrow$ (1). Since $\text{od}(I) = \tau < \kappa$, there exists a non-empty basic open subset $U$ of the space $<\kappa, I>$ with a dense subset $E$ of size $\tau = \text{od}(I) < \kappa$. Since there are at least $2^\kappa$-many distinct nonempty clopen sets inside $U$, the subspace topology on $E$ has cardinality at least $2^\kappa$. Since $|E| = \tau$, we then have $2^\kappa \leq 2^\tau$ and hence $2^\tau = 2^\kappa$. $\square$

Corollary 2.5. The following are equivalent.

(1) Luzin’s Hypothesis, $2^\omega = 2^{\omega_1}$.

(2) There exists an $(\omega,\omega_1)$-mif $I$ on $\omega_1$ of size $2^{\omega_1}$ which is not maximally independent and for which $\text{od}(I) = \omega$.

Problem 2.6. Is it true that Luzin’s Hypothesis is equivalent to this statement: there exists an $(\omega,\omega_1)$-mif of size $2^\omega$ on $\omega_1$?

Following the notation in [5], we use the symbol $S(X)$ to denote the smallest cardinal $\theta$ such that every pairwise disjoint family of nonempty open sets of the topological space $X$ has size less than $\kappa$. In the book [12], the same cardinal is denoted by $\hat{c}(X)$.

Let us call a space $<X, T > \kappa$-condensed if $X$ has a family $\mathcal{K}$ of nowhere dense subsets such that $|\mathcal{K}| = \kappa$ and every nowhere dense subset of $X$ is contained in some element of $\mathcal{K}$. It is shown in [10] that if a space $<X, T >$ has weight $\leq 2^\tau$ for some cardinal $\tau$, then it is $2^\tau$-condensed.
In [10], the following result was proved.

**Theorem 2.7.** Let \(<X, T>\) be a Tychonoff space. Let \(\tau\) be an infinite cardinal such that \(\tau < cf(S(<X, T>))\). Suppose that \(X\) is \(2^\tau\)-condensed and that \(X\) has a partition consisting of \(\tau\)-many OHI dense subsets. Then \(T\) has a Tychonoff expansion \(U \supset T\) with dispersion character \(\geq od(<X, T> \cdot \tau)\) which is \(\tau\)-resolvable but not \(S(<X, T>)\)-resolvable.

In particular, the following theorem holds.

**Theorem 2.8.** Assuming Luzin's Hypothesis. There exists an \(\omega\)-resolvable Tychonoff space of size \(\omega_1\) which is not maximally resolvable.

Hence, we have the following result.

**Theorem 2.9.** If there exists an \((\omega, \kappa)\)-mif on \(\kappa\) of size \(2^\kappa\) with open density \(<\kappa\), then there exists an \(\omega\)-resolvable Tychonoff space which is not maximally resolvable.

### 3. \((\omega, \kappa)\)-MIF OF SMALL SIZE

As usual, for any cardinal \(\kappa\), we let \(\log(\kappa) := \min\{\theta : 2^\theta \geq \kappa\}\). Let us call an \((\omega, \kappa)\)-independent family \(\mathcal{I}\) on \(\kappa\) of small size if \(\log(|\mathcal{I}|) < \kappa\). In this section, we prove that if there exists an \((\omega, \omega_1)\)-mif of small size, then there exists an \(\omega\)-resolvable Tychonoff space which is not maximally resolvable.

It is not known to us that whether there exists an \((\omega, \kappa)\)-mif of small size. In particular, we do not know whether there exists a dense irresolvable subset of size and open density \(\omega_1\) in the Cantor cube \(\{0, 1\}^\tau\). Note that it is showed, in [13] under the assumption of Martin’s Axiom, and later in [1] in ZFC, that there exist irresolvable dense subsets in \(\{0, 1\}^\tau\) and hence an irresolvable dense subset of size \(\omega_1\) by augmenting a discrete spaces.

**Lemma 3.1.** Let \(D\) be a dense irresolvable subset of \(\{0, 1\}^\tau\) with \(\Delta(D) = \kappa\). Then there exists a hereditarily irresolvable dense subset of \(\{0, 1\}^\tau\) such that \(od(D) \geq \kappa\).

**Proof.** Since \(D\) is an irresolvable dense subset of \(\{0, 1\}^\tau\), there is an \((\omega, \kappa)\)-mif \(\mathcal{I}\) on \(D\) which is also maximally independent.
Since $D$ is irresolvable, there is a non-empty open and hereditarily irresolvable subset $U$ of $D$. Without loss of generality, we can assume that $U$ is a basic open set, i.e., a set of the form $U = I_1 \cap \ldots \cap I_n \cap D \setminus J_1 \cap \ldots \cap D \setminus J_m$ for some $\{I_1, \ldots, I_n, J_1, \ldots, J_m\} \subseteq \mathcal{I}$. Certainly, $\Delta(U) = \kappa$. It remains to see that $od(U) = \kappa$.

Suppose $od(U) = \theta < \kappa$. Then for some basic open set $O$ of $U$, there is a dense subset $D$ of $O$ such that $|D| = d(O) = \theta < \kappa$. Since the dispersion character of $O$ is $\kappa$, the set $D$ cannot contain any open subset of $O$. Hence the set $O \setminus D$ is also dense in $O$. This implies that $O$ is not irresolvable, contrary to the assumption that $U \supseteq O$ is a hereditarily irresolvable space. \hfill \Box

From this lemma, we have the following direct corollary.

**Corollary 3.2.** Let $\mathcal{I}$ be a separated $(\omega, \kappa)$-independent family such that the induced topology is open-hereditarily irresolvable. Then $od(\mathcal{I}) = \kappa$.

**Lemma 3.3.** Suppose $\mathcal{I}$ is an $(\omega, \kappa)$-mif of size $\tau$ on $\kappa$. If $\mathcal{I}$ is not maximally independent, then there exists a cardinal $\theta < \kappa$ such that $\tau \leq 2^\theta \leq 2^\kappa$.

*Proof.* Since $\mathcal{I}$ is not maximally independent, there exists a subset $D \subseteq \kappa$ such that $\{D\} \cup \mathcal{I}$ is an independent family properly containing $\mathcal{I}$. Since $\mathcal{I}$ is already maximally $(\omega, \kappa)$-independent, there exists $\{I_1, \ldots, I_n, J_1, \ldots, J_m\}$ such that the set $W = I_1 \cap \ldots \cap I_n \cap (\kappa \setminus J_1) \cap \ldots \cap (\kappa \setminus J_m) \cap D$ has size $\theta$ for some cardinal $\theta < \kappa$. The fact that $\mathcal{I}$ is an independent family implies that on the subset $W \subseteq \kappa$, the induced topology inherited from $\mathcal{I}$ has weight $\tau$. Since $|W| = \theta$, we have $\tau \leq 2^\theta$. \hfill \Box

**Theorem 3.4.** Assume the existence of an $(\omega, \omega_1)$-mif of small size. Then there exists an $\omega$-resolvable Tychonoff space which is not maximally resolvable.

*Proof.* By Theorem 2.1, we know that on $\omega_1$ there exists a separated $(\omega, \omega_1)$-mif $\mathcal{I}$ of size $\tau$ which is not maximally independent.

By Lemma 3.3, we know that there exists a cardinal $\theta < \omega_1$ such that $\tau \leq 2^\theta = 2^\omega \leq 2^{\omega_1}$. Let $D_1 \subseteq \kappa$ be such that $\{D_1\} \cup \mathcal{I}$ is an independent family properly containing $\mathcal{I}$.

Since $\mathcal{I}$ is a maximal $(\omega, \omega_1)$-independent family, the space $<\omega_1, \mathcal{I}>$ is not an $\omega_1$-resolvable space.
We use the following process to define for $n < \omega$ a dense subset $E_n$ of the space $< \omega_1, I >$ and an independent family $I_n$ on $E_n$.

For $n < \omega$, repeat the following process until impossible. Let $E_1 = \omega_1 \setminus D_1$. On $E_1$, the family $I$ induces an $(\omega, \omega_1)$-mif $I_1$. Certainly, $I_1$ is still maximally $(\omega, \omega_1)$-independent, since $E_1$ is a dense subset of $< \omega_1, I >$. If $I_1$ is already maximally independent, then stop. If $I_1$ is not maximally independent, we let $D_2 \subset E_1$ be such that $I_1 \cup \{D_2\}$ is an independent family properly containing $I_1$. We let $E_2 = E_1 \setminus D_2$, and let $I_2$ be the independent family of $I_1$ restricted to $E_2$. The same reason shows that $I_2$ is also maximally $(\omega, \omega_1)$-independent.

If $E_n$ and $I_n$ have been defined for all $n < \omega$, then the family \{$D_n : n < \omega$\} is a disjoint family of dense subsets of the space $< \omega_1, I >$. In this case, the space $< \omega_1, I >$ is an $\omega$-resolvable Tychonoff space which is not maximally resolvable.

We will show, in the following, that if the process stops at a step $n$ for some $n < \omega$, i.e, $I_n$ is already maximally independent, then we can construct an $\omega$-resolvable, not maximal resolvable, Tychonoff space. Clearly, in this case, the space $< E_n, I_n >$ is an irresolvable Tychonoff space, and $|I_n| = |I| = \tau \leq 2^\omega \leq 2^{\omega_1}$.

By a standard result from the subject of resolvable space (see [9]), the space $< E_n, I_n >$ has an basic open set $U \subset E_n$ such that $U$ with the topology inherited from $I_n$ is hereditarily irresolvable. On the set $U$, the family $I_n$ induces an independent family $J$. Hence $< U, J >$ is a hereditarily irresolvable Tychonoff space with dispersion character $\omega_1$. Certainly, $|J| = |I_n| = \tau$. By Theorem 2.2(2), the space $< U, J >$ is homeomorphic to a dense subset $D$ of the Cantor cube $\{0, 1\}^\tau$. Certainly, $|D| = |U| = |E_n| = \omega_1$. Since $D$ with the topology inherited from $\{0, 1\}^\tau$ is hereditarily irresolvable and has dispersion character $\omega_1$, we can use the argument from Lemma 3.1 to show that $od(D) = \omega_1$.

Let $< D >$ be the subgroup generated by $D$, and let $X = \bigcup_{n < \omega} x_nD$ be such that the infinitely many cosets $x_n < D >$ are distinct. Then $X$ with the topology inherited from $\{0, 1\}^\tau$ is a Tychonoff space which is the union of $\omega$-many HI, dense subspaces with size and open density $\omega_1$. Since $\tau \leq 2^\omega$, the space $X$ is $2^{\omega_1}$-condensed. Hence, by Theorem 2.7, the space has a Tychonoff expansion that is $\omega$-resolvable but not maximally resolvable. □
We have shown that if either there exists an \((\omega, \omega_1)\)-mif of size \(2^{\omega_1}\) with open density \(\omega\), or there exists an \((\omega, \omega_1)\)-mif \(I\) of small size, i.e. \(|I| \leq 2^{\omega}\), then there exists an \(\omega\)-resolvable, not maximally resolvable, Tychonoff space of size \(\omega_1\). As to the remaining case and the relation between maximal \((\omega, \kappa)\)-independent families and examples of \(\omega\)-resolvable but not maximally resolvable spaces, the following question remains open.

**Problem 3.5.** Assume that every \((\omega, \omega_1)\)-mif on \(\omega_1\) is of open density \(\omega_1\) and of size \(\tau\) such that \(\log(\tau) = \omega_1\). Is there an \(\omega\)-resolvable, but not maximally resolvable, Tychonoff space?

**References**


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