COMPACTIFICATIONS OF BAIRE SPACES $\kappa^\omega$

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ABSTRACT. We show that the space of irrationals can be compactified in such a way that the remainder is the union of, apriori prescribed, countably many compact spaces each of weight not exceeding $\omega_1$. We show that any Baire space of an uncountable weight has a compactification such that its remainder is a $\sigma$-discrete space.

1. Compactifying Baire spaces of uncountable weight

The Cartesian product of countably many copies of an infinite discrete space of cardinality $\kappa$ is called the Baire space of weight $\kappa$. The Baire space of weight $\omega$ is homeomorphic to the space of irrational numbers.

No Baire space of any uncountable weight can have a compactification whose remainder is going to be the union of finitely many metrizable subspaces. We shall show that there is a one whose remainder is the union of countably many discrete (just metrizable) subspaces.

Throughout our discussion, we treat cardinals as von Neumann ordinals endowed with the discrete topology. Let $\kappa$ be an uncountable cardinal. The symbol $\leq^\omega \kappa$ denotes the complete tree of height $\omega + 1$, i.e.,

$$\leq^\omega \kappa = <^\omega \kappa \cup^\omega \kappa,$$

where

$$<^\omega \kappa = \{s : s is a function and Dom(s) \in \omega and Rng(s) \subseteq \kappa\}$$

and

$$\omega \kappa = \{s : s is a function and Dom(s) = \omega and Rng(s) \subseteq \kappa\}.$$

2000 Mathematics Subject Classification. 54A25, 54D30.

Key words and phrases. Compact space, the Bare space of weight $\kappa$, metrizability number.
If \( s \in ^{<\omega} \kappa \) and \( \alpha \in \kappa \), then \( s \upharpoonright \alpha \) denotes the concatenation of \( s \) by \( \alpha \).

For each \( n \in \omega \), let \( L_n = \{ t \in ^{<\omega} \kappa : |t| = n \} \) and \( T_n = \{ t \in ^{<\omega} \kappa : |t| \leq n \} \).

For each \( s \in ^{<\omega} \kappa \), let \( \text{Cone}(s) = \{ t \in ^{<\omega} \kappa : s \subseteq t \} \).

Let \( X_\kappa \) be the space whose underlying set is \( \omega^\kappa \) which is endowed with the tree topology, i.e., topology generated by sets of the form

\[
\text{Cone}(s) - (\text{Cone}(s \upharpoonright \alpha_1) \cup \text{Cone}(s \upharpoonright \alpha_2) \cup ... \cup \text{Cone}(s \upharpoonright \alpha_k)),
\]

where \( s \in ^{<\omega} \kappa \) and \( \alpha_i \in \kappa \) for each \( i = 1, 2, ..., k \). In the series of simple lemmas that follows we will verify the required properties for the space \( X_\kappa \) to be a required compactification of the Baire space of the uncountable weight \( \kappa \).

**Lemma 1.** If \( s, t \in ^{<\omega} \kappa \), \( s \neq t \), and \( t \in \text{Cone}(s) - (\text{Cone}(s \upharpoonright \alpha_1) \cup \text{Cone}(s \upharpoonright \alpha_2) \cup ... \cup \text{Cone}(s \upharpoonright \alpha_k)) \) then \( \text{Cone}(t) \subseteq \text{Cone}(s) - (\text{Cone}(s \upharpoonright \alpha_1) \cup \text{Cone}(s \upharpoonright \alpha_2) \cup ... \cup \text{Cone}(s \upharpoonright \alpha_k)) \).

**Lemma 2.** If \( s, t \in ^{<\omega} \kappa \), \( s \not\subseteq t \), and \( s \not\supseteq t \), then \( \text{Cone}(t) \cap \text{Cone}(s) = \emptyset \).

**Lemma 3.** If \( s \in ^{<\omega} \kappa \), then \( \text{Cone}(s) \cap ^{\omega} \kappa = \prod \{ C_i : i \in \omega \} \), where \( C_i = \{ s(i) \} \) for each \( i \in \text{Dom}(s) \), and \( C_i = \kappa \) for each \( i \notin \text{Dom}(s) \).

Thus the subspace \( ^{\omega} \kappa \) of \( X_\kappa \) is the Baire space of weight \( \kappa \).

**Lemma 4.** For each \( n \in \omega \), \( L_n \) is a discrete subspace of \( X_\kappa \) (and \( T_n \) is a closed subspace of \( X_\kappa \)).

**Proof.** If \( s \in L_n \), then \( L_n \cap \text{Cone}(s) = \{ s \} \). \( \square \)

**Theorem 5.** \( X_\kappa \) is a compactification of the Baire space \( ^{\omega} \kappa \).

**Proof.** The space \( X_\kappa \) is Hausdorff (use Lemma 2). By Lemma 3, the Baire space \( ^{\omega} \kappa \) is a dense subspace of the space \( X_\kappa \).

Suppose to the contrary that \( X_\kappa \) is not a compact space. Thus there exists an open cover \( \mathcal{P} \) of \( X_\kappa \) without a finite subcover. Without loss of generality we may assume that \( \mathcal{P} \) consists of the basic open sets. Let \( U_0 \in \mathcal{P} \) be a basic set containing \( \emptyset \in X_\kappa \). Since \( U_0 = \text{Cone}(\emptyset) \setminus (\text{Cone}(\emptyset \upharpoonright \alpha_1) \cup \text{Cone}(\emptyset \upharpoonright \alpha_2) \cup ... \cup \text{Cone}(\emptyset \upharpoonright \alpha_k)) \),
one of $\text{Cone}(\emptyset - \alpha_i)$, $i = 1, 2, ..., k$, cannot be covered by finitely many elements of the cover $\mathcal{P}$. Thus there exists a sequence $s_1$ of length 1 such that $\text{Cone}(s_1)$ cannot be covered by finitely many elements of the cover $\mathcal{P}$.

Suppose that we have defined sequences $s_1, s_2, ..., s_n$ satisfying the following conditions:

(i) For each $k \leq n$, $s_k \in \omega \kappa$ and $\text{Dom}(s_k) = k$;
(ii) $s_1 \subseteq s_2 \subseteq ... \subseteq s_n$;
(iii) For each $k \leq n$, the set $\text{Cone}(s_k)$ cannot be covered by finitely many elements of the cover $\mathcal{P}$.

Let $U_n \in \mathcal{P}$ be a basic set containing $s_n \in \omega \kappa$. Since $U_n = \text{Cone}(t) \setminus (\text{Cone}(t - \beta_1) \cup \text{Cone}(t - \beta_2) \cup ... \cup \text{Cone}(t - \beta_k))$, $s_n$ must be equal to $t$, by virtue of Lemma 1. Hence one of $\text{Cone}(s_n - \beta_j)$, $j = 1, 2, ..., k$, cannot be covered by finitely many elements of the cover $\mathcal{P}$. Thus there exists a sequence $s_{n+1}$ of length $n + 1$ such that $s_n \subseteq s_{n+1}$ and $\text{Cone}(s_{n+1})$ cannot be covered by finitely many elements of the cover $\mathcal{P}$.

By induction, there exists a sequence $s_0, s_1, ..., s_n, ...$ satisfying the following conditions:

(i) For each $k \in \omega$, $s_k \in \omega \kappa$ and $\text{Dom}(s_k) = k$;
(ii) $s_1 \subseteq s_2 \subseteq ... \subseteq s_n \subseteq ...$
(iii) For each $k > 0$, the set $\text{Cone}(s_k)$ cannot be covered by finitely many elements of the cover $\mathcal{P}$.

Let $x = \bigcup \{s_k : k \in \omega \}$. Since $x \in \omega \kappa$, there exists $U$ in $\mathcal{P}$ that contains the point $x$. Thus $x \in \text{Cone}(t) - (\text{Cone}(t - \beta_1) \cup \text{Cone}(t - \beta_2) \cup ... \cup \text{Cone}(t - \beta_k))$. It follows that $t \subseteq x$ and thus $t = s_n$ for some $n \in \omega$. Since $x \in \text{Cone}(t) - (\text{Cone}(t - \beta_1) \cup \text{Cone}(t - \beta_2) \cup ... \cup \text{Cone}(t - \beta_k))$, $s_{n+1} \in \text{Cone}(t) - (\text{Cone}(t - \beta_1) \cup \text{Cone}(t - \beta_2) \cup ... \cup \text{Cone}(t - \beta_k))$ too. By lemma 1, $\text{Cone}(s_{n+1}) \subseteq \text{Cone}(t) - (\text{Cone}(t - \beta_1) \cup \text{Cone}(t - \beta_2) \cup ... \cup \text{Cone}(t - \beta_k))$, a contradiction. \qed

2. Compactifying irrationals

We begin by proving an easy fact.

**Lemma 6.** Let $Y$ be a compact Hausdorff space and let $p \in Y$ be a non-isolated point. Suppose that $X$ is a compactification of the space $Y - \{p\}$ with remainder $Z$. Let $U$ be an open neighborhood of the point $p$ in the space $Y$ and let $V$ be an open neighborhood of a point $x \in Z$. Then $U \cap V \neq \emptyset$. 

Proof. The set $F = X - U$ is a compact subset of the space $Y - \{p\} \subset X$. So $V - F$ is an open neighborhood of the point $x$. Hence $\emptyset \neq (V - F) \cap (Y - \{p\}) \subseteq V \cap U$. \hfill \Box

Let $\mathcal{R}$ be the class of all compact Hausdorff spaces that can be used as a remainder of some compactification of the discrete countable space $\omega$. According to Parovičenko’s theorem (cf. [1]), any compact Hausdorff space of weight not exceeding $\omega_1$ is in $\mathcal{R}$.

**Lemma 7.** Let $Y = \oplus\{X_n : n \in \omega\}$ be the topological sum of compact Hausdorff spaces $X_n$. If $Z \in \mathcal{R}$, then there exists a compactification $X$ of the space $Y$ such that the remainder $X - Y$ is homeomorphic to $Z$.

Proof. Without loss of generality, we may assume that $Y$ and $Z$ are disjoint. Let $\tilde{X}$ be a compactification of the discrete space $\omega$ such that the remainder $\tilde{X} - \omega$ is homeomorphic to $Z$. For any open set $U$ of the space $\tilde{X}$ such that $U \cap Z \neq \emptyset$, let $e(U) = \oplus\{X_n : n \in U \cap \omega\} \cup (U \cap Z)$. We take $X$ to be the set $Y \cup Z$ with topology generated by the sets that are open subsets of the space $Y$ or of the form $e(U)$. \hfill \Box

**Lemma 8.** Let $Y$ be a compact Hausdorff space and let $p \in Y$ be a non-isolated point that has a countable base of closed-open subsets of $Y$. If $Z \in \mathcal{R}$, then there exists a compactification $X$ of the space $Y - \{p\}$ such that the remainder $X - (Y - \{p\})$ is homeomorphic to $Z$.

Let $C$ be a compact Hausdorff space and let $\{d_n : n \in \omega\}$ be an enumeration of a countable subset of $C$. Suppose further that each point $d_n$ is non-isolated and has a countable base of closed-open subsets of $C$. Let $Z_n \in \mathcal{R}$ for each $n = 1, 2, \ldots$. By induction, we define a sequence of spaces $\{C_n : n \in \omega\}$ as follows:

- $C_0 = C$;
- $C_{n+1} = \text{a compactification of the space } C_n - \{d_n\}$ such that the remainder $C^*_{n+1} = C_{n+1} - (C_n - \{d_n\})$ is homeomorphic to the space $Z_{n+1}$ (such a compactification exists by virtue of Lemma 8).

For $n = 1, 2, \ldots$, let $p_n$ be the natural projection from $C_n$ to $C_{n-1}$, i.e.,

$$p_n(x) = \begin{cases} d_{n-1}, & \text{if } x \in C^*_n \\ x, & \text{if } x \notin C^*_n \end{cases}.$$
Lemma 9. For \( n = 1, 2, \ldots \), \( p_n : C_n \to C_{n-1} \) is continuous.

**Proof.** Let \( U \) be an open neighborhood of the point \( d_{n-1} \) in the space \( C_{n-1} \). The set \( F = C_{n-1} - U \) is a compact subset of the space \( C_n \). Clearly \( p_{n-1}^{-1}(U) = (U - \{d_{n-1}\}) \cup Z = C_n - F \). In consequence, the set \( p_{n-1}^{-1}(U) \) is open in the space \( C_n \). □

Let us consider the inverse sequence

\[
C_0 \leftarrow^{p_1} C_1 \leftarrow^{p_2} C_2 \leftarrow^{p_3} \ldots \leftarrow^{p_{n-1}} C_n \leftarrow^{p_n} \ldots
\]

and its limit \( X \), i.e.,

\[
X = \left\{ (x_i) \in \prod \{C_i : i \in \omega \} : p_n(x_n) = x_{n-1} \text{ for } n = 1, 2, \ldots \right\}.
\]

**Lemma 10.** \( X \) is a compact Hausdorff space.

Let \( M_0 = \left\{ (x_i) \in \prod \{C_i : i \in \omega \} : x_i = x \text{ for } i \in \omega \text{ and } x \in C - \{d_n : n \in \omega \} \right\} ; \)

If \( n > 0 \), \( M_n = \left\{ (x_i) \in \prod \{C_i : i \in \omega \} : x_i = d_{n-1} \text{ for } i = 0, 1, 2, \ldots, n-1 \text{ and } x_i = x \text{ for } i \geq n \text{ and } x \in C_n^* \right\} \). The sets \( M_n, n \in \omega \), are pairwise disjoint.

**Lemma 11.** \( M_0 \) and \( C - \{d_n : n \in \omega \} \) are homeomorphic.

**Lemma 12.** For each \( n = 1, 2, \ldots \), \( M_n \) and \( Z_n \) are homeomorphic.

Both lemmas, above, follow immediately from the following one:

**Lemma 13.** Let \( \prod \{X_\alpha : \alpha \in S \} \) be the product of spaces \( X_\alpha \), where \( X_\alpha = X \) for each \( \alpha \in S \). Then the diagonal \( \Delta = \{(x_\alpha) \in \prod \{X_\alpha : \alpha \in S \} : x_\alpha = x \text{ for each } \alpha \in S \text{ and } x \in X \} \) and the space \( X \) are homeomorphic.

**Proof.** Let \( h : X \to \Delta \) be defined as follows:

\[
h(x) = (x_\alpha), \text{ where } x_\alpha = x \text{ for each } \alpha \in S.
\]

One can easily see that if \( A \subseteq X \) and \( \alpha \in S \) and \( \pi_\alpha : \prod \{X_\alpha : \alpha \in S \} \to X_\alpha \) is a natural projection, then \( h(A) = \Delta \cap \pi_\alpha^{-1}(A) \). □

**Lemma 14.** \( X = \bigcup \{M_n : n \in \omega \} \).
Proof. Let \((x_i) \in X\). Consider the following two cases:

Case (a) \(\forall i \; x_i = x_{i+1}\);

Case (b) \(\exists i \; x_i \neq x_{i+1}\).

In case (a), let \(x = x_i\) for each \(i\). Since \(x_0 = x, \; x \in C\). Clearly, \(x \neq d_n\) for each \(n \in \omega\) (for if \(x = d_n\), then \(p_{n+1}(x_{n+1}) = d_n = x\) and \(x = x_{n+1} \in C^*_n\); a contradiction). Hence \((x_i) \in M_0\).

In case (b), since \(p_{i+1}(x_{i+1}) = x_i, \; x_{i+1} \in C^*_n\) and \(x_i = d_i\). Thus \(x_j = d_i\) for each \(j \leq i\), and \(x_j = x_{i+1}\) for each \(j \geq i + 1\). Hence \((x_i) \in M_{i+1}\).

Lemma 15. Let \(U = U_0 \times U_1 \times \ldots \times U_n \times C_{n+1} \times C_{n+2} \times \ldots\) be an open basic subset of the product \(\prod \{C_i : i \in \omega\}\). If \(U \cap X \neq \emptyset\), then \((U_0 \cap U_1 \cap \ldots \cap U_n) \cap (C - \{d_0, d_1, \ldots, d_{n-1}\}) \neq \emptyset\).

Proof. By Lemma 14, \(U \cap M_k \neq \emptyset\) for some \(k \in \omega\). If \(k = 0\), then there exists \(x \in C - \{d_n : n \in \omega\}\) such that \((x_i) \in U\) and \(x_i = x\) for each \(i \in \omega\). Hence \(x \in U_0 \cap U_1 \cap \ldots \cap U_n\). Thus \((U_0 \cap U_1 \cap \ldots \cap U_n) \cap (C - \{d_0, d_1, \ldots, d_{n-1}\}) \neq \emptyset\). Let \(k > 0\) and let \((x_i) \in U \cap M_k\). Thus there exists \(x \in C^*_k\) such that

\[
    x_i = \begin{cases} 
    d_{k-1}, & \text{if } i < k \\
    x, & \text{if } i \geq k 
    \end{cases}
\]

If \(k > n\), then \(d_{k-1} \in U_0 \cap U_1 \cap \ldots \cap U_n\). Assume that \(k \leq n\). The set \(U = \bigcap \{U_i : i \leq k - 1\} \cap (C - \{d_0, d_1, \ldots, d_{k-2}\})\) is an open neighborhood of the point \(d_{k-1}\) in the subspace \((C - \{d_0, d_1, \ldots, d_{k-2}\})\).

The set \(V = \bigcap \{U_i : k \leq i \leq n\}\) an open neighborhood of the point \(x\) in the space \(C_k\). By Lemma 6, \(U \cap V \neq \emptyset\). \(\square\)

Lemma 16. \(M_0\) is a dense subset of \(X\).

Proof. Let \(U = U_0 \times U_1 \times \ldots \times U_n \times C_{n+1} \times C_{n+2} \times \ldots\) be an open basic subset of the product \(\prod \{C_i : i \in \omega\}\) such that \(U \cap X \neq \emptyset\).

By Lemma 15, \((U_0 \cap U_1 \cap \ldots \cap U_n) \cap (C - \{d_0, d_1, \ldots, d_{n-1}\}) \neq \emptyset\). In consequence, \((U_0 \cap U_1 \cap \ldots \cap U_n) \cap (C - \{d_n : n \in \omega\}) \neq \emptyset\). If \(x \in (U_0 \cap U_1 \cap \ldots \cap U_n) \cap (C - \{d_n : n \in \omega\})\) and \((x_i)\) is such that \(x_i = x\) for \(i \in \omega\), then \((x_i) \in U \cap M_0\). \(\square\)

Lemma 17. If \(\{d_n : n \in \omega\}\) is a dense subset of \(C\), then \(\{M_n : n \geq 1\}\) is a \(\pi\)-net in \(X\).

Proof. Let \(U = U_0 \times U_1 \times \ldots \times U_n \times C_{n+1} \times C_{n+2} \times \ldots\) be an open basic subset of the product \(\prod \{C_i : i \in \omega\}\) such that \(U \cap X \neq \emptyset\).
By Lemma 15, $\left( U_0 \cap U_1 \cap ... \cap U_n \right) \cap \left( C - \{d_0, d_1, ..., d_{n-1}\} \right) \neq \emptyset$. In consequence, $U_0 \cap U_1 \cap ... \cap U_n$ contains infinitely many elements among $\{d_n : n \in \omega\}$. Pick any $m$ such that $m > n$ and $d_m \in U_0 \cap U_1 \cap ... \cap U_n$. Then $M_m \subseteq U$. □

**Theorem 18.** There exists a compactification $X$ of the space of irrational numbers $\omega^\omega$ such that: (i) $X - \omega^\omega = \bigcup \{M_n : n \geq 1\}$, (ii) $\{M_n : n \geq 1\}$ is a $\pi$-net in $X$, (iii) For each $n = 1, 2, ..., M_n$ and $Z_n$ are homeomorphic.

### 3. Applications

The metrizability number $m(X)$ of a space $X$ is the smallest cardinal number $\kappa$ such that $X$ can be represented as a union of $\kappa$ many metrizable subspaces. In [2] we showed that compact Hausdorff spaces with finite metrizability number can be represented as follows:

**Theorem 19.** If $X$ is a (locally) compact Hausdorff space with $m(X) = n$, $2 \leq n < \omega$, then $X$ can be represented as $X = G \cup F$, where $G$ is an open dense metrizable subspace of $X$, $F \cap G = \emptyset$, and $m(F) = n - 1$.

A similar representation theorem may not hold for compact Hausdorff spaces with countable metrizability number.

**Theorem 20.** There exists a compact Hausdorff space $X$ with a countable $\pi$-base such that $m(U) = \omega$ for each non-empty open subset $U$ of $X$.

**Proof.** Let $X$ be the space constructed from the Cantor set $C$, an arbitrary countable dense subset $\{d_n : n \in \omega\}$, and from $Z_n$ that is e.g., the one-point compactification of a discrete space of cardinality $\aleph_1$, for each $n = 1, 2, ...$. □

**Theorem 21.** If $M$ is a zero-dimensional metrizable space, then $M$ has a compactification $Y$ such that $Y \setminus M$ is a union of countably many discrete subspaces of $Y$.

**Proof.** The space $M$ can be embedded into a Baire space $\kappa^\omega$. Let $X$ be the compactification of the Baire space $\kappa^\omega$ as given in Theorem 5. Then the closure of $M$ in the space $X$ gives the required compactification $Y$. □
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