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A NATURAL DOWKER SPACE

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Abstract. We will describe a plain version of a recursive technique to construct Dowker spaces.

1. Constructing the space

Let $X = \mathfrak{c} \times \omega$ and, for all $n \in \omega$, $\mathbb{W}_n = \mathfrak{c} \times n$. The collection $\mathcal{B}_0 = \{\mathbb{W}_n : n \in \omega\} \cup \{X \setminus \{x\} : x \in X\}$ is a subbase for an initial (T_1, not Hausdorff) topology. We will add more open sets in $2^{\mathfrak{c}}$ steps to $\mathcal{B}_0$ to make X normal but not countably paracompact.

For $U^0, U^1 \subset X$ we say $S = \langle U^0, U^1 \rangle$ is a *covering pair* if $U^0 \cup U^1 = X$. Fix a list $\mathcal{S} = \langle S_\xi \rangle_{\xi < 2^{\mathfrak{c}}}$ of all covering pairs mentioning each $2^{\mathfrak{c}}$ many times. $\mathcal{S}_\xi$ will denote the covering pair $\langle U^0_\xi, U^1_\xi \rangle$.

Inductively, we will define $H \subset 2^{\mathfrak{c}}$ and, for all $\xi \in H$, a pair $\langle B^0_\xi, B^1_\xi \rangle$ such that $B^0_\xi \subset U^0_\xi, B^1_\xi \subset U^1_\xi, B^0_\xi \cap B^1_\xi = \emptyset$ (and $B^0_\xi \cup B^1_\xi = X$). Then we'll set

$$\mathcal{B}_\xi = \mathcal{B}_0 \cup \{B^0_\eta, B^1_\eta : \eta \in H, \eta < \xi\}.$$

For a set to be *ξ-open*, we mean open in the topology generated by $\mathcal{B}_\xi$ as a subbase. If $A \in [X]^\omega$ and $S = \langle U^0, U^1 \rangle$ is a covering pair then the pair restricted to A is given by $S \restriction A = \langle U^0 \cap A, U^1 \cap A \rangle$. We let $\mathcal{S}_A = \{S \restriction A : S \text{ is a covering pair}\}$.

Key words and phrases. countably paracompact, Dowker space, elementary submodel, normal.

[1] Editor's note: Prepared by Dennis Burke, Miami University, from Zoli's handwritten notes, this paper is a "plain" version of the technique in [1] and may assist researchers hoping to apply Zoli's method to their own problems.

Definition. $\langle A, \mathcal{R}, u\rangle$ is a *control triple* if $A \in [X]^\omega$, $\mathcal{R} \in [\mathcal{S}_A]^\omega$ and u is a countably infinite function with $\mathrm{dom}(u) \subset A$ and

(C_1) $x \in \mathrm{dom}(u) \implies u(x) \in [\mathcal{S}_A \setminus \mathcal{R}]^{<\omega}$;

(C_2) $x \neq x'$ in $\mathrm{dom}(u) \implies u(x) \cap u(x') = \emptyset$.

Let $\langle A_\beta, \mathcal{R}_\beta, u_\beta\rangle_{\beta<\mathfrak{c}}$ be a list of all control triples mentioning each $\mathfrak{c}$ many times and where $A_\beta \subset \beta \times \omega$.

Suppose $\xi < 2^\mathfrak{c}$ and for every $\eta < \xi$ we have already decided whether $\eta \in H$ and, if so, what B^0_η, B^1_η are. We now decide if $\xi \in H$ and, if so, show how to define B^0_ξ, B^1_ξ.

Case 1. Suppose U^0_ξ, U^1_ξ are ξ-open and there is no $\eta < \xi$ such that $\langle U^0_\eta, U^1_\eta\rangle = \langle U^0_\xi, U^1_\xi\rangle$ and U^0_η, U^1_η are η-open. Then $\xi \in H$ and we need to define B^0_ξ, B^1_ξ.

Suppose $\beta < \mathfrak{c}$ and for every $\alpha < \beta, \forall k \in \omega$, we already decided $\langle\alpha, k\rangle(\xi)$ equals the unique $i \in \{0, 1\}$ with $\langle\alpha, k\rangle \in B^i_\xi$.

Subcase 1.1. If $S_\xi \restriction A_\beta \in \mathcal{R}_\beta$, then take the biggest $m \leq \omega$ such that there exists $i \in \{0, 1\}$ with $\{\beta\} \times m \subset U^i_\xi$. Fix an i with $\{\beta\} \times m \subset U^i_\xi$ and make sure $\{\beta\} \times m \subset B^i_\xi$. If this $m < \omega$, then, for $\omega > k \geq m$, pick any $i \in \{0, 1\}$ such that $\langle\beta, k\rangle \in U^i_\xi$ and set $\langle\beta, k\rangle \in B^i_\xi$.

Subcase 1.2. Suppose there exists $x \in \mathrm{dom}(u_\beta)$ with $S_\xi \restriction A_\beta \in u_\beta(x)$. Note that $S_\xi \restriction A_\beta \notin \mathcal{R}_\beta$ by (C_1) and there is only one such x by (C_2). Since $x \in A_\beta \subset \beta \times \omega$, $x(\xi)$ has been defined. For every $j \in \omega$, set $\langle\beta, j\rangle(\xi) = x(\xi)$ unless $\langle\beta, j\rangle \notin U^{x(\xi)}_\xi$ in which case put up with $\langle\beta, j\rangle \in B^{1-x(\xi)}_\xi$.

Subcase 1.3. Neither Subcase 1.1 nor Subcase 1.2. Then for every $j \in \omega$ pick one $i \in \{0, 1\}$ with $\langle\beta, j\rangle \in U^i_\xi$ and set $\langle\beta, j\rangle \in B^i_\xi$.

Case 2. Not Case 1. Then $\xi \notin H$ and B^0_ξ, B^1_ξ need not be defined.

The space $X = \mathfrak{c} \times \omega$ with

$$\mathcal{B} = \cup_{\xi<2^\mathfrak{c}} \mathcal{B}_\xi = \mathcal{B}_0 \cup \{B^0_\xi, B^1_\xi : \xi \in H\}$$

as a subbase for the topology is normal by construction. We need to prove that X is **not** countably paracompact.

2. Complete neighborhoods

Each basic open neighborhood for $x = \langle\alpha, k\rangle \in X$ is of the form

$$V_{t,K}(x) = \bigcap_{\xi \in t} B_\xi^{x(\xi)} \cap (\mathbb{W}_{k+1} \setminus K)$$

for some choice of $t \in [H]^{<\omega}$, $K \in [X \setminus \{x\}]^{<\omega}$.

Let us call $V_{t,K}(x)$ *complete* if for every $\xi \in t$,

$$V_{t \cap \xi, K}(x) \subset U_\xi^{x(\xi)}.$$

Lemma 2.1. *For every neighborhood $V_{t,K}(x)$, there are $t^* \supset t$, $K^* \supset K$ such that $V_{t^*,K^*}(x)$ is a complete neighborhood.*

Proof: For every incomplete neighborhood $V_{t',K'}(x)$ let $\xi_{t',K'}$ be the largest $\xi \in t'$ with

$$V_{t' \cap \xi, K'}(x) \not\subset U_\xi^{x(\xi)}.$$

CLAIM. For every incomplete neighborhood $V_{t,K}(x)$ there is a neighborhood $V_{t',K'}(x) \subset V_{t,K}(x)$ such that either $V_{t',K'}(x)$ is complete or $\xi_{t',K'} < \xi_{t,K}$.

Proof of Claim: Let $\eta = \xi_{t,K}$. Since $U_\eta^{x(\eta)}$ is η-open there exist $\overline{t} \in [H \cap \eta]^{<\omega}$, $\overline{K} \in [X]^{<\omega}$ with $V_{\overline{t},\overline{K}}(x) \subset U_\eta^{x(\eta)}$. Then $t' = t \cup \overline{t}$, $K' = K \cup \overline{K}$ is as required.

The proof of the Lemma now follows because otherwise we could construct an infinite decreasing sequence: $\xi_{t,K} > \xi_{t_1,K_1} > \xi_{t_2,K_2} > \cdots$. □

3. HOMOGENEITY: FINDING AND REFLECTING β

For $\beta \in \mathfrak{c}$, define $C_\beta = \{\beta\} \times \omega$. We say β is *ξ-homogeneous* if there exists $i \in \{0,1\}$ such that $C_\beta \subset B_\xi^i$.

Main Lemma 3.1. *If H' is any countable subset of H then there is some $\beta \in \mathfrak{c}$ such that β is ξ-homogeneous for every $\xi \in H'$.*

The proof of the Main Lemma will follow after the proofs of Lemma 3.2 (which "finds" β) and the Reflection Lemma 3.3. Express $H' = \{\xi_n : n \in \omega\}$. For every $x = \langle \beta, m \rangle \in X$ let $V(x) = V_{t(x),K(x)}(x)$ be a basic neighborhood such that

(T_0) $\{\xi_j : j \leq m\} \subset t(x)$, and thus $V(x) \subset \bigcap_{j \le m} B_{\xi_j}^{x(\xi_j)}$. (Recall the definition of the basic open neighborhood.) By passing to a smaller neighborhood, if needed, we will assume

(T_1) $j < m < \omega \implies t(\beta, j) \subset t(\beta, m)$;
(T_2) $V_{t(x),K(x)}(x)$ is a complete neighborhood.

Consider $t : X \to [H]^{<\omega}$ and $K : X \to [X]^{<\omega}$ to be the functions defined above for the conditions (T_0), (T_1), (T_2). Let

$$\mathfrak{c}, \langle S_\xi \rangle_{\xi < 2^{\mathfrak{c}}}, \langle \langle B_\xi^0, B_\xi^1 \rangle \rangle_{\xi \in H}, H, t, K, \langle \xi_m \rangle_{m \in \omega} \in M \in N \prec H((2^{2^{\mathfrak{c}}})^+)$$

(where M, N are countable elementary submodels and $H((2^{2^{\mathfrak{c}}})^+)$ is the collection of all sets having transitive closure of cardinality $\le 2^{2^{\mathfrak{c}}}$).

Let $A = N \cap X$ $(= (N \cap \mathfrak{c}) \times \omega)$ and $\mathcal{R} = \{S_\xi \restriction A : \xi \in M \cap H\}$. The next lemma will give a function u to complete a special control triple $\langle A, \mathcal{R}, u \rangle$.

Lemma 3.2. *There exists a countable function u with $\mathrm{dom}(u) \subset A$ satisfying (C_1) and (C_2) in the definition of a control triple such that whenever $v : X \to [H \setminus M]^{<\omega}$ is an infinite partial function, $v \in N$ and $x \ne x'$ in $\mathrm{dom}(v)$ implies $v(x) \cap v(x') = \emptyset$, then there are infinitely many $x \in \mathrm{dom}(v) \cap \mathrm{dom}(u)$ such that*

$$u(x) = \{S_\xi \restriction A : \xi \in v(x)\}.$$

Proof: Let $\langle v_j \rangle_{j \in \omega}$ enumerate all functions $v \in N$, as in the lemma, each listed infinitely many times. By induction on j pick distinct $\{x_j : j \in \omega\} \subset N \cap X$ such that $j < m < \omega$ implies $v_j(x_j) \cap v_m(x_m) = \emptyset$. Define

$$u : \{x_i : i \in \omega\} \to [\mathcal{S}_A \setminus \mathcal{R}]^{<\omega} : u(x_i) = \{S_\xi \restriction A : \xi \in v_i(x_i)\}.$$

For (C_1) to hold we need $u(x_j) \cap \mathcal{R} = \emptyset$ for all $j \in \omega$.

Assume indirectly that there exists $\xi \in v(x_j)$ and $\eta \in M \cap H$ such that $S_\xi \restriction A = S_\eta \restriction A$. Now, $\eta \in N$ since $\eta \in M$. To see $\xi \in N$ note that $v_j, x_j \in N$ implies $v_j(x_j) \in N$. Since $v_j(x_j)$ is finite, $\xi \in v_j(x_j) \subset N$. Since $\xi, \eta \in N$ and $S_\xi \restriction A = S_\eta \restriction A$ it follows that $S_\xi = S_\eta$. Since $\xi, \eta \in H$ it follows from minimality of Case 1 that $\xi = \eta$. Then $\xi \in v_j(x_i) \cap (M \cap H) = \emptyset$, a contradiction.

For (C_2), suppose indirectly that there exists $x_i \ne x_j$ in $\mathrm{dom}(u)$ with some $S_\sigma \restriction A \in u(x_i) \cap u(x_j)$. There would be $\xi \in v(x_i)$ and

$\eta \in v(x_j)$ such that $S_\xi \restriction A = S_\sigma \restriction A = S_\eta \restriction A$. As in the previous paragraph, $\xi, \eta \in H$ and $S_\xi \restriction A = S_\eta \restriction A$ gives $S_\xi = S_\eta$ and $\xi = \eta$. This contradicts $v(x_i) \cap v(x_j) = \emptyset$. □

Now, fix a u as in Lemma 3.2, and fix $\beta < \mathfrak{c}$ with $\beta \notin N$ and $\langle A, \mathcal{R}, u\rangle = \langle A_\beta, \mathcal{R}_\beta, u_\beta\rangle$. Notice that $\pi_1(A) \subset \beta$ since $A_\beta \subset \beta \times \omega$.

Reflection Lemma 3.3. *Let $k \in \omega$. Then there exists $x = \langle \alpha, k\rangle \in \mathrm{dom}(u)$ such that*

(R_1) $t(x) \cap M = t(\beta, k) \cap M$;
(R_2) $\forall \xi \in t(\beta, k) \cap M,\ \langle \beta, k\rangle(\xi) = x(\xi)$;
(R_3) $u(x) = \{S_\xi \restriction A : \xi \in t(x) \smallsetminus M\}$.

Proof: Let $r = t(\beta, k) \cap M \in M$, $\mathrm{dom}(f) = r$, $f(\xi) = \langle \beta, k\rangle(\xi)$. Let $\phi(\alpha)$ be the statement:

"$t(\alpha, k) \supset r$ and, $\forall \xi \in r,\ \langle \alpha, k\rangle(\xi) = f(\xi)$."

All parameters of $\phi(\alpha)$ are in M and $\phi(\beta)$ is true. Let E be maximal so that $\phi(E)$ holds and such that $t(\alpha, k) \smallsetminus r, \alpha \in E$ are pairwise disjoint. Since all parameters of $\phi(\alpha)$ are from M we may assume $E \in M$. Suppose indirectly that E is countable. Then $E \subset M$. If $E' = E \cup \{\beta\}$ then $E \subsetneq E'$. Now, $\phi(E')$ holds because $\phi(\beta)$ clearly holds and, for all $\alpha \in E$, $(t(\alpha, k) \smallsetminus r) \cap (t(\beta, k) \smallsetminus r) \subset M \cap (t(\beta, k) \smallsetminus M) = \emptyset$. This contradicts the maximality of E so E, in fact, must be uncountable.

Let $v(x) = t(\alpha, k) \smallsetminus r$. The argument above shows there are uncountably many $\alpha \in \mathfrak{c}$ with $\phi(\alpha)$ true such that the sets $v(x) = t(\alpha, k) \smallsetminus r$ are pairwise disjoint. Hence, there exist uncountably many with $v(x) \cap M = \emptyset$.

Let $\psi(v)$ stand for:

"v is an infinite function, $\mathrm{dom}(v) \subset \mathfrak{c} \times \{k\}$, $\forall \langle \alpha, k\rangle \in \mathrm{dom}(v), \phi(\alpha)$ is true and $v(\alpha, k) = t(\alpha, k) \smallsetminus M$, and $x \neq x'$ in $\mathrm{dom}(v)$ implies $v(x) \cap v(x') = \emptyset$."

All parameters of $\psi(v)$ are in N so there exists such $v \in N$. By Lemma 3.2, there exists $x \in \mathrm{dom}(v) \cap \mathrm{dom}(u)$ satisfying (R_3). The argument above shows that (R_1) and (R_2) are true. □

Conclusion of the Proof of the Main Lemma: Continue with the $\beta < \mathfrak{c}$ as given after the proof of 3.2. Aiming for a contradiction to the Main Lemma let $\theta \in H' \cap M = H'$ be the smallest counterexample for which β is not θ-homogeneous. Notice that earlier condition (T_1) insures that it is possible to pick large enough $k \in \omega$ such that for $y[k] = \{\langle \beta, j \rangle : j \leq k\}$,

(1_k) $y[k] \cap B^0_\theta \neq \emptyset$ and $y[k] \cap B^1_\theta \neq \emptyset$,

and

(2_k) $\theta \in t(\beta, k)$.

Let $x = \langle \alpha, k \rangle$ be as in the Reflection Lemma. Notice that $x \in N$ implies $K(x) \in N$, and $K(x)$ is finite so $K(x) \subset N$. Since $\beta \notin N$ it must be true that $C_\beta \cap K(x) \subset C_\beta \cap N = \emptyset$. So, $y[k] \cap K(x) = \emptyset$. This is needed to help with the following claim.

Claim. $y[k] \subset V_{t(x)\cap\theta, K(x)}(x)$.

Proof of Claim: Let $\xi \in t(x) \cap \theta$ and suppose for all $\eta \in t(x) \cap \xi$ we have

(I_η) $y[k] \subset B^{x(\eta)}_\eta$.

We'll show

(I_ξ) $y[k] \subset B^{x(\xi)}_\xi$.

Certainly, by completeness and the fact that $y[k] \cap K(x) = \emptyset$,

(3_k) $y[k] \subset V_{t(x)\cap\xi, K(x)} \subset U^{x(\xi)}_\xi$.

Case (a). $\xi \in M \cap t(x)$. By minimality of θ it follows that there exists $i \in \{0, 1\}$ such that $C_\beta \subset B^i_\xi$. By (R_1), $\xi \in t(\beta, k) \cap M$. By (R_2), this $i = \langle \beta, k \rangle(\xi) = x(\xi)$. So $y[k] \subset C_\beta \subset B^{x(\xi)}_\xi$.

Case (b). $\xi \in t(x) \setminus M$. Then by Subcase 1.2 and (3_k), $\langle \beta, j \rangle(\xi) = x(\xi)$ for every $j \leq k$. That is, $y[k] \subset B^{x(\xi)}_\xi$.

Now, (I_ξ) true for all $\xi \in t(x) \cap \theta$ implies that the claim is true.

From condition (R_1), of the Reflection Lemma, $\theta \in t(x)$. Now, by the Claim above and the fact that $V_{t(x),K(x)}(x)$ is a complete neighborhood we see that $y[k] \subset U^{x(\theta)}_\theta$. Hence, by Subcase 1.1, there exists $i \in \{0, 1\}$ such that $y[k] \subset B^{x(\theta)}_\theta$ which contradicts (1_k). That concludes the proof of the Main Lemma. □

4. X is not countably paracompact

To finish the proof that X is not countably paracompact let us indirectly take closed sets $Z_m \subset \mathbb{W}_m$ such that $\cup_{m\in\omega} Z_m = X$. For every $m \in \omega$, let ξ_m be the unique element of H such that $\langle U^0_{\xi_m}, U^1_{\xi_m} \rangle = \langle \mathbb{W}_m, X \setminus Z_m \rangle$ (uniqueness by Case 1).

Proposition 4.1. *For every $\beta \in \mathfrak{c}$ there exists $m \in \omega$ such that β is not ξ_m-homogeneous.*

Proof: Since $\cup_{n\in\omega} Z_n = X$, we can pick m with $Z_m \cap C_\beta \neq \emptyset$. For such m, $C_\beta \not\subset B^1_{\xi_m} \subset \mathbb{W}_m$ and $C_\beta \not\subset B^1_{\xi_m} \subset X \smallsetminus Z_m$. □

Using $H' = \{\xi_n : n \in \omega\}$, the previous proposition contradicts the Main Lemma and concludes our argument that X is not countably paracompact.

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