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ON DENSITY AND THE NUMBER OF G_δ-POINTS IN SOMEWHAT LINDELÖF SPACES

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ABSTRACT. We use the κ-pseudocharacter to bound density and the number of G_κ-points in somewhat Lindelöf T_1-spaces. We also extend a recent result of A. Arhangel'skii and R. Buzyakova on the cardinality of linearly Lindelöf spaces.

1. INTRODUCTION

The goal of this paper is to present two cardinality theorems. The first, Theorem 2.3, says that if X is a T_1-space with $F(X) \leq \kappa$ (i.e., without free sequences of length κ^+) and for every $S \in [X]^{\leq\kappa}$, $\overline{S}$ is a G_{2^κ}-set, then both the density and the number of G_κ-points in X are bounded by 2^κ. This shortens, somewhat, the proof of the second result (Theorem 3.2; see also Corollary 3.7) which extends and somewhat improves of a recent result of A. Arhangel'skii and R. Buzyakova [1] and [2] on the cardinality of linearly Lindelöf spaces. It also follows that in Arhangel'skii's classical theorem that if X is Lindelöf and $\psi(X)t(X) = \omega$, then $|X| \leq 2^\omega$, one can weaken "Lindelöf" to "initially 2^ω-linearly Lindelöf." It is an open problem whether "initially ω_1-Lindelöf" would suffice in ZFC.

Our terminology and notation follow the standards of set-theoretic topology as used in [5]. In particular, $[X]^{\leq\kappa} = \{A \subset$

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$X : |A| \leq \kappa\}$. We also make the convention that $^-$ denotes closure in the space X, whatever space X is in the proof.

2. Bounds by the κ-pseudocharacter

Definition 2.1. Let $\kappa \geq 1$ be a cardinal. The κ-pseudocharacter $\psi_\kappa(X)$ of a T_1-space X is the smallest infinite cardinal λ such that for every $S \in [X]^{\leq\kappa}$, $\overline{S}$ is a G_λ-set (i.e., $\overline{S}$ is the intersection of $\leq \lambda$ open sets).

Remark 2.2. The pseudocharacter $\psi(X)$ is the same as the 1-pseudocharacter $\psi_1(X)$. Clearly, $\psi(X) \leq \psi_\kappa(X)$ for every $\kappa \geq 1$. For example, in the Michael Line M, $\psi(M) = \omega < \psi_\omega(M)$, since the rationals form a non-G_δ closed separable set in M.

Recall that a sequence $\langle x_\alpha \rangle_{\alpha<\lambda}$ of points is a **free sequence** if for every $\alpha < \lambda$, $\overline{\{x_\beta : \beta < \alpha\}} \bigcap \overline{\{x_\beta : \beta \geq \alpha\}} = \phi$. The notation $F(X) \leq \kappa$ means that X has no free sequences of length κ^+.

Theorem 2.3. *Let κ be an infinite cardinal. Suppose that X is a T_1-space satisfying $F(X) \leq \kappa$ and $\psi_\kappa(X) \leq 2^\kappa$. Then*
(a) $d(X) \leq 2^\kappa$;
(b) X has $\leq 2^\kappa -$ many $G_\kappa -$ points.

Proof: Let M be an elementary submodel of $H(\theta)$ for a "big enough" θ such that M is closed under κ-sequences (i.e., $M^\kappa \subset M$), $|M| = 2^\kappa \subset M$ and $2^\kappa, X$, and the family $\mathcal{T}$ of all open subsets of X are elements of M. Set $Y = X \cap M$. □

Claim 1. If $S \in [Y]^{\leq\kappa}$ and $x \notin \overline{S}$, then there is a $U \in \mathcal{T} \cap M$ such that $\overline{S} \subset U \subset X \setminus \{x\}$. To prove Claim 1, note first that by $\psi_\kappa(X) \leq 2^\kappa$ there is a family $\mathcal{U} \in [\mathcal{T}]^{\leq 2^\kappa}$ with $\overline{S} = \cap\mathcal{U}$. Since $X, \mathcal{T} \in M$ and $S \in M^\kappa \subset M$, it follows that $\overline{S} \in M$. As $2^\kappa \in M$, we may assume that $\mathcal{U} \in M$. By $|\mathcal{U}| \leq 2^\kappa$ it follows that $\mathcal{U} \subset M$. Thus, picking a $U \in \mathcal{U}$ with $x \notin U$ is as required.

Claim 2. If $Z \subset Y$ and $x \notin \overline{Z}$, then there is a $\mathcal{W} \in [\mathcal{T} \cap M]^{\leq\kappa}$ such that $Z \subset \cup\mathcal{W} \subset X \setminus \{x\}$. To prove Claim 2, suppose indirectly that there is no such $\mathcal{W}$. Then we can build a sequence $\langle z_\alpha, U_\alpha \rangle_{\alpha<\kappa^+}$ such that for every $\alpha < \kappa^+$,
(i) $\overline{\{z_\beta : \beta < \alpha\}} \subset U_\alpha \in \mathcal{T} \cap M$ and $x \notin U_\alpha$;
(ii) $z_\alpha \in Z \setminus \bigcup_{\beta\leq\alpha} U_\beta$.

(i) is possible by Claim 1. (ii) is possible by our indirect assumption of the nonexistence of $\mathcal{W}$.

Since for every $\alpha < \kappa^+$, $\overline{\{z_\beta : \beta < \alpha\}} \cap \overline{\{z_\beta : \beta \geq \alpha\}} \subset U_\alpha \cap (X \setminus U_\alpha) = \phi$, it follows that $\langle z_\alpha \rangle_\alpha < \kappa^+$ is a free sequence, contradicting $F(X) \leq \kappa$. Having proved Claim 2, we can turn to the proofs of (a) and (b) in Theorem 2.3.

To prove (a), suppose indirectly that $\overline{Y} \neq X$ and pick a point $x \in X \setminus \overline{Y}$. Let $Z = Y$ and take $\mathcal{W}$ as in Claim 1. By $M^\kappa \subset M$ it follows that $\mathcal{W} \in M$. Since $Y = X \cap M \subset \cup\mathcal{W}, M \models X \subset \cup\mathcal{W}$. By elementarity, $V \models X \subset \cup\mathcal{W}$, contradicting $x \notin \cup\mathcal{W}$.

Now assume indirectly that (b) fails. Then there is a G_κ-point $x \in X \setminus Y$. Let $\langle V_\alpha \rangle_{\alpha<\kappa}$ be open sets with $\{x\} = \cap_{\alpha<\kappa} V_\alpha$. Apply Claim 2 to $Z_\alpha = Y \setminus V_\alpha$ to get a $\mathcal{W}_\alpha \in [\tau \cap M]^{\leq\kappa}$ with $Z_\alpha \subset \cup\mathcal{W}_\alpha \subset X \setminus \{x\}$. Then $\mathcal{W} = \bigcup_{\alpha<\kappa} \mathcal{W}_\alpha \in M$, $\mathcal{W}$ covers $X \cap M$, but does not cover X, contradicting elementarity as in the previous paragraph.

A space is called initially ω_1-Lindelöf if every open cover of size ω_1 has a countable subcover.

Corollary 2.4. *Suppose that X is an initially ω_1-Lindelöf T_1-space with $t(X) = \omega$ and $\psi_\omega(X) \leq 2^\omega$. Then*

(a) $d(X) \leq 2^\omega$;

(b) X has $\leq 2^\omega$ many G_δ-points.

Proof: Since every subset of size ω_1 in an initially ω_1-Lindelöf space has a complete accumulation point, by $t(X) = \omega$ it follows that $F(X) = \omega$. Thus, Theorem 2.3 is applicable. □

Example 2.5. V. Fedorcuk [3] constructed a hereditarily separable compact T_2-space X with $|X| = 2^{2^\omega}$ from the combinatorial principle $\diamond$. By hereditarily separability, $t(X) = \omega$ and $\psi_\omega(X) \leq 2^\omega$. This example thus shows that we can't conclude $|X| \leq 2^\omega$ in Corollary 2.4 even in compact T_2-spaces.

Corollary 2.6. *Suppose that X is an initially ω_1-Lindelöf T_1-space with $\psi(X)t(X) = \omega$ and $\psi_\omega(X) \leq 2^\omega$. Then $|X| \leq 2^\omega$.*

Corollary 2.7. (A. Gryzlov [4]). *Suppose that X is a first countable, Lindelöf T_1-space with $\psi_\omega(X) \leq 2^\omega$. Then $|X| \leq 2^\omega$.*

Corollary 2.8. (Arhangel'skii, see [5, p. 19]). *Suppose that X is a Lindelöf T_2-space with $\psi(X)t(X) = \omega$. Then $|X| \leq 2^\omega$.*

Proof: In Lindelöf Hausdorff spaces, $\psi(X) \cdot t(X) = \omega$ implies $\psi_\omega(X) \leq 2^\omega$. $\square$

3. On a result of Arhangel'skii and Buzyakova

All spaces in this section are Tychonoff.

Definition 3.1. Let $\lambda > \kappa \leq \omega$ be cardinals. We will say that a space X is $[\kappa, \lambda]$-linearly Lindelöf, if every cover of X by open sets increasing in well-order type $\leq \lambda$ has a subcover of cardinality $\leq \kappa$.

The following is the main result of this section. After its proof, we'll list some corollaries and discuss their relationship to the main results in [1], [2].

Theorem 3.2. *Let κ be an infinite cardinal. Suppose that X is a $[\kappa, 2^\kappa]$-linearly Lindelöf (Tychonoff) space such that $t(X) \leq \kappa, \psi(X) \leq 2^\kappa$ and $(*)|\overline{S}| \leq 2^\kappa$ for every $S \in [X]^{\leq\kappa}$. Then $|X| \leq 2^\kappa$.*

For the proof of Theorem 3.2, we will need several lemmas. The first two of these lemmas are proved in [2] for $\kappa = \omega$, and the extension to higher cardinals is straightforward. We will write out a proof of Lemma 3.4 which we believe to be shorter here than in [2].

Lemma 3.3. ([2], in essence). *Let κ be an infinite cardinal. Suppose $X \subset X^*$ are spaces, $F(X) \leq \kappa, A \subset X$, and $x \in X^* \setminus X$. Then either (a) there is a closed G_κ-set $F \ni x$ of X^* with $A \cap F = \emptyset$; or (b) there is a $S \in [A]^{\leq\kappa}$ with $x \in cl_{X^*}(S)$.*

Lemma 3.4. *Let κ be an infinite cardinal. Suppose that X is a $[\kappa, 2^\kappa]$-linearly Lindelöf space, $|X| \leq 2^\kappa, F(X) \leq \kappa, X \subset X^*$, and $w(X^*) \leq 2^\kappa$. Then $X^* \setminus X$ is the union of $\leq 2^\kappa$ closed subsets of X^*.*

Proof: Let $\mathcal{B}$ be a base of X^* with $|\mathcal{B}| \leq 2^\kappa$. Let $\mathcal{H}$ denote the set of all closed subsets H of X^* such that H is the intersection of $\leq \kappa$ many members of $\mathcal{B}$ or of $\leq \kappa$ many members of $\{cl_{X^*}(S) : S \in [X]^{\leq\kappa}\}$. Clearly, $|\mathcal{H}| \leq 2^\kappa$, and we'll show that $\cup\{H \in \mathcal{H} : H \subset X^* \setminus X\} = X^* \setminus X$. To prove this, let $x \in X^* \setminus X$. Let λ be the smallest infinite cardinal such that there is a family $\mathcal{U}$ of λ many open subsets of X^* with $X \subset \cup\mathcal{U}$ and $x \notin \overline{U}$ for every $U \in \mathcal{U}\cdot$

If $\lambda \leq \kappa$, then there is an $H \in \mathcal{H}$ with $x \in H \subset X^* \setminus X$ that is the intersection of $\leq \kappa$ many members of $\mathcal{B}$. So suppose $\lambda > \kappa$. Since X is $[\kappa, 2^\kappa]$-linearly Lindelöf and $w(X^*) \leq 2^\kappa$, it follows that $cf(\lambda) \leq \kappa$. Thus, there is an increasing sequence $\langle \mathcal{U}_\alpha \rangle_{\alpha<\kappa}$ such that $\mathcal{U} = \cup_{\alpha<\kappa} \mathcal{U}_\alpha$ and $|\mathcal{U}_\alpha| < \lambda$. Let $A_\alpha = X \setminus \cup \mathcal{U}_\alpha$. By $\lambda > \kappa$ and the minimality of λ, there is a $\delta < \kappa$ such that for every α with $\delta < \alpha < \kappa$, there is no closed G_κ-set $F \ni x$ of X^* with $F \cap A_\alpha = \emptyset$. Thus, for every α with $\delta < \alpha < \kappa$, Lemma 3.3 gives an $S_\alpha \in [A_\alpha]^{\leq\kappa}$ such that $x \in cl_{X^*}(S_\alpha)$. Since $cl_{X^*}(S_\alpha) \cap (\cup \mathcal{U}_\alpha) = \emptyset, x \in H = \cap_{\delta<\alpha<\kappa} cl_{X^*}(S_\alpha) \subset X^* \setminus X$ and $H \in \mathcal{H}$. □

Lemma 3.5. *Let κ be an infinite cardinal. Suppose that $X \subset X^*$ are spaces, $F(X) \leq \kappa$, and $C \subset X^* \setminus X$ is a compact subspace. Then there is a closed G_{2^κ}-subset F of X^* such that $C \subset F \subset X^* \setminus X$.*

Proof: Suppose indirectly that there is no such F. Then by induction we can build a sequence $\langle z_\alpha, F_\alpha \rangle_{\alpha<\kappa^+}$ such that for every $\alpha < \kappa^+$,

(i) $F_\alpha \subset \cap_{\beta<\alpha} F_\beta$ is a closed G_{2^κ}-subset of X^* with $F_\alpha \supset C$ and $\overline{\{z_\beta : \beta < \alpha\}} \cap F_\alpha = \emptyset$;

(ii) $z_\alpha \in F_\alpha \cap X$.

(i) is possible, because the weight of $Z_\alpha = \overline{\{z_\beta : \beta < \alpha\}}$ is $\leq 2^\kappa$ and thus, Z_α can be covered $\leq 2^\kappa$ many open subsets U of X^* such that $\overline{U} \cap C = \emptyset$. (ii) is possible by our indirect assumption. Note that for every $\alpha < \kappa^+$, $\overline{\{z_\beta : \beta < \alpha\}} \cap \overline{\{z_\beta : \beta \leq \alpha\}} \subset \overline{\{z_\beta : \beta < \alpha\}} \cap F_\alpha = \emptyset$. Thus, $\langle z_\alpha \rangle_{\alpha<\kappa^+}$ is a free sequence contradicting $F(X) \leq \kappa$. □

Proof of Theorem 3.2. By $t(X) \leq \kappa$ and $(*)$, it is enough to prove that $d(X) \leq 2^\kappa$. Since X is $[\kappa, 2^\kappa]$-linearly Lindelöf, every subset of size κ^+ of X has a complete accumulation point. Since $t(X) \leq \kappa$, it follows that $F(X) \leq \kappa$. Thus, by Theorem 2.3, it is enough to pick an arbitrary $S \in [X]^{\leq\kappa}$ and prove that $\overline{S}$ is a G_{2^κ}-set.

To do so, consider the subspace $X^* = cl_{\beta X}(S) \cup X$ of βX. We are done if we can show that $cl_{\beta X}(S)$ is a G_{2^κ}-set in X^*. Since $cl_{\beta X}(S)$ is compact, it is enough to show that $cl_{\beta X}(S)$ is the union of $\leq 2^\kappa$ many G_{2^κ}-sets in X^*. For $cl_{\beta X}(\overline{S}) \setminus \overline{S} = X^* \setminus X$ this follows by lemmas 3.4 and 3.5. For $\overline{S}$, this follows from $|\overline{S}| \leq 2^\kappa$. (Note by $w(\beta(\overline{S})) \leq 2^\kappa$ and $\psi(X) \leq 2^\kappa, \psi(X^*) \leq 2^\kappa$.)

For the next corollary, let's call an $[\omega, 2^\omega]$-linearly Lindelöf space initially 2^ω-linearly Lindelöf.

Corollary 3.6. *Suppose that X is initially 2^ω-linearly Lindelöf, $t(X) = \omega, \psi(X) \leq 2^\omega$ and $(*)|\overline{S}| \leq 2^\omega$ for every $S \in [X]^{\leq\omega}$. Then $|X| \leq 2^\omega$.*

Corollary 3.7. *Suppose that X is initially 2^ω-linearly Lindelöf, sequential, and $\psi(X) \leq 2^\omega$. Then $|X| \leq 2^\omega$.*

Remark 3.8. Corollary 3.7 is an improvement on the main result in [2] in that it weakens "linearly Lindelöf" to "initially 2^ω-linearly Lindelöf." Indeed, the conclusion of Corollary 3.7, $|X| \leq 2^\omega$, says that there are no strictly increasing open covers of length $(2^\omega)^+$ in X. Yet, to prove $|X| \leq 2^\omega$, [2] makes use of the assumption that if there were any such covers, then they would contain a countable subcover. The need for this assumption is eliminated by our Lemma 3.5.

Corollary 3.9. *Suppose that X is initially 2^ω-linearly Lindelöf and $\psi(X) \cdot t(X) = \omega$. Then $|X| \leq 2^\omega$.*

Proof: (*) follows because a (regular T_1) separable space X with $\psi(X) = \omega$ has cardinality $\leq 2^\omega$. □

Question 3.10. Is it true in ZFC that a first countable initially ω_1-Lindelöf space has cardinality $\leq 2^\omega$? (By Corollary 3.9, this is true if CH holds.)

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