Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 27, No. 1, 2003
Pages 15–25

EQUIVALENCE OF STAR-PRODUCTS ON SYMPLECTIC MANIFOLDS

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Abstract. Besides properties related to the Hochschild cohomology of a symplectic manifold model in analytical dynamics with applications in quantum theory, this paper also shows the equivalence of two differential star-products, more specifically, that every differential star-product of two functions u and v on a symplectic manifold is equivalent to one whose linear term is half of the Poisson bracket of these functions, i.e., $\frac{1}{2}\{u, v\}$.

1. Introduction

Star-products were introduced in [2] to consider a particular deformation of the space $C^\infty(M)$ of smooth functions on a symplectic manifold equipped with its double structure of associative algebra according to the usual pointwise multiplication of functions and the Poisson Lie algebra structure for a new approach of quantum mechanics. That is, a star-product is a formal deformation of these two algebraic structures. J. Vey [7] proved the existence of such deformations assuming that the third De Rham cohomology group of the manifold vanishes. Then, in 1983, M. De Wilde and P. B. A.

2000 *Mathematics Subject Classification.* 53D05, 53D55, 57R19.

Key words and phrases. coboundary operator, cochain annihilating constants, differential equaivalent star-products, Hochschild cohomology, symplectic manifolds.

Financial assistance was provided to the first author by the Postgraduate Funding Service of the University of Cape Town.

Lecomte [4] proved the existence of a star-product on a symplectic manifold. In this paper, we will only be concerned with differential star-products, that is, star-products defined by a series of bidifferential operators on a symplectic manifold. A particular type of such star-products is defined.

2. Preliminaries

Definition 2.1. A *symplectic structure* on a manifold M is a closed and nondegenerate 2-form ω on M. The pair (M, ω) is called symplectic manifold.

Definition 2.2. Let (M, ω) be a symplectic manifold. A vector field X is said to be *locally Hamiltonian* if $i_X\omega$ is an exact 1-form on M, where i is the interior product on M.

Notation. We shall denote by X_u the unique Hamiltonian vector field such that $i_{X_u}\omega = du$ for $u \in C^\infty(M)$.

Remarks:

(1) The space of symplectic vector fields modulo the space of Hamiltonian vector fields is isomorphic to the space of closed 1-forms modulo the space of exact 1-forms. That is, it is isomorphic to the first group of De Rham cohomology of M denoted by $H^1(M, \mathbb{R})$.
(2) It is a consequence of the Poincare Lemma that every symplectic vector field is locally Hamiltonian.

Definition 2.3. Let (M, ω) be a symplectic manifold, and let u and v be two smooth functions on M. The *Poisson bracket* of u and v, denoted by $\{u, v\}$, is defined by

$$\{u, v\} = X_u(v) = \omega(X_v, X_u).$$

Note that the Poisson bracket of u and v can also be defined in terms of the Lie derivative as follows:

$$\{u, v\} = -L_{X_u}(v) = L_{X_v}(u) = -i_{X_u} \circ i_{X_v}(\omega).$$

Defined in this way, the Poisson bracket can be viewed as a derivation on the space $C^\infty(M)$ of smooth functions on the manifold M. The vector space $C^\infty(M)$ equipped with the Poisson bracket is a

Lie algebra. The Poisson tensor $\bigwedge$ is a 2-alternative vector field such that:

$$\{u, v\} = i\bigwedge(\mathrm{du} \wedge \mathrm{dv})$$

whose local co-ordinates $\wedge^{ij}$ define a square matrix $(\wedge^{ij})$ so that the inverse of this matrix is (ω_{ij}), where ω_{ij} are the components of the symplectic 2-form ω. In what follows, we consider formal deformations of the associative structure of the algebra $N = (C^\infty(M), \{,\})$, that is, the deformations defined on the space $N[\nu]$ of formal series in the formal parameter ν with coefficients in N. The linear (or multilinear) map T on $N[\nu]$ is formal if $N[\nu] = \{f_\nu / f_\nu = \sum_{k=0}^{\infty} \nu^k f_k;\ f_k \in N\}$, and the linear (or multilinear) map T on $N[\nu]$ is formal if $T(\nu f_\nu) = \nu T(f_\nu)$ (satisfied by each argument in the multilinear case).

3. Star-product on a symplectic manifold (M, ω)

Unless otherwise indicated, the manifold that we consider will be assumed paracompact. Furthermore, to set the following definition, we consider on $C^\infty(M)$ the functions C_r (r a natural number) defined by $C_r : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$ such that

(1) C_r is a differential operator in both variables and annihilates constants;
(2) $C_0(u, v) = u \cdot v$ where $\cdot$ is the pointwise multiplication of functions;
(3) $C_1(u, v) - C_1(v, u) \equiv \{u, v\}$ is the Poisson bracket on (M, ω);
(4) $C_k(u, v) = (-1)^k C_k(v, u)$ for $k \geq 2$;
(5) $C_k(1, u) = C_k(u, 1) = 0$.

Definition 3.1. A *star product* (*$\star$-product*) on a symplectic manifold (M, ω) is a formal deformation M_ν defined on $N[\nu]$ such that for all u and v in N,

$$M_\nu(u, v) = u \star v = u \star_\nu v = \sum_{r \geq 0} \nu^r C_r(u, v)$$

where C_r is defined as above and the constant function 1 is the unity. A $\star$-product is said to be *of order* k if $u \star_\nu v = \sum_{r=0}^{k} C_r(u, v)$.

The star-commutator on a symplectic manifold (M, ω) is defined in terms of a star-product as follows

$$[u, v]_\star = u \star v - v \star u.$$

This is closely related to the commutator of observables in quantum theory. The symplectic manifold (M, ω) is said to admit a deformation quantization provided a star-product always exists on M.

Note that the set $N[\nu]$ is a Lie algebra whose adjoint representation is given by

$$ad_\star(u)(v) = [u, v]_\star \ .$$

From the definition of the star-product, we have

$$[u, v]_\star = \nu\{u, v\} - 2\sum_{r=1}^{\infty} C_{2r+1}(u, v)\nu^{2r+1} + ...$$

Definition 3.2. Two star-products, $\star$ and $\star'$, on the symplectic manifold (M, ω) are equivalent if there exists a series

$$T = \sum_{r=0}^{\infty} \nu^r T_r$$

where for each natural number r, T_r is a linear operator on N, $T_0 = id_N$ is the identity function on N, and T is a linear bijection on $N[\nu]$ which satisfies

$$T(u \star v) = T(u) \star' T(v).$$

The study of a star-product on its star-product algebra $N[\nu] = C^\infty(M)[\nu]$ modulo this equivalence relation requires M. Gerstenhaber's deformation theory [5] which is based on the Hochschild cohomology of the algebra N.

Definition 3.3. A Hochschild p-*cochain* on the commutative algebra N is the p-linear map from $N \times N \times ... \times N$ (p copies of N) into N, and a Hochschild coboundary operator on the algebra N is a map ∂ satisfying the following property:
for all p-cochains C on N,

$$\begin{aligned}(\partial C)(u_0, ..., u_p) \ &= \ u_0 \star_\nu C(u_1, ..., u_p) + \\ &\sum_{r=1}^{p}(-1)^r C(u_0, ..., u_{r-1} \star_\nu u_r, ..., u_p) + \\ &(-1)^{p+1} C(u_0, ..., u_{p-1}) \star_\nu u_p \ .\end{aligned}$$

Remark: In this paper, p-cochains, p-coboundaries, and the coboundary operator are related to Hochschild cohomology.

Example. A 1 or 2-coboundary ∂ is given by

$$(\partial C_1)(u,v) = uC_1(v) - C_1(u \cdot v) + C_1(u) \cdot v$$

$$(\partial C_2)(u,v,w) = \\ uC_2(v,w) - C_2(u \cdot v, w) + C_2(u, v \cdot w) - C_2(u,v) \cdot w$$

where C_1 and C_2 are cochains.

Definition 3.4. A p-cochain C is said to be a *p-cocycle* if $\partial C = 0$, and it is said to be a coboundary if $C = \partial B$ for some $(p-1)$-cochain B.

Definition 3.5. A p-cochain C is said to be

(1) differential if its variables are all differential operators;
(2) k-differential if each variable is a k-differential operator.

A p-cochain is said to annihilate constants if it vanishes for any constant among its variables.

It can be seen immediately that annihilating differential 1-cochains are cocycles, and that 1-cocycles are derivations on $N = C^\infty(M)$. In fact, from

$$0 = (\partial C)(u,v) = uC(v) - C(u \cdot v) + C(u) \cdot v,$$

one deduces that

$$C(u \cdot v) = uC(v) + C(u) \cdot v,$$

so that if the 1-cocycles define vector fields on M, then they define the 1-differential annihilating cochains.

Definition 3.6. The differential p-group of Hochschild cohomology of N is the quotient of the space of differential p-cocycles by the space of differential p-coboundaries.

From a p-cochain C and a q-cochain D, a $(p+q)$-cochain is defined by

$$(C \otimes D)(u_1, ..., u_p, u_{p+1}, ..., u_{p+q}) = C(u_1, ..., u_p) + D(u_{p+1}, ..., u_{p+q})$$

and the coboundary operator is a graded derivation, that is,

$$\partial(C \otimes D) = (\partial C) \otimes D + (-1)^p C \otimes (\partial D).$$

When the cochain C is expressed in terms of local co-ordinates, its support $suppC$ is the union of supports of coefficients.

Proposition 3.1. *If C is a differential p-cocycle on $C^{\infty}(\mathbb{R}^n)$, then there exists a $(p-1)$-cochain B and a differential p-cocycle A such that $C = \partial B + A$. If C annihilates constants, then A and B can be chosen such that they also annihilate constants and their supports are contained in the support of C.*

Proof: Let $p = 1$. Then every 1-cocycle is a vector field and so, the proposition holds trivially.

Now let us suppose that this result holds for a differential r-cocycle $C(u_1, ..., u_r)$ with $1 \geq r \leq (p-1)$ and u_1 an operator of order k $(k > 1)$. There exists a coboundary of first order as we can let

$$C(u_1, ..., u_p) = \sum_{i_1,...,i_k} \frac{\partial^k u_1}{\partial x_{i_1} ... \partial x_{i_k}} D_{i_1...i_k}(u_2, ..., u_p) + ...$$

where $D_{i_1...i_k}$ are the symmetric cochains with respect to $i_1, ..., i_k$; or, using the multi-index notation, it follows that

$$C = \sum_{|i|=k} \partial_i \otimes D_i + ...$$

Thus, we can use the identities (1) and (2) above and the fact that $\partial^2 = 0$ to get

$$\partial C = -\sum_{|i|=k} \partial_i \otimes \partial D_i + ...$$

That is, if C is a p-cocycle, the coefficients of higher degree derivatives with respect to u_s $(s = 1, ..., k)$ are $(p-1)$-cocycles. By induction hypothesis, $D_i = \partial E_i + F_i$, where F_i are 1-differential and $suppF_i \subset suppC$ for $1 \leq i \leq p-1$. If

$$G = \sum_{|i|=k} \partial_i \otimes E_i + F_i \circ (\partial_i \otimes Id_{p-2}),$$

then a short calculation leads to

$$\partial G = -\sum_{|i|=k} \partial_i \otimes D_i + ...$$

where ... stands for the terms in which the derivative of the first variable is of order less than k. Hence, the order of $C + \partial G$ is at most $k - 1$ with respect to the first argument. By iteration, this order can be reduced to 1 with respect to the first argument. Suppose that

$$C = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \otimes D_i \ .$$

Then,

$$\partial C = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \otimes \partial D_i,$$

and C is a cocycle if and only if D_i are cocycles, i.e.,

$$D_i = \partial E_i + F_i$$

where F_i are 1-differential and

$$C + \partial \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \otimes E_i = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \otimes F_i.$$

□

Proposition 3.2. *If C is a 1-differential p-cochain on $\mathbb{R}^n$ whose alternating part is A, then*

$$C = \partial B + A$$

where B is 2-differential and is determined by C, and $suppB \subset suppC$.

Proof: Let C be a 1-differential p-cochain annihilating constants. Then C has the following form:

$$C(u_1, ..., u_p) = \sum_{i_1, i_2, ..., i_p} C_{i_1 i_2 ... i_p} \frac{\partial u_1}{\partial x_{i_1}} \cdots \frac{\partial u_p}{\partial x_{i_p}},$$

where the coefficients $C_{i_1 i_2 ... i_p}$ are given by

$$C_{i_1 ... i_p} = C(x_{i_1}, ..., x_{i_p}).$$

For a permutation σ on p elements $\{1, ..., p\}$, let

$$(\sigma C)(u_1, ..., u_p) = C(u_{\sigma^{-1}(1)}, ..., u_{\sigma^{-1}(p)})$$

define an action of the group $\sigma \in \mathfrak{S}_p$ of all permutations of p elements on the p-cochains. For each permutation $\sigma \in \mathfrak{S}_p$, we define a 2-differential $(p-1)$-cochain $\phi_\sigma(C)$ by

$$\partial \phi_\sigma(C) := C - \epsilon(\sigma)\sigma \cdot C$$

where $\epsilon(\sigma)$ is the signature for $\sigma \in \mathfrak{S}_p$. In particular, if τ is a transposition of consecutive integers, and if we consider a fixed i such that $i \leq p-1$, the $(p-1)$-cochain $\phi_\tau(C)$

$$\phi_\tau(C)(u_1, ..., u_{p-1}) =$$

$$(-1)^i \sum_{r,s} C(u_1, ..., u_{i-1}, x_r, x_s, u_{i+1}, ..., u_{p-1}) \frac{\partial^2 u_i}{\partial x_r \partial x_s}$$

satisfies

$$\partial \phi_\tau(C) = C + \tau \cdot C.$$

Then, we make two transpositions, τ_1 and τ_2, of two consecutive integers and we have

$$\partial[\phi_{\tau_1}(\tau_2 C) - \phi_{\tau_2}(C)] = C - \tau_1 \tau_2 C.$$

For each $\sigma \in \mathfrak{S}_p$, we define a 2-differential $(p-1)$-cochain $\phi_\sigma(C)$ by letting

$$\partial \phi_\sigma(C) := C - \epsilon(\sigma)\sigma \cdot C$$

where $\epsilon(\sigma)$ is the signature of σ. Now, let

$$\phi(C) = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \phi_\sigma(C).$$

Then,

$$C = \partial \phi(C) + \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \epsilon(\sigma)\sigma \cdot C$$

up to $p!$. Clearly, C is cohomologous to its antisymmetric part and that $supp\phi(C) \subset suppC$. □

Theorem 3.1. *A 1-differential p-cocycle C on a manifold M is the sum of a coboundary of a differential $(p-1)$-cochain B and a 1-differential skew-symmetric p-cocycle, A, i.e.,*

$$C = \partial B + A.$$

If C annihilates constants, then B can also be chosen to annihilate constants.

Proof: Let $\{U_\lambda\}_{\lambda\in\bigwedge}$ be a locally finite covering subordinated to a partition of unity ρ_λ of the manifold M. Then, a p-cocycle C can be written as a locally finite sum of p-cocycles

$$C = \sum_{\lambda\in\bigwedge} \rho_\lambda C_\lambda.$$

From propositions 3.1 and 3.2 above, we can write

$$\rho_\lambda C_\lambda = \partial B_\lambda + A_\lambda$$

where $suppB_\lambda \subset U_\lambda$. Let $B = \sum_{\lambda\in\bigwedge} B_\lambda$ and $A = \sum_{\lambda\in\bigwedge} A_\lambda$, then $C = \partial B + A$, where A and B are locally finite, and are globally defined. □

Remark: Note that in the symplectic case, the skew-symmetric 1-differential p-cocycle A can be written in terms of Hamiltonian vector fields as $A(u_1, ..., u_p) = \alpha(X_{u_1}, ..., X_{u_p})$ where α is therefore a one p-form and

$$C(u_1, ..., u_p) = (\partial B)(u_1, ..., u_p) + \alpha(X_{u_1}, ..., X_{u_p}).$$

Definition 3.7. A star-product, $\star$, on a symplectic manifold (M, ω) is said to be differential if the 2-cochains $C_\tau(u, v)$ defining it are bi-differential operators.

Definition 3.8. Two star-products, $\star$ and $\star'$, on a symplectic manifold (M, ω) are said to be differentially equivalent if there exists a series

$$T = \sum_{r=0}^{\infty} \nu^r T_r$$

where T_r are differential operators on $N = C^\infty(M)$ and such that

$$T(u \star v) = T(u) \star' T(v).$$

Note that star-products that are differentially equivalent are equivalent. The following theorem asserts that when star-products are differential, they are differentially equivalent if and only if they are equivalent.

Theorem 3.2. *Let $\star$ and $\star'$ be two differential star-products and let*

$$T = \sum_{r\geq 0} \nu^r T_r,$$

where $T_0 = id$ *be an equivalence with* $T(u \star v) = T(u) \star' T(v)$*, then* T_r *are differential operators on* N*.*

Proof: Let us suppose that the first k operators T_1, ..., T_k in T are differential and define $T' = \sum_{r=0}^{k} \nu^r T_r$ and $T'' = {T'}^{-1} \circ T$. It is obvious that the form of T'' is

$$T''(u) = u + \nu^{k+1} T''_{k+1}(u) + ...$$

Let us define a star-product $\star''$ such that

$$u \star'' v = T^{-1}(T'(u) \star' T'(v))$$

and $\star$ satisfies the following relation

$$u \star v = {T'}^{-1}(T''(u) \star'' T''(v)).$$

Then, $T'' = {T'}^{-1} \circ T$ is an equivalence between two differential star-products, $\star$ and $\star''$. If we consider only the (k+1)-th degree's terms in $u \star v = {T'}^{-1}(T''(u) \star'' T''(v))$, it follows that

$$(\partial T''_{k+1})(u, v) = T''_{k+1}(u)v + uT''_{k+1}(v) - T''_{k+1}(u \cdot v)$$

is a 2-cocycle bidifferential and symmetric. Thus, from Theorem 3.1 above, $\partial T''_{k+1}$ is the coboundary of a differential skew-symmetric 1-cochain plus a skew-symmetric differential 1-cocycle. The exact terms are symmetric so that the skew-symmetric part is equal to zero. Thus, there exists a differential 1-cochain B such that

$$\partial(T''_{k+1} - B) = 0.$$

Hence, $X = T''_{k+1} - B$ is a vector field on M, that is a derivation on $N = C^\infty(M)$. Thus, $T''_{k+1} = B + X$ is differential. Now T_{k+1} is a linear combination of T_1,...,T_k and T''_{k+1}, which are differential, and T_{k+1} is also. By induction, one concludes that T is differential. □

The next proposition is an immediate application of Theorem 3.1 above.

Proposition 3.3. *Every star-product is equivalent to a star-product such that its linear term in* ν *is given by* $\frac{1}{2}\{u, v\}$*.*

Proof: Let

$$u \star v = uv + \nu C_1(u, v) + ...$$

be a star-product on a symplectic manifold (M, ω); then, we need to show that $C_1(u, v)$ is a Hochschild cocycle such that its skew-symmetric part is $\frac{1}{2}\{u, v\}$. That is, from Theorem 3.1, we should have

$$C_1(u, v) = uB(v) - B(u \cdot v) + B(u) \cdot v + \tfrac{1}{2}\{u, v\}$$

where B is a differential 1-cochain. Let $T(u) = u + \nu B(u)$ and $u \star' v = T(T^{-1}(u) \star T^{-1}(v)$. Then

$$u \star' v = uv + \tfrac{1}{2}\nu\{u, v\} + \dots .$$

Thus, T is a differential equivalence and so, $\star'$ is a differential star-product. □

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