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**SELECTED PROBLEMS IN CONTINUUM THEORY**

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**ABSTRACT.** Some open problems in metric continuum theory are discussed. They concern 1) generalized homogeneity, in particular monotone homogeneity of dendrites; 2) planability of curves; 3) means on continua; 4) contractibility of continua; and 5) selectibility of continua.

## INTRODUCTION

A *continuum* means a compact connected metric space. The aim of this paper is to present a number of problems pertinent to continuum theory, especially connected with continuous mappings of continua. The background and the motivation for these problems come from the author's investigations in continuum theory. The formulation of these problems is restricted to the area of metric continua, although many of them can be posed for wider classes of spaces (e.g., for Hausdorff continua or for metric connected – not necessarily compact – spaces). Special attention is paid to topology of curves (i.e., one-dimensional continua), to their structural as well as mapping properties. For other collections of continuum theory problems see historically the first such set [36], and also [35], [59], [60] and [72].

The reader is referred to [71] for concepts not defined here.

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## 1. GENERALIZED HOMOGENEITY

A space  $X$  is said to be *homogeneous* provided that for every two points  $p$  and  $q$  in  $X$  there is a homeomorphism  $h : X \rightarrow X$  such that  $h(p) = q$ . Homogeneity of topological spaces, in particular of continua, has received much attention from specialists since the 1920s (see e.g. [26, Section 8, p. 733]). Many homogeneity problems are recalled in survey articles by W. Lewis [62], [63], [64]. Since they are widely known, there is no need to repeat them here.

The concept of homogeneity was generalized in many ways, see [27]. One of these concepts, due to D. P. Bellamy, is the following. Let a class  $\mathcal{M}$  of mappings between continua be given such that it contains all homeomorphisms and has the composition property, i.e., the composition of any two mappings belonging to  $\mathcal{M}$  is also in  $\mathcal{M}$ . A space  $X$  is said to be *homogeneous with respect to  $\mathcal{M}$*  (more concisely  *$\mathcal{M}$ -homogeneous*) provided that for every two points  $p$  and  $q$  in  $X$  there is a mapping  $f : X \rightarrow X$  such that  $f(p) = q$  and  $f \in \mathcal{M}$ . If, in particular,  $\mathcal{M}$  denotes the class of homeomorphisms, then  $X$  is homogeneous in the previously considered sense. A general problem, which can be considered as a research program rather than a particular question, is to verify what results concerning homogeneity can be strengthened so that the usual concept of homogeneity is replaced by homogeneity with respect to a wider (thus less restrictive than homeomorphisms) class of mappings. Below we recall several particular questions of this kind.

## 1.A. CIRCLE-LIKE CONTINUA

Some results on characterizations of solenoids in the realm of circle-like continua via generalized homogeneity are collected in [13, Theorem, p. 173]. The following question remains open [13, Question 2, p. 173].

**Question 1.1.** Let a circle-like open-homogeneous continuum that is not  $S^1$  contain an arc. Is it then a solenoid?

In connection with circle-like continua, recall that in 1951, R. H. Bing constructed a hereditarily indecomposable circle-like continuum called the *pseudo-circle* (see [9, p. 48] for the definition) and asked about its homogeneity. The pseudo-circle was shown to be unique [39] and not homogeneous [40] and [79]. Thus, natural

questions arise (see [15, Problem 7, p. 5] and [25, problems 1 and 2, p. 10]), which are (as far as I know) still open.

**Question 1.2.** Is the pseudo-circle a) open-homogeneous, b) confluent-homogeneous?

1.B. MENGER'S UNIVERSAL CONTINUA AND URYSOHN'S CONTINUA

Given a collection  $\mathcal{C}$  of subsets of a space, we let  $\mathcal{C}^*$  denote the union of all elements of  $\mathcal{C}$ . For a positive integer  $n$ , let  $\mathcal{C}$  be a collection of  $n$ -dimensional cubes in the Euclidean  $n$ -space. For each  $k \in \{0, \dots, n\}$ , denote by  $\mathcal{C}^{(k)}$  the collection of all  $k$ -dimensional faces of cubes from  $\mathcal{C}$ , and let  $\text{par } \mathcal{C}$  stand for the collection of  $n$ -dimensional cubes which is obtained by partitioning each cube from  $\mathcal{C}$  into  $3^n$  congruent cubes. Let  $k \in \{0, \dots, n\}$  be fixed. Take the collection  $\mathcal{C}_{0,k} = \{[0, 1]^n\}$  consisting only of the unit  $n$ -cube  $[0, 1]^n$ , and define inductively, for each positive integer  $i$ , a collection  $\mathcal{C}_{i,k}$  by the formula

$$\mathcal{C}_{i,k} = \{C \in \text{par } \mathcal{C}_{i-1,k} : C \cap (\mathcal{C}_{i-1,k}^{(k)})^* \neq \emptyset\}.$$

Then the *Menger universal continua*  $M_k^n$  are defined by

$$M_k^n = \bigcap \{(\mathcal{C}_{i,k})^* : i \in \{0, 1, 2, \dots\}\}.$$

They were characterized in [8].

It can be shown that for each  $n \in \mathbb{N}$  the set  $M_0^n$  is homeomorphic to the Cantor middle-third set, so it is homogeneous as a topological group.  $M_1^2$  is the Sierpiński universal plane curve which is not homogeneous as a locally connected plane continuum distinct from the simple closed curve. However, it is known to be homogeneous with respect to the class of mappings which are open and monotone simultaneously; thus, it is both open- and monotone-homogeneous, (see [78, Theorem 23, p. 38] and compare also [80]).  $M_1^3$  is the *Menger universal curve*, which is homogeneous, (see [1] and [2]). Further, it is shown in [8] that the continua  $M_n^{2n+1}$  are homogeneous for each  $n \in \{0, 1, 2, \dots\}$ , whence it follows that all  $M_n^m$  are for  $m \geq 2n+1$  (note that for  $m \geq 2n+1$  all  $M_n^m$  are topologically stabilized). Finally, so called intermediate Menger compacta  $M_n^m$  for each  $n \in \{1, 2, \dots\}$  and  $m \in \{1, 2, \dots, 2n\}$  are not homogeneous [61]. In

connection with the latter result, recall the following problem (see [25, Problem 3, p. 11] and [27, Problem 4.17, p. 76]).

**Problem 1.3.** For what classes  $\mathfrak{M}$  of mappings are the intermediate Menger compacta  $M_n^m$  for each  $n \in \mathbb{N}$  and  $m \in \{1, 2, \dots, 2n\}$   $\mathfrak{M}$ -homogeneous?

Constructions are known, due to P. S. Urysohn, of locally connected plane curves,  $X(\omega)$  and  $X(\aleph_0)$  having the property that each of their points is of Menger-Urysohn order  $\omega$  or  $\aleph_0$ , respectively (see [22]). These constructions are not unique, so we can consider some classes or types of curves of constructions in a prescribed way rather than any particular examples. Obviously, no such curve is homogeneous (because the only plane homogeneous locally connected continuum is the simple closed curve [68]), while each of them is homogeneous with respect to the class of all mappings [52, Theorem 1, p. 347]. Homogeneity of these curves with respect to other classes of mappings (as open, monotone, etc.) is not known. So, we have the following question.

**Question 1.4.** Can the above mentioned curves  $X(\omega)$  and/or  $X(\aleph_0)$  be constructed so that the resulting continuum is a) open-, b) monotone-, c) confluent-homogeneous?

This question is a particular case of a more general problem concerning locally connected continua.

**Problem 1.5.** Characterize locally connected continua that are a) open-, b) monotone-, c) confluent-homogeneous.

#### 1.C. DENDROIDS AND DENDRITES

A continuum  $X$  is said to have the *property of Kelley* provided that for each point  $p \in X$ , for each subcontinuum  $K$  of  $X$  containing  $p$ , and for each sequence of points  $p_n$  converging to  $p$ , there exists a sequence of subcontinua  $K_n$  of  $X$  containing  $p_n$  and converging to the continuum  $K$ . It is known [14, Statement, p. 380] that each open-homogeneous continuum has the property of Kelley. This result cannot be extended to confluent-homogeneous continua. Answering the author's question, H. Kato has constructed in [48] two examples of continua (one contractible and 2-dimensional, and the other 1-dimensional) which are confluent-homogeneous and which

do not have the property of Kelley. Even monotone homogeneity does not imply the property of Kelley. A. Illanes has constructed a plane continuum which is monotone-homogeneous and which does not have the property of Kelley [44], as well as a *dendroid*, i.e., an arcwise connected and hereditarily unicoherent continuum, with the same properties (unpublished). Since each *dendrite* (i.e., a locally connected continuum containing no simple closed curve) has the property of Kelley, the following problem is natural.

**Problem 1.6.** Characterize monotone-homogeneous dendroids that have the property of Kelley.

It is shown in [48, Example 2.4, p. 59] and [49, Proposition 2.4, p. 223] that the standard universal dendrite  $D_3$  of order 3 is monotone-homogeneous. After some generalizations, (see [19, Theorem 7.1, p. 186] and [24, Theorem 3.3, p. 292 and Corollary 3.8, p. 293]), the strongest result in this direction says that if a dendrite  $X$  has the set of its ramification points  $R(X)$  dense in  $X$ , then  $X$  is monotone-homogeneous [28, Proposition 15, p. 364]. The converse is not true and, moreover, it can be seen that the condition  $\text{cl } R(X) = X$  is far from being necessary for a dendrite  $X$  to be monotone-homogeneous. Namely, recall the *Omiljanowski dendrite*  $L_0$  which is constructed as follows (see [19, Example 6.9, p. 182] and also [28, p. 365]).

Let  $L_1$  be the unit interval in the plane. Divide it into three equal parts, and in the middle of them,  $M$ , locate a thrice diminished copy of the standard Cantor ternary set. At the mid point of each contiguous interval  $K$  to  $C$  (i.e., of a component  $K$  of  $M \setminus C$ ), erect perpendicularly to  $L_1$  a straight line segment whose length equals the length of  $K$ . Denote by  $L_2$  the union of  $L_1$  and of all erected segments (there are countably many of them). Perform the same construction on each of the added segments: divide such a segment into three equal subsegments, locate in the middle subsegment  $M$  a copy  $C$  of the Cantor set properly diminished, and at the mid point of any component  $K$  of  $M \setminus C$ , construct a perpendicular to  $K$  segment as long as  $K$  is, and denote by  $L_3$  the union of  $L_2$  and of all attached segments. Continuing in this manner we get an increasing sequence of dendrites  $L_n$ . Then

$$L_0 = \text{cl} \left( \bigcup \{L_n : n \in \mathbb{N}\} \right).$$

Note that the set  $R(L_0)$  of ramification points of  $L_0$  is discrete (thus nowhere dense in  $L_0$ ). For the proof of monotone homogeneity of  $L_0$  see [19, Example 6.9, p. 182]. It is shown in [28, Proposition 20, p. 366] that if a dendrite contains a homeomorphic copy of  $L_0$ , then it is monotone-homogeneous. The converse is not known, and the following two questions are still open (see [19, Question 7.2, p. 186], and [28, Question 21, p. 366]).

**Question 1.7.** Does every monotone-homogeneous dendrite contain a homeomorphic copy of the dendrite  $L_0$  (equivalently, does it admit any monotone mapping onto  $D_3$ )?

**Question 1.8.** What is an internal (structural) characterization of monotone-homogeneous dendrites?

The reader is referred to Section 3 of [29] for a summary of known results on monotone-homogeneous dendrites.

A larger class of mappings than that of monotone ones is the class of confluent mappings. For dendrites, monotone homogeneity and confluent homogeneity are equivalent, but it is not known if this equivalence is valid for wider classes of continua, e.g., for (smooth) dendroids [28, p. 363 ff].

## 2. PLANABILITY OF DENDROIDS

At the end of the 1950s, B. Knaster formulated (and published in [51]) the following problem.

**Problem 2.1.** Give necessary and sufficient (structural) conditions under which a dendroid can be embedded in the plane.

Very little is known about this. The general problem of planability of dendroids is discussed in [16] and in [23, Section 6, p. 25], where references to some partial answers can be found.

There are some partial results about preservation of planability of dendroids under mappings satisfying some special conditions. Monotone ones preserve (even for  $\lambda$ -dendroids, i.e. hereditarily uncoherent and hereditarily decomposable continua) while confluent do not, see [23, Section 6, (6.4) and (6.5), pp. 27-28]. The following question of T. Maćkowiak is still open, see [65, §5, p. 266].

**Question 2.2.** Is an open image of a planable dendroid always a planable dendroid?

We remark that open mappings do not preserve planability for graphs [83, Example, p. 189].

### 3. MEANS ON CONTINUA

Given a Hausdorff space  $X$ , a *mean* on  $X$  is a mapping  $m : X \times X \rightarrow X$  such that  $m(x, x) = x$  and  $m(x, y) = m(y, x)$  for every  $x, y \in X$ . Means on continua have been studied by many authors. Some basic information is contained in [70, p. 285], [46, p. 374], [21], and [30, Section 5] and in references given there.

#### 3.A. GENERALITIES

The basic problem related to the considered topic is to know what spaces admit a mean (see [70, Theorem 6.17 and Question 6.17.1, p. 285]).

**Problem 3.1.** Find any structural (intrinsic) characterization of spaces (of continua) that admit a mean.

A mean  $m : X \times X \rightarrow X$  is said to be *associative* provided that

$$m(x, m(y, z)) = m(m(x, y), z) \quad \text{for every } x, y, z \in X.$$

A dendroid  $X$  is said to be *smooth* provided that there is a point  $p \in X$  such that for each point  $x \in X$  and for each sequence of points  $x_n$  tending to  $x$ , the arc  $px$  is the limit of the sequence of the arcs  $px_n$ .

Among many other results concerning means, the following ones are known.

- (3.2) If a continuum admits a mean, then it is unicoherent [4, Theorem 1.1, p. 211]. In particular, the circle  $S^1$  does not admit any mean (compare [70, 0.71.1, p. 50]).
- (3.3) A one-dimensional locally connected continuum admits a mean if and only if it is a dendrite [81, p. 85] (compare also [30, Theorem 5.31, p. 22]).
- (3.4) Let a continuum  $X$  be either one-dimensional or hereditarily unicoherent. If  $X$  admits an associative mean, then  $X$  is a smooth dendroid [30, Theorem 5.21, p. 20].
- (3.5) Each plane smooth dendroid admits a mean [31, Theorem 6.6, p. 497]. Since each smooth fan is embeddable in the Cantor fan (i.e., the cone over the Cantor ternary set), it is planable, whence it follows that each smooth fan admits a

mean [30, Corollary 5.41, p. 24] (also [6, Proposition 4.2, p. 43]). There is a (nonplanable) smooth dendroid that admits no mean [30, Example 5.52, p. 25] (also [46, Example 76.12, p. 376, and Exercise 76.33, p. 380]). There exists a non-smooth fan that admits a mean (see [6, Example 4.8, p. 45]; for a picture, see [46, Figure 30, p. 195]).

(3.6) The  $\sin(1/x)$ -curve admits no mean [3].

In connection with (3.2)-(3.5) the following problems are of a special interest [30, Problems 5.50 and 5.56, p. 25 and 28].

**Problem 3.7.** Find any structural (intrinsic) characterization (a) of dendroids and (b) of smooth dendroids that admit a mean.

A continuum  $X$  is said to be *uniformly arcwise connected* provided that it is arcwise connected and that for each  $\varepsilon > 0$  there is a  $k \in \mathbb{N}$  such that every arc in  $X$  contains  $k$  points that cut it into subarcs of diameters less than  $\varepsilon$ . By [54, Theorem 3.5, p. 322], each uniformly arcwise connected continuum is a continuous image of the Cantor fan (but not conversely; the equivalence holds for uniquely arcwise connected continua, in particular for dendroids [54, p. 316]).

As mentioned in (3.5), smoothness of dendroids neither implies nor is implied by admitting a mean. Smoothness of dendroids is an intermediate property between the property of Kelley and uniform arcwise connectedness in the sense that for dendroids the property of Kelley implies smoothness [38, Corollary 5, p. 730], which in turn implies uniform arcwise connectedness [32, Corollary 16, p. 318]. It is interesting to know if admitting a mean also is such an intermediate property. In other words, we have the following two questions (see [30, questions 5.48 and 5.49, p. 25]).

**Question 3.8.** Let a dendroid have the property of Kelley. Must it then admit any mean?

Note that for  $\lambda$ -dendroids the property of Kelley does not imply existence of a mean, by (3.6).

**Question 3.9.** Let a dendroid admit a mean. Must it then be uniformly arcwise connected?

## 3.B. NO-MEAN RESULTS AND QUESTIONS

Concerning other classes of continua, not necessarily arcwise connected curves, attention was paid to arc-like continua. Let us recall that the simplest arc-like continuum, that is an arc, admits a mean according to (3.3). Moreover, the arc is the only *known* arc-like continuum admitting a mean. More than thirty years ago, P. Bacon, in [3, p. 13], asked if it is the only possible one:

**Question 3.10.** Is the arc the only arc-like continuum that admits a mean?

The question is still open. Some partial results are known, from which we conclude that in many particular cases arc-like continua distinct from an arc do not admit any mean. For example, it is shown in [3, Theorem, p. 11] that there is no mean on the  $\sin 1/x$ -curve. This result is extended in [6, Theorem 3.5, p. 42] by showing that if a continuum  $X$  contains an arc  $A$  and two sequences of arcs that are folded in opposite directions with respect to  $A$ , and moreover, if these sequences converge to  $A$  0-regularly, then  $X$  admits no mean. A further extension is made in [50]. To present it, and also for other purposes, recall some auxiliary notions.

Chains of (compact) disks in the plane  $\mathbb{R}^2$  (called *links* of the chain) will be considered. For a fixed  $\varepsilon > 0$ , a chain  $\mathcal{C}$  is called an  $\varepsilon$ -chain provided that each link of  $\mathcal{C}$  has diameter less than  $\varepsilon$ .

Let chains  $\mathcal{C}$  and  $\mathcal{C}'$  be given. We say that the chain  $\mathcal{C}'$  *refines* the chain  $\mathcal{C}$  if each link  $L'$  of the chain  $\mathcal{C}'$  is contained in the interior of at least one link  $L$  of the chain  $\mathcal{C}$ ; i.e., for every  $L' \in \mathcal{C}'$  there exists a link  $L \in \mathcal{C}$  such that  $L' \subset L$  and the disk  $L'$  has no common points with the boundary of the disk  $L$ .

Let  $m$  and  $m'$  be numbers of links of the chains  $\mathcal{C}$  and  $\mathcal{C}'$ , respectively, such that  $\mathcal{C}'$  refines  $\mathcal{C}$ , and let  $\varphi : \{1, 2, \dots, m'\} \rightarrow \{1, 2, \dots, m\}$  be a function defined by

$$\varphi(i) = \min\{j : L_i \subset L'_j \text{ for } L_i \in \mathcal{C} \text{ and } L'_j \in \mathcal{C}'\}.$$

Consider chains  $\mathcal{C}$  and  $\mathcal{C}'$  and points  $a$  and  $b$ , such that  $a \in L'_u \subset L_s$  and  $b \in L'_v \subset L_t$ , where  $L'_u, L'_v \in \mathcal{C}'$  and  $L_s, L_t \in \mathcal{C}$ . We say that  $\mathcal{C}'$  is *straight in  $\mathcal{C}$  between points  $a$  and  $b$*  provided that

- (a)  $\mathcal{C}'$  refines  $\mathcal{C}$ ;
- (b)  $|s - t| > 2$ ;

- (c) the partial function  $\varphi\{u, \dots, v\} : \{u, \dots, v\} \rightarrow \{s, \dots, t\}$  is either nondecreasing or nonincreasing.

Let  $k \geq 2$  be an integer. Take a set  $S_k = \{e_1, \dots, e_k\}$  of  $k$  distinct points in the plane  $\mathbb{R}^2$ . For each positive integer  $n$  let  $\mathcal{C}_n$  be a chain, the union of all links of which (being a continuum) is denoted by  $K_n$ , such that, for each  $n \in \mathbb{N}$ , the following conditions are satisfied:

- (d)  $\mathcal{C}_n$  is a  $\frac{1}{2^n}$ -chain;  
 (e)  $\mathcal{C}_{n+1}$  refines  $\mathcal{C}_n$ ;  
 (f)  $S_k \subset K_n$ ;  
 (g) for every two indices  $i, j \in \{1, 2, \dots, k\}$  there exists an integer  $m \geq n$  such that the chain  $\mathcal{C}_m$  contains the point  $e_i$  in its first, and the point  $e_j$  in its last link.

Put

$$(3.11) \quad X^{(k)} = \bigcap \{K_n : n \in \mathbb{N}\}.$$

Thus,  $X^{(k)}$  is an arc-like continuum. It can be verified (see [55, §48, V, Example 3, Fig. 5, p. 205] for a picture and details in the case of  $k = 2$ ; and [71, 1.10, p. 8] and [42, p. 142] for details in the case of  $k = 3$ ; for  $k > 3$  the argument is similar) that the continuum  $X^{(k)}$  is irreducible between any pair of points of  $S_k$ , and thus indecomposable, and that the points of  $S_k$  are in distinct components of  $X^{(k)}$ .

A point  $p$  of an arc-like continuum is called an *end point* of the continuum provided that for each  $\varepsilon > 0$  there is an  $\varepsilon$ -chain covering the continuum such that only the first link of the chain contains  $p$ . By (g) it follows that each of the  $k$  points of  $S_k$  is an end point of  $X^{(k)}$ .

If we additionally assume that

- (3.12) for each  $n \in \mathbb{N}$  the chain  $\mathcal{C}_{n+1}$  is straight in  $\mathcal{C}_n$  between any two points  $e_i, e_j \in S_k$  such that if  $e_i \in L'_u$  and  $e_j \in L'_v$ , where  $L'_u, L'_v \in \mathcal{C}_{n+1}$ , then the union of all links of  $\mathcal{C}_{n+1}$  lying between  $L'_u$  and  $L'_v$  does not contain any other points of  $S_k$ ; i.e.,  $(L'_u \cup \dots \cup L'_v) \cap S_k = \{e_i, e_j\}$ ,

then it can be shown that each of the proper subcontinua of  $X^{(k)}$  is an arc. The continuum  $X^{(3)}$  can also be seen as the inverse limit of an inverse sequence of closed unit intervals  $X_n = [0, 1]$  with the

same piecewise linear bonding mappings  $f_n = f : [0, 1] \rightarrow [0, 1]$  determined by

$$f(0) = \frac{1}{2}, \quad f\left(\frac{1}{2}\right) = 1, \quad f(1) = 0$$

and being linear on  $[0, \frac{1}{2}]$  and  $[\frac{1}{2}, 1]$ . It is proved in [5, Theorem 8 and Remarks, p. 168] that each proper subcontinuum of  $X^{(3)}$  is an arc. But note that even with condition (3.12), the continuum  $X^{(k)}$  is not uniquely determined in general.

**Definition 3.13.** Let  $A$  be an arc-like subcontinuum of a continuum  $X$  and let  $a \in A$  be an end point of  $A$ . A sequence  $\{A_n : n \in \mathbb{N}\}$  of subcontinua of  $X$  is called a *folding sequence with respect to  $a$*  (see [50, Definition 2.1, p. 99]), provided that for each  $n \in \mathbb{N}$  there are subcontinua  $E_n$  and  $F_n$  of  $A_n$  such that

- (3.13.1)  $A_n = E_n \cup F_n$ ;
- (3.13.2)  $\text{Lim}E_n = \text{Lim}F_n = A$ ;
- (3.13.3)  $\text{Lim}(E_n \cap F_n) = \{a\}$ .

We say that a chain  $\mathcal{C} = \{D_1, \dots, D_m\}$  of compact disks  $D_i$  in  $\mathbb{R}^2$  is a chain from a point  $a$  to a point  $b$  provided that  $a \in \text{int}D_1 \setminus D_2$  and  $b \in \text{int}D_m \setminus D_{m-1}$ . The points  $a$  and  $b$  of an arc-like continuum  $A$  are called *opposite end points* of  $A$  if for each  $\varepsilon > 0$  there is an  $\varepsilon$ -chain from  $a$  to  $b$  covering  $A$  (see [10, p. 661]).

Observe that in the above construction (3.11) of the continuum  $X^{(k)}$  each two points of  $S_k$  are opposite end points of  $X^{(k)}$ .

The following result is shown in [50, Theorem 2.2, p. 99].

**Theorem 3.14.** *Let a continuum  $X$  be hereditarily unicoherent. If*

- (3.14.1) *there is an arc-like subcontinuum  $A \subset X$  with points  $a, b \in A$  as opposite end points of  $A$ , and there are sequences  $\{A_n\}$  and  $\{B_n\}$  of subcontinua of  $X$  which are folding sequences with respect to the points  $a$  and  $b$ , respectively,*

*then  $X$  admits no mean.*

**Corollary 3.15.** *Let  $k \geq 2$  be given. If the continuum  $X^{(k)}$  defined by (3.11) is such that*

- (3.15.1) *each proper subcontinuum of  $X^{(k)}$  is an arc,*

*then it admits no mean.*

*Proof:* Take  $A = X^{(k)}$ . Let  $C_1$  be the component of  $X$  containing the point  $e_1$ . Choose two sequences of points  $e_m^1, e_m^2 \in C_1$  such that

$$e_m^2 \in e_1 e_m^1 \subset C_1, \quad d(e_2, e_m^2) < \frac{1}{m}, \quad \text{and} \quad d(e_1, e_m^1) < \frac{1}{m}.$$

Then  $\lim e_m^1 = e_1$  and  $\lim e_m^2 = e_2$ . For each  $m \in \mathbb{N}$  define

$$E_m = e_1 e_m^2, \quad F_m = e_m^2 e_m^1, \quad A_m = E_m \cup F_m = e_1 e_m^1,$$

and note that (taking convergent sequences, if necessary) the limit continuum  $\text{Lim} E_m$  contains the points  $e_1$  and  $e_2$ , which are opposite end points of  $A = X^{(k)}$ . Since  $A$  is irreducible between  $e_1$  and  $e_2$ , it follows that  $\text{Lim} E_m = A$ . Similarly,  $\text{Lim} F_m = A$ . Further,

$$(3.15.2) \quad E_m \cap F_m = \{e_m^2\},$$

and therefore conditions (3.13.1)-(3.13.3) are satisfied. Hence,  $\{A_m\}$  is a folding sequence with respect to  $e_2$ . A folding sequence with respect to  $e_1$  can be defined in the same way. So, the corollary follows from Theorem 3.14.  $\square$

**Corollary 3.16.** *Let  $k \geq 2$  be given. If the continuum  $X^{(k)}$  defined by (3.11) satisfies condition (3.12), then it admits no mean.*

**Remark 3.17.** Condition (3.15.1) in Corollary 3.15 is needed to get (3.15.2) which implies (3.13.3).

**Question 3.18.** Is condition (3.15.1) essential in Corollary 3.15?

**Remark 3.19.** Answering the author's question [21, Question 20, p. 172], Illanes has shown in [45] that the simplest indecomposable continuum  $B$  with exactly one end point, (i.e., the bucket-handle continuum, see e.g., [71, 2.9, p. 22]), admits no mean. The following questions are related to this result.

**Question 3.20.** a) Let  $X$  be an arbitrary arc-like continuum with exactly one end point. Is it true that  $X$  admits no mean?

b) Let  $Y$  be the union of two copies of  $B$  with the end points identified. Then  $Y$  is an arc-like continuum without any end point. Is it true that  $Y$  admits no mean?

c) Let  $Z$  be an arbitrary arc-like continuum without any end point. Is it true that  $Z$  admits no mean?

Note that the arc, the  $\sin(1/x)$ -curve and the bucket-handle continuum  $B$  are all arc-like and each of them contains a one-to-one image of the real half line  $[0, \infty)$  as a dense subset. Of the three

examples, the arc is the only one that admits a mean, (see (3.6) and Remark 3.19). Thus, the following questions are natural.

**Question 3.21.** Is the arc the only arc-like continuum admitting a mean that contains (a) a one-to-one image of the real half line  $[0, \infty)$  as a dense subset? (b) a dense arc component?

**Remark 3.22.** Note that the assumption that the continuum under consideration in Question 3.21 is arc-like is essential and it cannot be replaced by “circle-like.” Indeed, the dyadic solenoid is circle-like, has dense arc components, and, contrary to  $\mathbb{S}^1$  (see (3.2) above), admits a mean (see [50, (3), p. 97], [30, Example 5.47, p. 24] and [46, Example 76.6, p. 374]). In general, any solenoid  $\Sigma$  different from a circle is determined by a sequence  $(n_1, n_2, \dots)$  of integers greater than 1 so that  $\Sigma(n_1, n_2, \dots) = \varprojlim \{S_k, f_k\}$ , where  $S_k = \mathbb{S}^1$  and  $f_k = S_{k+1} \rightarrow S_k$  is given by  $f_k(z) = z^{n_k}$ .

P. Krupski [53, Theorem 2] has shown the following result.

(3.22.1) *A solenoid  $\Sigma(n_1, n_2, \dots)$  admits a mean if and only if there are infinitely many even numbers in the sequence  $(n_1, n_2, \dots)$ .*

**Remark 3.23.** The concept of a folding sequence of continua with respect to a point and the criterion of the non-existence of a mean formulated in Theorem 3.14 are related to some other concepts similarly defined and used to show the non-existence of such properties as means, contractibility, selectibility, and some others. The mentioned concepts are:

(3.23.1) of containing a *zigzag*, see [74, p. 837] and [41, p. 78];

(3.23.2) to be of *type N*, see [74, p. 837] and [75, p. 393]; compare also the *property (\*)* in [37, Theorem, p. 121].

It follows from the definitions that if a continuum contains a zigzag, then it is of type N, which in turn implies condition (3.14.1). A common property of these three conditions is that the considered (limit) continuum  $A$  is arc-like, as in (3.14.1), or even an arc, as in (3.23.1) and (3.23.2). This restriction is connected with *methods* used in proofs (of the non-existence of a mean) rather than that of the concept of a mean itself.

A less restrictive condition is of being of generalized type N; (see [17, p. 96]), in which the arc  $A = xy$  (mentioned in the definition

of type N, see [74, p. 837]) is replaced by an arbitrary continuum. The definition runs as follows.

**Definition 3.24.** A continuum  $X$  is said to be of *generalized type N* (between points  $x$  and  $y$ ) provided that there exist in  $X$ : a continuum  $A$  containing the points  $x$  and  $y$ , two sequences of arcs  $x_n x'_n$  and  $y_n y'_n$ , and points  $x''_n \in y_n y'_n \setminus \{y_n, y'_n\}$  and  $y''_n \in x_n x'_n \setminus \{x_n, x'_n\}$ , such that

- (3.24.1)  $A = \text{Lim} x_n x'_n = \text{Lim} y_n y'_n$ ;
- (3.24.2)  $x = \lim x_n = \lim x'_n = \lim x''_n$ ;
- (3.24.3)  $y = \lim y_n = \lim y'_n = \lim y''_n$ .

Recall that a continuum  $X$  is said to be of *type N* (between points  $x$  and  $y$ ) provided that the continuum  $A$  in the above definition is an arc from  $x$  to  $y$ . If  $X$  is a dendroid, then  $A = xy$  is unique.

**Question 3.25.** Let a hereditarily unicoherent continuum  $X$  be of generalized type N. Does it follow that  $X$  admits no mean?

A more general concept than that of a folding sequence in Definition 3.13 is the one of a bend set, introduced in [66, p. 548].

**Definition 3.26.** A subset  $B$  of a subcontinuum  $A$  of a continuum  $X$  is said to be a *bend set of A* provided that there are two sequences of subcontinua  $E_n$  and  $F_n$  of  $X$ , where  $n \in \mathbb{N}$ , such that:

- (3.26.1)  $E_n \cap F_n \neq \emptyset$  for each  $n \in \mathbb{N}$ ;
- (3.26.2)  $A = \text{Lim} E_n = \text{Lim} F_n$ ;
- (3.26.3)  $B = \text{Lim}(E_n \cap F_n)$ .

A continuum  $X$  is said to have the *bend intersection property* provided that for each subcontinuum  $A \subset X$  the intersection of all bend sets of  $A$  is nonempty, see [66, p. 548]; compare also [57] and [58].

Observe that in a particular case when the continuum  $A$  in the above definition is arc-like with an end point  $a$  and if a folding sequence  $A_n$  with respect to  $a$  does exist, then the singleton  $\{a\}$  is a bend set of  $A$ .

The following question, related to Theorem 3.14, is of great importance; a positive answer to it would be a generalization of Theorem 3.14.

**Question 3.27.** Let a hereditarily unicoherent continuum (in particular a dendroid)  $X$  admit a mean. Does it follow that  $X$  has the bend intersection property?

A result related to this question is the following [58, Theorem 5, p. 124].

**Theorem 3.28.** *A dendroid  $X$  is not of type N if and only if for each arc  $A \subset X$  the intersection of all bend sets of  $A$  is empty.*

Consequently, we have a partial answer to Question 3.27.

**Proposition 3.29.** *Let a dendroid  $X$  admit a mean. Then for each arc  $A \subset X$  the intersection of all bend sets of  $A$  is empty.*

*Proof:* It follows from [6, Theorem 3.5, p. 42] (or from [50, Theorem 2.2, p. 99]; compare also [30, Corollary 5.40, p. 23]) that a dendroid admitting a mean is not of type N. Thus, the conclusion follows from Theorem 3.28.  $\square$

Let us recall the following example from [66, Example 3, p. 549] and consider some questions related to it.

**Example 3.30.** There is a dendroid  $D$  in the plane being the closure of the unions of two countable families of arcs, each of which approximates a triod. Further,  $D$  is not of (generalized) type N.

*Proof:* Let  $(\rho, \varphi)$  denote a point of the Euclidean plane having  $\rho$  and  $\varphi$  as its polar coordinates. Put, for  $n \in \mathbb{N}$ ,

$$p = (0, 0), \quad a = (1, 0), \quad b = (1, \frac{\pi}{2}), \quad c = (1, \pi),$$

$$a_n = (1, \frac{1}{n}), \quad b_n = (1 + \frac{1}{n}, \frac{\pi}{2}), \quad p_n = (\frac{1}{n}, \frac{\pi}{4}), \quad p'_n = (\frac{1}{n}, \frac{3\pi}{4}),$$

and denote by  $\overline{xy}$  the straight line segment joining points  $x$  and  $y$ . Define

$$D_1 = \overline{ac} \cup \overline{pb} \cup \bigcup \{ (\overline{a_n p_n} \cup \overline{p_n b_n} \cup \overline{b_n p'_n} \cup \overline{p'_n c}) : n \in \mathbb{N} \}.$$

Denote by  $h$  the reflection mapping about the origin, and put

$$(3.30.1) \quad D = D_1 \cup h(D_1).$$

Then the 4-od  $\overline{ac} \cup \overline{bh(b)}$  is the limit continuum in  $D$ , and

- the arcs  $\overline{ca_n} = \overline{a_n p_n} \cup \overline{p_n b_n} \cup \overline{b_n p'_n} \cup \overline{p'_n c}$  approximate the triod  $\overline{ac} \cup \overline{pb}$ ,

- the arcs  $ah(a_n) = \overline{ah(p'_n) \cup h(p'_n)h(b_n) \cup h(b_n)h(p_n) \cup h(p_n)h(a_n)}$  approximate the triod  $\overline{ac} \cup \overline{ph(b)}$ .

Note that  $D$  is not of (generalized) type N by its construction.  $\square$

Observe further that since each arc-like subcontinuum of  $D$  is an arc, it follows that  $D$  does not satisfy condition (3.14.1). Therefore, there is no general criterion from which it follows that  $D$  admits no mean. So we have a question.

**Question 3.31.** Does the dendroid  $D$  defined by (3.30.1) admit any mean?

The author conjectures that the answer to Question 3.31 should be negative. Some other (mapping) properties related to the dendroid  $D$  will be discussed in the next subsection.

**Remark 3.32.** Example 3.30 can serve as a good illustration of the concepts in Definition 3.26. Namely, taking the limit 4-od  $\overline{ca} \cup \overline{bh(b)}$  as  $A$  and defining  $E_n = h(b)a_n$  and  $F_n = bh(a_n) = h(E_n)$ , we have  $E_n \cap F_n = \{p\}$ , whence  $B = \{p\}$  is a bend set of  $A$ . Note that  $D$  has the bend intersection property.

It can be observed that all the previously considered concepts of a folding sequence used in (3.14.1), of a (generalized) type N, and of a bend set used in the bend intersection property are (in some sense) too restrictive, so they can be applied rather to a very limited number of cases. Looking for a more general condition that could imply non-existence of a mean for a continuum, we can consider the following one which is a common modification of the conditions considered above.

**Definition 3.33.** Let  $A$  be a subcontinuum of a continuum  $X$  and let  $a \in A$ . A pair of sequences  $\{E_n : n \in \mathbb{N}\}$  and  $\{F_n : n \in \mathbb{N}\}$  of subcontinua of  $X$  is called a *pair of surrounding sequences for  $A$  with respect to  $a$*  provided that

- (3.33.1)  $E_n \cap F_n \neq \emptyset$  for each  $n \in \mathbb{N}$ ;
- (3.33.2)  $A \subset \text{Lim}E_n \cup \text{Lim}F_n$ ;
- (3.33.3)  $\text{Lim}(E_n \cap F_n) = \{a\}$ .

**Question 3.34.** Let a hereditarily unicoherent continuum  $X$  contain a subcontinuum  $A$ , and let two pairs of surrounding sequences  $(\{E_n\}, \{F_n\})$  and  $(\{G_n\}, \{H_n\})$  of  $A$  with respect to distinct points  $a$  and  $b$ , correspondingly, be given. Assume that the irreducible continuum between the points  $a$  and  $b$  is contained in the intersections  $\text{Lim}E_n \cap \text{Lim}F_n$  and  $\text{Lim}G_n \cap \text{Lim}H_n$ . 1) Does it follow that then  $X$  admits no mean? 2) If not, under what additional conditions does it admit no mean?

Another concept, introduced in [7] and known from investigations of various phenomena in dendroids (see e.g., [20], [41], or [75]), is the one of a Q-point. We redefine it here to apply it to a larger class of continua.

**Definition 3.35.** A point  $p$  of a hereditarily unicoherent continuum  $X$  is said to be a *Q-point* provided that there are in  $X$  a sequence of points  $p_n$  converging to  $p$  and a sequence of continua  $I(p, p_n)$  irreducible from  $p$  to  $p_n$  such that  $\text{Ls}I(p, p_n) \neq \{p\}$ ; and, if for each  $n \in \mathbb{N}$ , a point  $q_n$  is defined so that the irreducible continuum  $I(p_n, q_n)$  from  $p_n$  to  $q_n$  is irreducible between the point  $p_n$  and the continuum  $\text{Ls}I(p, p_n)$ , then the sequence of points  $q_n$  also converges to  $p$ .

**Question 3.36.** Let a hereditarily unicoherent continuum  $X$  contain a Q-point. Does it follow that  $X$  admits no mean?

### 3.C. MEANS AND MAPPINGS

Let us come back to the dendroid  $D$  defined by (3.30.1) in Example 3.30. Let  $f : D \rightarrow E = \overline{f(D)}$  be a monotone mapping that shrinks in  $D$  the vertical segment  $\overline{bh(b)}$  to the origin  $p$  and is a homeomorphism on the rest. The obtained dendroid admits no mean since it is of type N between the points  $f(a)$  and  $f(c)$ . The following questions are related to this example.

**Question 3.37.** Let a hereditarily unicoherent continuum  $X$  admit a mean, and let a mapping  $f : X \rightarrow f(X)$  be monotone. Does it follow that  $f(X)$  also admits a mean? If not, is the implication true under an additional assumption that  $X$  is a dendroid?

Note that an affirmative answer to either of these questions implies a negative answer to Question 3.31, as conjectured above.

Let  $g : D \rightarrow F = g(D)$  be a mapping on  $D$  which identifies the points  $x$  and  $h(x)$  for each  $x \in D$  (here  $h$  is the reflection about the origin  $p$  as defined in Example 3.30). Then the mapping  $g$  is open, and the obtained dendroid  $F$  admits no mean again for the same reason: it is of type N between points  $g(b)$  and  $g(c)$ . The next questions arise.

**Question 3.38.** Let a hereditarily unicoherent and hereditarily decomposable continuum  $X$  (i.e., a  $\lambda$ -dendroid) admit a mean, and let a mapping  $g : X \rightarrow g(X)$  be open. Does it follow that  $g(X)$  also admits a mean? If not, is the implication true under an additional assumption that  $X$  is a dendroid? If not, is it true under any additional (nontrivial) condition?

Note that an affirmative answer to either the first or the second question 3.38 again implies a negative answer to Question 3.31.

Observe also that open mappings do not preserve admitting a mean for hereditarily unicoherent continua in general: the dyadic solenoid  $\Sigma_2$  is hereditarily unicoherent and admits a mean (compare Remark 3.22), while the circle (that is, its open image) admits no mean according to (3.2).

Questions 3.37 and 3.38 are particular cases of the following.

**Question 3.39.** What kinds of mappings between continua preserve admitting a mean? What if the domain and/or the range spaces are a hereditarily unicoherent continuum?

The following property (C) is a modification of a property (\*) defined in [37, p. 121].

**Definition 3.40.** Let a surjective mapping  $f : X \rightarrow Y$  between continua  $X$  and  $Y$  be given. We say that the triad  $(X, f, Y)$  has the *property (C)* provided that

- (3.40.1) there is an arc-like subcontinuum  $A \subset X$  with points  $a, b \in A$  as opposite end points of  $A$  and there are sequences  $\{A_n\}$  and  $\{B_n\}$  of subcontinua of  $X$  which are folding sequences with respect to the points  $a$  and  $b$ , respectively; we assume that for each  $n \in \mathbb{N}$  we have  $A_n = E_n \cup F_n$  and  $B_n = G_n \cup H_n$ , respectively, according to Definition 3.13 (in other words, the continuum  $X$  satisfies condition (3.14.1) for the points  $a$  and  $b$  of  $X$ );

- (3.40.2) the continuum  $Y$  is hereditarily unicoherent;
- (3.40.3)  $f(a) \neq f(b)$ ;
- (3.40.4)  $f(E_n) \cap f(F_n) = f(E_n \cap F_n)$ ;
- (3.40.5)  $f(G_n) \cap f(H_n) = f(G_n \cap H_n)$ .

**Question 3.41.** Let a continuum  $Y$  be the image of a continuum  $X$  under a mapping  $f$  such that the triad  $(X, f, Y)$  has the property (C). Does it follow that  $Y$  admits no mean?

### 3.D. MEANS WITH SPECIAL PROPERTIES

Means with some special properties, such as monotone, open, and confluent means, have very recently started to be studied, see [47]. Among many others, the following results have been obtained in [47].

- (3.42) Each dendrite admits a monotone mean, while the harmonic fan admits no monotone mean.
- (3.43) Each  $n$ -cell, as well as the dyadic solenoid, admits a mean that is monotone and open, simultaneously.
- (3.44) Each simple  $n$ -od (for  $n \in \mathbb{N}$ ), as well as the Cantor fan, admits an open mean, while the harmonic fan admits no open mean.
- (3.45) The harmonic fan admits a confluent mean.

In the light of the above mentioned results, the authors of [47] asked the following questions. Let me underline that answering these questions seems to be basic and very important for a further study in the area.

**Question 3.46.** Let a dendroid admit a monotone mean. Is it then a dendrite?

**Question 3.47.** Does each tree admit an open mean?

**Question 3.48.** Does there exist a dendrite that is not a tree and that admits an open mean?

**Question 3.49.** Does there exist a continuum which admits a mean but not a confluent one?

**Remark 3.50.** Recall that a mean  $m : X \times X \rightarrow X$  on a dendroid  $X$  is said to be *internal* provided that  $m(x, y) \in xy$  for every two points  $x, y \in X$ . It is known that each smooth fan admits an associative and internal mean [30, Corollary 5.41, p. 24]. Therefore,

it follows from (3.42) and (3.44) that an associative and an internal mean has to be neither open nor monotone.

#### 4. CONTRACTIBILITY OF CONTINUA

Given a space  $X$ , a mapping  $h : X \times [0, 1] \rightarrow X$  is called a *homotopy*. If for each point  $x \in X$ , the condition  $h(x, 0) = x$  holds, then the homotopy is called a *deformation* of  $X$ . A space  $X$  is said to be *contractible* provided that there exists a deformation  $h$  of  $X$  to a point  $p \in X$ , i.e., such that  $h(x, 1) = p$  for each point  $x \in X$ . The following facts are known.

- (4.1) *if a continuum is 1-dimensional and contractible, then it is a dendroid* (see [11, Proposition 1, p. 73]);
- (4.2) *each contractible dendroid is uniformly arcwise connected* (see [11, Proposition 4, p. 73] and compare [33, Theorem 3, p. 94]);
- (4.3) *each smooth dendroid is contractible* (see [33, Corollary, p. 93]);
- (4.4) *a locally connected curve is contractible if and only if it is a dendrite* (this is a consequence of (4.1) and (4.3) above).

During the last three decades, contractibility of continua, in particular of curves, was studied by a number of authors. Many conditions were considered which either imply or are implied by contractibility of a continuum. These conditions were formulated using various techniques and were expressed in different ways, so sometimes it is not easy to compare them. Some relations between them have been investigated in [20]. But the main problem in the area is still open.

**Problem 4.5.** Give an internal (structural) characterization of contractible curves (equivalently, by (4.1), of contractible dendroids).

Recall that the above problem has been solved in a particular case of fans in [75, Theorem 3.4, p. 393] and in [20, Theorem 3.4, p. 573].

The following proposition gives a general sufficient condition for a space to be noncontractible [34, Proposition 1, p. 230].

**Proposition 4.6.** *If a space  $X$  contains some two subsets  $A$  and  $B$  such that*

(4.6.1)  $\emptyset \neq A \subset B \neq X$ ,

(4.6.2) for each deformation  $h : X \times [0, 1] \rightarrow X$  we have  $h(A \times [0, 1]) \subset B$ ,

then  $X$  is not contractible.

The next questions were asked in [20, pp. 561 and 562].

**Question 4.7.** Does every noncontractible dendroid  $X$  contain (a) two subsets  $A$  and  $B$  satisfying (4.6.1) and (4.6.2); (b) a non-empty subset  $A$  such that for each deformation  $h : X \times [0, 1] \rightarrow X$  and for each  $t \in [0, 1]$  we have  $A \subset h(X \times \{t\})$ ?

It is known that if a fan contains a Q-point (see Definition 3.35), then it is not contractible (see [75, theorems 3.2 and 3.4, p. 393], [41, Theorem 2.3, p. 81] and [20, Theorem 3.4, p. 573]), but it is conjectured in [20, Question 2.24, p. 568] that the implication should be true for all dendroids.

**Question 4.8.** Does it follow that a dendroid containing a Q-point is not contractible?

It is known that each contractible fan is locally connected at its top [77, Theorem 6.1, p. 394], which implies in turn that the fan can be embedded in the plane [76, Theorem 5.2, p. 502]. Thus, each contractible fan is embeddable in the plane. This result cannot be extended to arbitrary dendroids [20, Example 2.32, p. 571]. Therefore, the next question is natural.

**Question 4.9.** For which dendroids does contractibility imply planability?

A continuum  $X$  is said to be *pseudo-contractible* provided that there is a continuum  $Y$ , points  $a, b \in Y$ , and a mapping  $h : X \times Y \rightarrow X$  such that for each point  $x \in X$  the conditions  $h(x, a) = x$  and  $h(x, b) = p$  hold, where  $p$  is a constant point of  $X$ . Bellamy asked the following question:

**Question 4.10.** Is every pseudo-contractible dendroid also contractible?

## 5. SELECTIBILITY OF CONTINUA

Given a continuum  $X$ , we denote by  $2^X$  the hyperspace of all nonempty closed subsets of  $X$  equipped with the Vietoris topology

or, equivalently, with the Hausdorff metric; the hyperspace of all closed and connected subsets of  $X$  is denoted by  $C(X)$  (see [70] or [46] for more information). The term “*hyperspace*” denotes any nonempty subspace of  $2^X$ . A (continuous) *selection* for a hyperspace  $\mathcal{H} \subset 2^X$  means a mapping  $s : \mathcal{H} \rightarrow X$  such that  $s(A) \in A$  for each  $A \in \mathcal{H}$ . If, additionally, the condition  $s(A) \in A \subset B$  implies  $s(A) = s(B)$  for every  $A, B \in \mathcal{H}$ , then the selection  $s$  is said to be *rigid*, see [82, p. 1041]. A continuum  $X$  is said to be *selectible* provided that there exists a selection for  $C(X)$ . The reader is referred to [46, Section 75, p. 363] for more information on selections.

The following results are known for a continuum  $X$ :

- (5.1) *A selection for  $2^X$  does exist if and only if  $X$  is an arc* (see [69, Theorem 1.9, p. 155 and Proposition 2.7, p. 158] and [56, Theorem 1, p. 5]; compare [46, Theorem 75.2, p. 363]).
- (5.2) *If  $X$  is selectible, then  $X$  is a dendroid* (see [73, Lemma 3, p. 370]); further, *each selectible dendroid is an image of the Cantor fan; thus, it is uniformly arcwise connected* (see [12, Proposition 2, p. 110]).
- (5.3) *A continuum  $X$  admits a rigid selection for  $C(X)$  if and only if  $X$  is a smooth dendroid* (see [82, Theorem 2, p. 1043]).
- (5.4) *A locally connected continuum is selectible if and only if it is a dendrite* (see [73, Corollary, p. 371]).

In connection with (5.2) and (5.3) the following problem is very natural and important. It seems to be the main problem in the area considered (see [70, Question 5.11, p. 259]).

**Problem 5.5.** Give an internal (structural) characterization of selectible continua (equivalently, by (5.2), of selectible dendroids).

Various conditions that imply nonexistence of any selection for  $C(X)$  are discussed in [12], [17], [18], [46, Section 75, pp. 363-371], [66], and [73]. Some of them are known to imply noncontractibility and/or nonexistence of a mean. Such is, e.g., the condition of being of type N (see above, Definition 3.24; compare [46, Definition 75.10, p. 367 and Exercise 75.25, p. 369]). Moreover, if a dendroid is of generalized type N (see again Definition 3.24) then it is nonselectible [17, (25), p. 96]. This is a particular case of the following general theorem [66, Theorem, p. 547].

**Theorem 5.6.** *Let a dendroid  $X$  contain two sequences of subcontinua  $E_n$  and  $F_n$  of  $X$ , where  $n \in \mathbb{N}$ , such that:*

$$(5.6.1) \quad E_n \cap F_n \neq \emptyset \quad \text{for each } n \in \mathbb{N};$$

$$(5.6.2) \quad \text{Lim}E_n \subset A = \text{Lim}F_n;$$

$$(5.6.3) \quad B = \text{Lim}(E_n \cap F_n).$$

*If  $X$  admits a selection  $s : C(X) \rightarrow X$  such that  $s(E_n \cup F_n) \in E_n$  for each  $n \in \mathbb{N}$ , then  $s(A) \in B$ .*

The following result is a consequence of Theorem 5.6 (compare Definition 3.26; see also [66, Corollary, p. 548] ).

(5.7) *Each selectable dendroid has the bend intersection property.*

The opposite implication to (5.7) is not true [66, Example 1, p. 548] ).

Since the hyperspace  $F_1(X)$  of singletons of a continuum  $X$  is homeomorphic to  $X$ , and since obviously  $F_1(X) \subset C(X)$ , the continuum  $X$  can be seen as a subspace of  $C(X)$ . Therefore, a selection  $s : C(X) \rightarrow X$  can be treated as a special kind of a retraction  $r : C(X) \rightarrow X$ . So, the next two questions arise in a natural way.

**Question 5.8.** Can the selection  $s : C(X) \rightarrow X$  in Theorem 5.6 be replaced by an arbitrary retraction  $r : C(X) \rightarrow X$ ?

**Question 5.9.** Let a dendroid  $X$  admit a retraction  $r : C(X) \rightarrow X$ . Must then  $X$  have the bend intersection property?

The reader is referred to [46, Section 75, questions 75.11-75.17, pp. 367-368] for other questions pertinent to selectibility, related mainly to special kinds of dendroids (fans, for example) as well as to special kinds of mappings between them (such as monotone, open, or confluent).

We close this set of problems with a short discussion that concerns three conditions (related to dendroids  $X$ ) that were considered in the previous sections: existence of a mean, of a contraction, and of a selection for  $C(X)$ .

**Remarks 5.10.** (a) Hereditary contractibility *and* rigid selectibility do not imply the existence of a mean. Indeed, there exists a smooth dendroid  $X$  in the 3-space which admits no mean (see [30, Example 5.52, p. 25] or [46, Example 76.12, p. 376]). Being smooth, it is hereditarily contractible [34, Proposition 14, p. 235], and it admits a rigid selection for  $C(X)$  [82, Theorem 2, p. 1043].

No such example exists in the plane because each plane smooth dendroid admits a mean [31, Theorem 6.6, p. 497].

(b) The existence of a mean *and* selectibility does not imply contractibility, even for fans. Indeed, there exists a fan in the plane which is non-contractible and selectible [12, Proposition 4, p. 111 and Figure 2, p. 112], and which admits a mean [6, Example 4.9, p. 46]. But if we assume that the selection is rigid, which is equivalent to smoothness [82, Theorem 2, p. 1043], then (hereditary) contractibility follows by [34, Proposition 14, p. 235].

(c) The existence of a mean *and* contractibility does not imply selectibility. Indeed, the Illanes-Maćkowiak dendroid, constructed in [67, Example, p. 321] and in [43, Section 4, p. 70] (see also [30, Theorem 5.78, p. 31] and [46, Example 75.9, p. 365]), admits a mean [6, Example 4.10, p. 47], is contractible (but not hereditarily contractible), and is not selectible. The example is not planable.

In connection with Remark 5.10 (c), one can ask the following questions (compare [17, Questions 11, p. 94], [23, Question 8.7, p. 34] and [46, Questions 75.12, p. 367]).

**Question 5.11.** Does there exist a planar, contractible and non-selectible dendroid?

**Question 5.12.** Does there exist a dendroid, as in 5.11, admitting a mean?

**Question 5.13.** Does hereditary contractibility imply selectibility of dendroids?

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