A NOTE ON JÓNSSON CARDINALS

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Abstract. We use elementary submodels to prove a few facts about Jónsson cardinals.

Definition 1. A cardinal \( \lambda \) is a Jónsson cardinal if \( \lambda \rightarrow [\lambda]_{<\aleph_0}^\lambda \). This means that for any function \( f : [\lambda]_{<\aleph_0} \rightarrow \lambda \), there is \( H \in [\lambda]^{\lambda} \) such that the range of \( f \upharpoonright H_{<\aleph_0} \) is a proper subset of \( \lambda \).

Jónsson cardinals have been extensively studied in the literature. A. Kanamori’s book [2] has an excellent survey of what is known and how Jónsson cardinals are related to large cardinals.

Proposition 2 (Folklore). A cardinal \( \lambda \) is a Jónsson cardinal if and only if for every large enough regular \( \chi \) and every \( x \in H(\chi) \), we can find \( M \prec H(\chi) \) such that

- \( \{\lambda, x\} \in M \)
- \( |M \cap \lambda| = \lambda \)
- \( \lambda \notin M \).

We open this paper with an application of Jónsson cardinals to topology. Recall that if \( M \prec H(\chi) \) and \( X \in M \) is a topological space, then \( X_M \) is the topological space with underlying set \( M \cap X \) and base \( \{M \cap U : U \in M, U \text{ open in } X\} \).

Theorem 1. The following statements are equivalent:

(1) There is a Jónsson cardinal.

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(2) There is a topological space $X$ and $M \prec H(\chi)$ (for $\chi$ some large regular cardinal) with $X \in M$ such that $X_M$ is homeomorphic to $X$ but $X \neq X_M$.

Proof: The proof that (1) implies (2) is due to L. Junqueira and F. Tall [1]; it suffices to observe that if $\lambda$ is a Jónsson cardinal, then the discrete space of cardinality $\lambda$ works — we just take $M$ witnessing that $\lambda$ is Jónsson.

The proof that (2) implies (1) is more involved; we show that (2) implies that at least one of $|X|$ and $w(X)$ is a Jónsson cardinal.

Suppose that we are given $M \prec H(\chi)$ and $X \in M$ such that $X_M$ is homeomorphic to $X$ but not equal to $X$. Further suppose that $|X|$ is not a Jónsson cardinal.

Since $X_M$ is homeomorphic to $X$, we know $|M \cap X| = |X_M| = |X|$. Also, $|X| \in M$ because $X$ is. Since $|X|$ is not a Jónsson cardinal, we are forced to conclude that $|X| \subseteq M$, and hence $X \subseteq M$.

In $M$, let us fix a base $\{U_\alpha : \alpha < w(X)\}$ for the topology of $X$. The cardinal $w(X)$ is in $M$ because $X$ is. Now $\{U_\alpha : \alpha \in M \cap w(X)\}$ is a base for the topology of $X_M$. Since $X_M$ and $X$ are homeomorphic, we know $w(X_M) = w(X)$ and therefore, $|M \cap w(X)| = w(X)$. Since $X_M \neq X$, we note that $w(X)$ cannot be a subset of $M$. Putting all these facts together, we arrive at the conclusion that $w(X)$ is Jónsson. \[\square\]

Remarks.

(1) P. Welch has shown [7] that the existence of a model $M \prec H(\epsilon^+)$ such that $\mathbb{R}_M$ is homeomorphic to $\mathbb{R}$ but not equal to $\mathbb{R}$ is equiconsistent with the existence of a Jónsson cardinal.

(2) Junqueira and Tall come close to establishing Theorem 1 in [1]. As mentioned before, they prove that (1) implies (2), and show that (2) fails if $\theta^2$ exists. It is well-known that the existence of a Jónsson cardinal implies that $\theta^2$ exists, so Theorem 1 simply closes the gap.

(3) We have not investigated the “non-constructive” nature of Theorem 1. It should be clear that the existence of an example witnessing (2) implies the existence of a discrete example. It is plausible that if there is a Jónsson cardinal then there exists a space $X$ and model $M$ such that $X \cong X_M$, $X \subseteq M$ (as a set), but $X \neq X_M$.
Our next application of elementary submodels is to give a short proof of a result due independently to J. Tryba [6] and H. Woodin (unpublished).

**Theorem 2.** If $\lambda$ is a Jónsson cardinal, then every stationary subset of $\lambda$ reflects.

**Lemma 3.** Suppose $M \prec H(\chi)$, $\lambda \in M$, $|M \cap \lambda| = \lambda$, and $\lambda \notin M$. If $S \in M$ is a stationary subset of $\lambda$, then $S \setminus M$ is stationary.

**Proof:** Suppose $S$ and $M$ are counterexamples. There is a closed unbounded set $E \subseteq \lambda$ such that $E \cap S \subseteq M$.

In $M$, we can fix a partition of $S$ into $\lambda$ stationary subsets, i.e., there is a function $f : S \rightarrow \lambda$ in $M$ such that $S_\alpha := f^{-1}(\{\alpha\})$ is stationary for each $\alpha < \lambda$.

Fix $\alpha < \lambda$ such that $\alpha \notin M$. Since $S_\alpha$ is stationary, we know that $E \cap S_\alpha$ is non-empty. Since $S_\alpha \subseteq S$, we have $E \cap S_\alpha \subseteq M$. Fix $\beta \in E \cap S_\alpha$. Then, since $f \in M$ and $\beta \in M$, $\alpha = f(\beta)$ is in $M$, a contradiction. \[\square\]

**Proof of Theorem 2.** Let $S$ be a stationary subset of $\lambda$. We must produce $\beta < \lambda$ such that $S \cap \beta$ is stationary in $\beta$.

Since $\lambda$ is a Jónsson cardinal, we can find $M \prec H(\chi)$ such that

- $\{S, \lambda\} \in M$
- $|M \cap \lambda| = \lambda$
- $\lambda \notin M$.

By our lemma, we can find $\delta \in S \setminus M$ such that $\delta = \sup(M \cap \delta)$ (as the set $\{\delta < \lambda : \delta = \sup(M \cap \delta)\}$ is club in $\lambda$). Let $\beta_\delta = \min(M \cap \lambda \setminus \delta)$; clearly, $\delta < \beta_\delta$.

**Claim 4.** $S \cap \beta_\delta$ is a stationary subset of $\beta_\delta$.

**Proof:** The proof is by contradiction. If this fails, then there is a closed unbounded $C \subseteq \beta_\delta$ disjoint from $S$. Since $S$ and $\beta_\delta$ are both in $M$, we may assume that $C \subseteq M$.

Given $\alpha < \delta$, we can find $\beta \in M$ such that $\alpha < \beta < \delta$ because $\delta = \sup(M \cap \delta)$. Since $M \models "C$ is unbounded in $\delta"$, we can find $\gamma \in M \cap C$ such that $\beta < \gamma$. By choice of $\beta_\delta$, we see that $\gamma < \delta$. Since $\alpha$ was an arbitrary ordinal $< \delta$, we have shown that $\delta$ is a limit point of $C$. As $C$ is closed, we have $\delta \in C$, a contradiction as $C \cap S$ was supposed to be empty. \[\square\]

The proof of Lemma 3 can be easily generalized to other ideals.
Lemma 5. Suppose $M \prec H(\chi)$ with $\lambda \in M$. Let $I \in M$ be an ideal on $\lambda$ such that there is a function $f : \lambda \to \lambda$ with $f^{-1}(\{\alpha\}) \notin I$ for each $\alpha < \lambda$. If $\lambda \setminus M \in I$, then $\lambda \subseteq M$.

Proof: Without loss of generality, the function $f$ is in $M$. Given $\alpha < \lambda$, the set $f^{-1}(\{\alpha\})$ is not in $I$. Since $\lambda \setminus M \in I$, this means that there is $\beta < \lambda \setminus M$ such that $f(\beta) = \alpha$. Since $f$ and $\beta$ are in $M$, $\alpha$ must be in $M$ as well. As $\alpha < \lambda$ was arbitrary, we conclude $\lambda \subseteq M$. □

We now exploit this lemma by connecting the question of whether the successor of a singular cardinal can be Jónsson to a question on whether a certain ideal possesses a weak form of saturation. This approach is implicit in much of S. Shelah’s work in [4]. We note that the question of whether the successor of a singular cardinal can be Jónsson is still very much an open question (see [5] for example).

For the remainder of the paper, assume that $\lambda = \mu^+$ for some singular cardinal $\mu$, and we let $S$ be a stationary subset of $\lambda \setminus \mu$ such that $\sup\{\cf(\delta) : \delta \in S\} < \mu$. We let $\bar{C} = \langle C_\delta : \delta \in S \rangle$ be such that $C_\delta$ is club in $\delta$ with order–type $\cf(\delta)$. For $\delta \in S$, we define an ideal $I_\delta$ of subsets of $C_\delta$ by

$$A \text{ is not in } I_\delta \iff \forall \alpha < \delta \forall \beta < \mu (\exists \gamma \in \text{nacc}(C_\delta)) [\gamma > \alpha \text{ and } \cf(\gamma) > \beta].$$

Here $\text{nacc}(C_\delta)$ is the set of non–accumulation points of $C_\delta$, i.e., those $\alpha \in C_\delta$ such that $\sup(\alpha \cap C_\delta) < \alpha$. It is not hard to see that $I_\delta$ is an ideal of subsets of $C_\delta$, and we let $\bar{I} = \langle I_\delta : \delta \in S \rangle.$

Definition 6. The ideal $\idp(\bar{C}, \bar{I})$ is defined by putting $A \in \idp(\bar{C}, \bar{I})$ if and only if there is a closed unbounded $E \subseteq \lambda$ such that for every $\delta \in S \cap E$, $A \cap E \cap C_\delta \in I_\delta$.

Said another way, if $A \notin \idp(\bar{C}, \bar{I})$, then for every club $E \subseteq \lambda$ there is $\delta \in S \cap E$ such that $A \cap E \cap \text{nacc}(C_\delta)$ is large in the sense that it is not in the ideal $I_\delta$. Shelah’s work in [4] shows that in many cases the ideal $\idp(\bar{C}, \bar{I})$ is non–trivial — given $S$, we can find $\bar{C}$ such that $\lambda \notin \idp(\bar{C}, \bar{I})$.

The proposition we state next is new, although it lurks in the background throughout much of Chapter IV of [3]. It ties together many of the proofs there.
Proposition 7. Let $\lambda = \mu^+$ where $\mu$ is singular, and let $S$ be a stationary subset of $S^\lambda_\kappa$ for some $\kappa < \lambda$. Let $M \prec H(\chi)$ with $\{\lambda, S, C\} \in M$, and assume $|M \cap \lambda| = \lambda$. Then $\lambda \setminus M \in \text{id}_p(C, \bar{I})$.

Proof: Suppose this is not the case. Let $E = \{\delta < \lambda : \delta = \sup(M \cap \delta)\}$; since $|M \cap \lambda| = \lambda$, we know that $E$ is closed unbounded in $\lambda$. Since we assume $\lambda \setminus M /\in \text{id}_p(C, \bar{I})$, there is a $\delta \in S \cap E$ with $(\lambda \setminus M) \cap E \cap C_\delta \notin I_\delta$. This means that we can find points in $\text{nacc}(C_\delta) \cap E$ with cofinality arbitrarily large beneath $\mu$ that are not in $M$.

Note that we have no guarantee that $\delta$ and $C_\delta$ are in $M$; to get around this, let us define $\beta_\delta := \min(M \cap \lambda \setminus C_\delta)$. (So $\beta_\delta = \delta$ if $\delta \in M$.) In $M$, we can fix $C$ such that $C$ is club in $\beta_\delta$ and $\text{otp}(C) = \text{cf}(\beta_\delta)$. Note that since $S \subseteq \lambda \setminus \mu$, we know that $\beta_\delta$ is singular and $\text{cf}(\beta_\delta) < \mu$. We define

$$C^* = \bigcup_{\beta \in C \cap S} C_\beta.$$ 

Since $C$ and $S$ are in $M$, the set $C^*$ is in $M$ as well. Also, note that $C_\delta$ is a subset of $C^*$. By our assumption on $S$, there is some $\gamma < \mu$ such that $|C_\delta| < \gamma$ for each $\delta \in S$. This together with the fact that $|C| < \mu$ is enough to guarantee that $|C^*| < \mu$.

Since $(\lambda \setminus M) \cap E \cap C_\delta \notin I_\delta$, there is $\alpha \in E \cap \text{nacc}(C_\delta)$ such that $\text{cf}(\alpha) > |C^*|$ and $\alpha \notin M$. Since $\text{cf}(\alpha) > |C^*|$, we know that $\alpha \in \text{nacc}(C^*)$ as well. This means that there is a $\beta \in M \cap \lambda$ such that

$$\text{sup}(C^* \cap \alpha) < \beta < \alpha.$$ 

This implies that $\alpha$ can be defined as the least member of $C^*$ that is above $\beta$; since $C^*$ and $\beta$ are in $M$, we conclude $\alpha \in M$. This is a contradiction of our choice of $\alpha$. □

We can now draw some conclusions about the possibility of the successor of a singular cardinal being Jónsson. For example, if $\lambda = \mu^+$ and $M \prec H(\chi)$ satisfies

- $|M \cap \lambda| = \lambda$, and
- $\lambda \notin M$,

then $M$ will contain a stationary set $S \subseteq \lambda$ and an $S$–club system $C$ such that the ideal $\text{id}_p(C, \bar{I})$ is non–trivial. Since $\text{id}_p(C, \bar{I}) \subseteq M$ and $\lambda \setminus M \subseteq \text{id}_p(C, \bar{I})$, Lemma 5 tells us that whenever we partition $\lambda$ into $\lambda$ sets, at least one of the pieces of the partition must be in
id\(_p(\overline{C}, \overline{I})\). The power of this lies in our ability to prove that in certain situations, it is possible to partition \(\lambda\) into \(\lambda\) disjoint sets, none of which are in id\(_p(\overline{C}, \overline{I})\), and thus show that \(\lambda\) is not a Jónsson cardinal — this is the essence of many results in Shelah's book [3].

References