AN EXPANSIVE HOMEOMORPHISM ON A TWO-DIMENSIONAL PLANAR CONTINUUM

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Abstract. A homeomorphism $h : X \to X$ is called expansive provided that for some fixed $c > 0$ and every $x, y \in X$ there exists an integer $n$, dependent only on $x$ and $y$, such that $d(h^n(x), h^n(y)) > c$. A two-dimensional planar continuum that admits an expansive homeomorphism is constructed.

1. Introduction

A continuum is a nondegenerate compact connected metric space. A homeomorphism $h : X \to X$ is called expansive provided that for some fixed $c > 0$ and every distinct $x, y \in X$ there exists an integer $n$, dependent only on $x$ and $y$, such that $d_X(h^n(x), h^n(y)) > c$. Expansive homeomorphisms exhibit chaotic behavior in that no matter how close two points are either their forward or reverse images will eventually be a certain distance apart. The Plykin attractor [4] and the dyadic solenoid [5] are examples of continua that admit expansive homeomorphisms.

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A continuum is 1-dimensional if for every $\epsilon > 0$ there exists a finite open cover $U$ whose mesh is less than $\epsilon$ such that for every $y \in X$, $y$ is in at most 2 elements of $U$. If $X$ is a plane continuum, then $X$ is 2-dimensional if $X$ contains an open disk and 1-dimensional otherwise. A planar continuum $X$ is a non-separating plane continuum provided that $\mathbb{R}^2 - X$ is connected. The Plykin attractor is a Lakes of Wada continuum that separates the plane into 4 components and is the most widely known 1-dimensional planar continuum that admits an expansive homeomorphism. It has been shown by the author that 1-dimensional non-separating plane continua do not admit expansive homeomorphisms. (In fact, tree-like continua do not admit expansive homeomorphisms [3].) This paper gives an example of a 2-dimensional planar continuum that admits an expansive homeomorphism and separates the plane. However, the following question remains open: Does there exist a 2-dimensional non-separating plane continuum that admits expansive homeomorphism?

2. INVERSE LIMITS AND THE PLYKIN ATTRACTOR

A useful method of constructing continua is through inverse limits. Let $\{G_i\}_{i=1}^{\infty}$ be a sequence of topological spaces. For each $i < j$, let $f_{ij} : G_j \rightarrow G_i$ be a continuous function called a bonding map. If $f_{kj} = f_{ki} \circ f_{ij}$ for each $k < i < j$, then the collection $\{G_i, f_{ij}\}_{i=1}^{\infty}$ is called an inverse system. Each of the spaces $G_i$ is called a factor space of the inverse system.

Since each bonding map $f_{ij}$ is determined by the collection of one step bonding maps $f_{ij}^{k+1} = f_i$ for $k \leq i < j$, it is sufficient to consider only these maps. The inverse system $\{G_i, f_i\}_{i=1}^{\infty}$ is sometimes written as

$$G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2} G_3 \ldots \xleftarrow{f_{i-1}} G_i \xleftarrow{f_i} \ldots .$$

Every inverse system $\{G_i, f_i\}_{i=1}^{\infty}$ determines a topological space $X$ called the inverse limit of the system and is written $\lim\{G_i, f_i\}_{i=1}^{\infty}$. The space $X$ is the subspace of the Cartesian product $\prod_{i=1}^{\infty} G_i$ given by

$$X = \lim\{G_i, f_i\}_{i=1}^{\infty} = \{(x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} G_i | f_i(x_{i+1}) = x_i \text{ for each } i\}.$$
$X$ has the subspace topology induced on it by $\prod_{i=1}^{\infty} G_i$. If $x = (x_i)_{i=1}^{\infty}$ and $y = (y_i)_{i=1}^{\infty}$ are two points of the inverse limit, we define distance to be

$$d(x, y) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i},$$

where $d_i$ is the metric on $G_i$. If each of the factor spaces $G = G_i$ and each of the bonding maps $f = f_i$ are the same, then there is a natural homeomorphism on the inverse limit $\hat{f} : X \rightarrow X$ defined by

$$\hat{f}(x_1, x_2, x_3...) = (f(x_1), f(x_2), f(x_3), ...) = (f(x_1), x_1, x_2, ...).$$

We call $\hat{f}$ the shift homeomorphism on $X$. Notice that

$$\hat{f}^{-1}(x_1, x_2, x_3...) = (x_2, x_3, x_4, ...).$$

The construction of the Plykin attractor will be through the inverse limit of the common factor space $P$ with common bonding map $f$. Let $P = [0, 4]/\{A, B\}$ where $A$ is the identification of 0, 1, and 2 and $B$ is the identification of 3 and 4 in the interval $[0, 4]$ (see Figure 1). Define

$$d_a(x) = \begin{cases} 
\min\{|x|, |x - 1|, |x - 2| & x \in [0, 3] \\
\min\{|x - 3| + 1, |x - 4| + 1, & x \in (3, 4) \}
\end{cases}$$

and

$$d_b(x) = \begin{cases} 
\min\{|x| + 1, |x - 1| + 1, |x - 2| + 1 & x \in [0, 2] \\
\min\{|x - 3|, |x - 4|, & x \in (2, 4) \}
\end{cases}.$$
\[ f(x) = \begin{cases} 
2.5 - 2x & 0 \leq x < .25 \\
1 + 2(x - .25) & .25 \leq x < .75 \\
2.5 - 2(1 - x) & .75 \leq x < 1 \\
2.5 - 4(x - 1) & 1 \leq x < 1.125 \\
1 + 8(x - 1.125) & 1.125 \leq x < 1.375 \\
3 + 4(x - 1.375) & 1.375 \leq x < 1.625 \\
1 + 8(1.875 - x) & 1.625 \leq x < 1.875 \\
2.5 - 4(2 - x) & 1.875 \leq x < 2 \\
2.5 + 2(x - 2) & 2 \leq x < 2.75 \\
3 - 2(x - 2.75) & 2.75 \leq x < 3 \\
2.5 - 2(x - 3) & 3 \leq x < 3.25 \\
2(x - 3.25) & 3.25 \leq x < 3.75 \\
2.5 - 2(4 - x) & 3.75 \leq x < 4 
\end{cases} \]

Motivation for \( f(x) \) comes from Figure 5.6 on page 210 of [2]. For a pictorial representation of \( f \) see Figure 2. Let \( P = \overline{\lim\{P, f_i\}}_{i=1}^\infty \) where \( f_i = f \) for each \( i \). Then \( x = \{x_1, x_2, \ldots\} \) is an element of \( P \) provided that \( f(x_{i+1}) = f_i(x_{i+1}) = x_i \) for each \( i \). For \( x, y \in P \), define distance as \( d_P(x, y) = \sum_{i=1}^\infty 2^{-i}(d_P(x_i, y_i)) \). Let \( \hat{f} : P \to P \) be the shift homeomorphism induced by \( f \).

**Proposition 1.** Suppose \( x = (x_1, x_2, \ldots) \in P \), then the following are true:

1. If \( x_i \in (0, 1) \) then \( x_{i+1} \in (3.25, 3.75) \).
2. If \( x_i \in (1, 2) \) then \( x_{i+1} \in (.25, .75) \cup (1.125, 1.25) \cup (1.75, 1.875) \).
3. If \( x_i \in (2, 2.5) \) then \( x_{i+1} \in (0, .25) \cup (1, 1.125) \cup (1.875, 2) \cup (1.25, 1.3125) \cup (1.6875, 1.75) \cup (3, 3.25) \cup (3.75, 4) \).
4. If \( x_i \in (2.5, 3) \) then \( x_{i+1} \in (1.3125, 1.375) \cup (1.625, 1.6875) \cup (2, 2.25) \cup (2.75, 3) \).
5. If \( x_i \in (3, 4) \) then \( x_{i+1} \in (1.375, 1.625) \cup (2.25, 2.75) \).

The next two lemmas show how \( f \) expands the distances between points.

**Lemma 2.** Suppose \( x_{i-1} = y_{i-1} \) and \( x_i \neq y_i \), then either \( d_P(x_i, y_i) \geq .0625 \) or \( d_P(x_{i+1}, y_{i+1}) \geq .0625 \).
Proof. The proof of this lemma contains a large number of cases of which 3 are shown. The proof of the other cases are similar.

**Case 1.** Suppose \(x_i, y_i \in (0, 1)\). Since \(f\) maps the interval \((0, .75)\) one-to-one and onto \((1, 2.5)/A\), \(f\) maps \(.25, 1\) one-to-one and onto \((1, 2.5)/A\) and \(x_{i-1} = y_{i-1}\), we may take \(x_i \in (0, .25)\) and \(y_i \in (.75, 1)\). Then \(x_{i+1} \in (3.25, 3.375)\) and \(y_{i+1} \in (3.625, 3.75)\). Hence,
\[ d_P(x_{i+1}, y_{i+1}) \geq 0.25 > 0.0625. \]

**Case 2.** Suppose \( x_i \in (0,1) \) and \( y_i \in (1,2) \). Then \( x_{i+1} \in (3.25,3.75) \) and \( y_{i+1} \in (-0.25, -0.75) \cup (1.125,1.25) \cup (1.75,1.875) \). Hence,

\[ d_P(x_{i+1}, y_{i+1}) \geq 0.375 > 0.0625. \]

**Case 3.** Suppose \( x_i \in (0,1) \) and \( y_i \in (2,2.5) \). Then \( x_{i+1} \in (1,2.5) \cup A \) and \( y_{i+1} \in (2.5,3.5) \). Thus, \( x_{i+1} \neq y_{i+1} \) which is a contradiction.

**Lemma 3.** Suppose \( d_P(x_i, y_i) < 0.0625 \) for all \( i \geq 1 \) then

\[ d_P(f(x_1), f(y_1)) \geq 2d_P(x_1, y_1). \]

**Proof.** The proof of this lemma also contains a large number of cases of which 2 are shown. Again, the proof of the other cases are similar. From Lemma 2 we may assume that \( x_1 \neq y_1 \).

**Case A.** Suppose \( x_1, y_1 \in (3,4) \). There are 8 subcases to consider:

**Case A.1.** Suppose \( x_1, y_1 \in (3,3.25) \), then \( d_P(x_1, y_1) = |y_1 - x_1| \).

Here, \( f(x_1), f(y_1) \in (2,2.5) \). So, \( d_P(f(x_1), f(y_1)) = |f(y_1) - f(x_1)| = |2.5 - (y_1 - 3) - (2.5 - 2(x_1 - 3))| = 2|y_1 - x_1| = 2d_P(x_1, y_1) \).

**Case A.2.** Suppose \( x_1 \in (3,3.25) \), \( y_1 \in (3.25,3.5) \), then \( d_P(x_1, y_1) = y_1 - x_1 \).

Here, \( f(x_1) \in (2,2.5) \), \( f(y_1) \in (0,.5) \). So, \( d_P(f(x_1), f(y_1)) = f(x_1) - 2 + f(y_1) = 2.5 - 2(x_1 - 3) - 2 + 2(y_1 - 3.25) = 2(y_1 - x_1) = 2d_P(x_1, y_1) \).

**Case A.3.** Suppose \( x_1, y_1 \in (3.25,3.5) \), then \( d_P(x_1, y_1) = |y_1 - x_1| \).

Here, \( f(x_1), f(y_1) \in (0,.5) \). So, \( d_P(f(x_1), f(y_1)) = |f(y_1) - f(x_1)| = |2(y_1 - 3.25) - (2(x_1 - 3.25))| = 2|y_1 - x_1| = 2d_P(x_1, y_1) \).

**Case A.4.** Suppose \( x_1 \in (3.375,3.5) \), \( y_1 \in (3.5,3.625) \) then \( d_P(x_1, y_1) = y_1 - x_1 \).

Here, \( f(x_1) \in (.25,.5) \), \( f(y_1) \in (.5,.75) \). So,

\[ d_P(f(x_1), f(y_1)) = f(y_1) - f(x_1) = 2(y_1 - 3.25) - (2(x_1 - 3.25)) = 2(y_1 - x_1) = 2d_P(x_1, y_1). \]

**Case A.5.** Suppose \( x_1 \in (3.5,3.75) \), \( y_1 \in (3.5,3.75) \) then \( d_P(x_1, y_1) = |y_1 - x_1| \).

Here, \( f(x_1), f(y_1) \in (.5,1) \). So,

\[ d_P(f(x_1), f(y_1)) = |f(y_1) - f(x_1)| = |2(y_1 - 3.25) - (2(x_1 - 3.25))| = 2|y_1 - x_1| = 2d_P(x_1, y_1). \]
Case A.6. Suppose \( x_1 \in (3.625, 3.75) \), \( y_1 \in (3.75, 3.875) \) then \( d_P(x_1, y_1) = y_1 - x_1 \). Here, \( f(x_1) \in (5, 1) \), \( f(y_1) \in (2, 2.25) \). So, \( d_P(f(x_1), f(y_1)) = f(y_1) - 2 + 1 - f(x_1) = 2.5 - 2(4 - y_1) - 2 + 1 - (2(x_1 - 3.25)) = 2(y_1 - x_1) = 2d_P(x_1, y_1) \).

Case A.7. Suppose \( x_1, y_1 \in (3.75, 4) \) then \( d_P(x_1, y_1) = |y_1 - x_1| \). Here, \( f(x_1), f(x_2) \in (2, 2.5) \). So, \( d_P(f(x_1), f(y_1)) = |f(y_1) - f(x_1)| = |2.5 - 2(4 - y_1) - (2.5 - 2(4 - x_1))| = 2|y_1 - x_1| = 2d_P(x_1, y_1) \).

Case A.8. Suppose \( x_1 \in (3, 3.25) \), \( y_1 \in (3.75, 4) \) then \( x_2 \in (2.25, 2.375) \cup (1.375, 1.4375) \) and \( y_2 \in (2.625, 2.75) \cup (1.5625, 1.625) \). So, \( d_P(x_2, y_2) \geq .125 > .0625 \) which is a contradiction.

Case B. Suppose \( x_1 \in (2, 3) \) and \( y_1 \in (3, 4) \). Since \( d_P(x_1, y_1) < .0625 \), there are 2 subcases to consider:

Case B.1. Suppose \( x_1 \in (2.9375, 3) \) and \( y_1 \in (3, 3.0625) \) then \( d_P(x_1, y_1) = |y_1 - x_1| \). Here, \( f(x_1) \in (2.5, 2.625) \) and \( f(y_1) \in (2.375, 2.5) \). So, \( d_P(f(x_1), f(y_1)) = |f(y_1) - f(x_1)| = |2.5 - 2(3 - 2(2.625))| = 2|y_1 - x_1| \).

Case B.2. Suppose \( x_1 \in (2.9375, 3) \) and \( y_1 \in (3.9375, 4) \) then \( d_P(x_1, y_1) = |7 - x_1 - y_1| \). Here, \( f(x_1) \in (2.5, 2.625) \) and \( f(y_1) \in (2.375, 2.5) \). So, plane \( d_P(f(x_1), f(y_1)) = |f(y_1) - f(x_1)| = |2.5 - 2(4 - y_1) - (3 - 2(2.625))| = 2|y_1 - x_1| \).

The next theorem states that the shift homeomorphism on the Plykin attractor is expansive.

**Theorem 4.** \( \hat{f} : \mathcal{P} \to \mathcal{P} \) is an expansive homeomorphism with expansive constant .0625.

**Proof.** Suppose \( x = \{x_1, x_2, \ldots\} \), \( y = \{y_1, y_2, \ldots\} \) are distinct points of \( \mathcal{P} \). Let \( i \) be the smallest index such that \( x_i \neq y_i \).

**Case 1.** Suppose \( i = 1 \) and \( d_P(x_k, y_k) \geq .0625 \) for some \( k \geq 1 \). Then \( \hat{f}^{i-k}(x) = \{x_k, x_{k+1}, \ldots\} \) and \( \hat{f}^{i-k}(y) = \{y_k, y_{k+1}, \ldots\} \). Hence,

\[
d_P(\hat{f}^{i-k}(x), \hat{f}^{i-k}(y)) \geq d_P(x_k, y_k) \geq .0625.
\]

**Case 2.** Suppose \( i = 1 \) and \( d_P(x_k, y_k) < .0625 \) for all positive integers \( k \). Let \( n \) be an integer such that

\[
\log_2(0.0625/d_P(x_1, y_1)) \leq n < \log_2(0.0625/d_P(x_1, y_1)) + 1.
\]
Then it follows from Lemma 3 that
\[ d_P(\hat{f}_n^*(x), \hat{f}_n^*(y)) \geq d_P(f_n^*(x_1), f_n^*(y_1)) \geq 2^n d_P(x_1, y_1) \geq .0625. \]

**Case 3.** Suppose \( i > 1 \). Then by Lemma 2, either \( d_P(x_i, y_i) \geq .0625 \) or \( d_P(x_{i+1}, y_{i+1}) \geq .0625 \). This is similar to Case 1) by letting \( k = i \) or \( k = i + 1 \). \( \square \)

3. **Construction of the 2-dimensional plane continuum \( X \) that admits an expansive homeomorphism.**

The 2-dimensional planar continuum \( X \) that admits an expansive homeomorphism that is constructed is the compactification of a disk minus two points, \( D - \{a, b\} \), whose boundary contains \( P_1 \) and \( P_2 \) (see figure 5). Let \( \hat{f}_1 \) and \( \hat{f}_2 \) be shift homeomorphisms on Plykin attractors \( P_1 \) and \( P_2 \). The expansive homeomorphisms \( F \) on \( X \) uses the shift homeomorphism \( \hat{f}_1 \) when restricted to \( P_1 \) and the inverse of the shift homeomorphism \( \hat{f}_2^{-1} \) when restricted to \( P_2 \). If two distinct points \( x, y \in D - \{a, b\} \) have different 1st coordinates, then it is shown that there is a positive integer \( n \) such that \( d_X(F^n(x), F^n(y)) > c \). On the other hand if \( x, y \) have different 2nd coordinates, then it is shown that there is a positive integer \( n \) such that \( d_X(F^{-n}(x), F^{-n}(y)) > c \), where \( c \) is the expansive constant.

The construction of \( X \) begins by creating 2 homeomorphic continua \( Y_1, Y_2 \) with inverse limits that use single bonding maps \( h_1, h_2 \) on factor spaces \( G_1, G_2 \). To construct the factor space, define
\[ T = \{(x, y)| x \geq 0, y \geq 0, x + y \leq 2 \} \]
\[ L_1 = \{(x, 0)| 2 \leq x \leq 2.5 \}, \quad L_2 = \{(0, y)| 2 \leq y \leq 2.5 \} \]
\[ P_1 = \{(x, p_1)| x \in P \}, \quad P_2 = \{(p_2, y)| y \in P \}. \]

Then let \( G_1 = T \cup L_1 \cup P_1 \), where the point \((2.5, 0)\) in \( L_1 \) is identified to the point \((A, p_1)\) in \( P_1 \) and let \( G_2 = T \cup L_2 \cup P_2 \), where the point \((0, 2.5)\) in \( L_2 \) is identified to the point \((p_2, A)\) in \( P_2 \) (see Figures 3 and 4).

To define distance on \( G_1 \) and \( G_2 \), let \( \pi_{G_1}^k, \pi_{G_2}^k \) be the projection maps on the \( k \)th coordinate for \( G_1 \) and \( G_2 \), where \( k \in \{1, 2\} \). Also for \( k \in \{1, 2\} \) and \( x, y \in P_k \) define
\[ d_{G_k}(x, y) = d_{P_k}(x, y) = d_P(\pi_{G_k}^k(x), \pi_{G_k}^k(y)). \]
If \( x, y \in T \cup L_k \), define
\[
\text{d}_{G_k}(x, y) = \text{d}_{T \cup L_k}(x, y) = |\pi_{G_k}^1(x) - \pi_{G_k}^1(y)| + |\pi_{G_k}^2(x) - \pi_{G_k}^2(y)|.
\]
If \( x \in T \cup L_1 \) and \( y \in P_1 \), define
\[
\text{d}_{G_1}(x, y) = \text{d}_{T \cup L_1}(x, (2.5, 0)) + \text{d}_{P_1}((A, p_1), y).
\]
If \( x \in T \cup L_2 \) and \( y \in P_2 \), define
\[
\text{d}_{G_2}(x, y) = \text{d}_{T \cup L_2}(x, (0, 2.5)) + \text{d}_{P_2}((p_2, A), y).
\]
To define the bonding maps \( h_1, h_2 \), let \( \phi, \psi : [0, 2] \rightarrow [0, 2] \) by
\[
\phi(z) = \begin{cases} 
2z & 0 \leq z < .5 \\
z + .5 & .5 \leq z < 1 \\
1 + .5z & 1 \leq z \leq 2
\end{cases}
\]
\[
\psi(z) = \begin{cases} 
.5z & 0 \leq z < 1 \\
z - .5 & 1 \leq z < 1.5 \\
2(z - 1) & 1.5 \leq z \leq 2
\end{cases}
\]
Define \( g_1 : T \cup L_1 \rightarrow G_1 \) by
\[
g_1(x, y) = \begin{cases} 
(\phi(x), \psi(y)) & x \leq 1.5 \\
(x + .25, (1.75 - x)\psi(y)) & 1.5 < x \leq 1.75 \\
(2(x - 1.75) + 2, 0) & 1.75 < x \leq 2 \\
(x, p_1) & 2 < x \leq 2.5
\end{cases}
\]
Define \( g_2 : T \cup L_2 \rightarrow G_2 \) by
\[
g_2(x, y) = \begin{cases} 
(\psi(x), \phi(y)) & y \leq 1.5 \\
((1.75 - y)\phi(x) + .25, y) & 1.5 < y \leq 1.75 \\
(0, 2(x - 1.75) + 2) & 1.75 < y \leq 2 \\
(p_2, y) & 2 < y \leq 2.5
\end{cases}
\]
Let \( h_1 : G_1 \rightarrow G_1 \) be defined by
\[
h_1(w) = \begin{cases} 
g_1(\pi_{G_1}^1(w), \pi_{G_1}^2(w)) & w \in T \cup L_1 \\
(f(\pi_{G_1}^1(w)), p_1) & w \in P_1
\end{cases}
\]
Let \( h_2 : G_2 \rightarrow G_2 \) be defined by
\[
h_2(w) = \begin{cases} 
g_2(\pi_{G_2}^1(w), \pi_{G_2}^2(w)) & w \in T \cup L_2 \\
(p_2, f(\pi_{G_2}^1(w))) & w \in P_2
\end{cases}
\]
The following 2 lemmas examines the movement of points under the bonding maps \( h_1 \) and \( h_2 \).
Lemma 5. Suppose that \( k \in \{1, 2\} \) and that \((r, s) \in T \cup L_k\) such that \( r \neq 0 \). Then there exists a positive integer \( n \) such that \( h_k^{n-1}(r, s) \in T \cup L_k \) and \( h_k^n(r, s) \in P_k \).

Proof. The proof is shown for \( k = 1 \). The proof is similar for \( k = 2 \). Here \( h_1((r, s)) = g_1((r, s)) \). So it suffices to find an \( n \) such that \( g_1^n(r, s) \in P_1 \).
Case 1. Suppose \( (r, s) \in L_1 \). Then \( g_1((r, s)) = (r, p_1) \in P_1 \). So take \( n = 1 \).

Case 2. Suppose \( (r, s) \in T \) such that \( 1.75 < r \leq 2 \). Then \( g_1((r, s)) \in L_1 \). Now Case 1 applies. Here take \( n = 2 \).

Case 3. Suppose \( (r, s) \in T \) such that \( 1.5 < r \leq 1.75 \). Then \( 1.75 < \pi^1_{G_1}(g_1((r, s))) \leq 2 \), so Case 2 applies. Now take \( n = 3 \).

Case 4. Suppose \( (r, s) \in T \) such that \( 1 < r \leq 1.5 \). Then \( \pi^1_{G_1}(g_1((r, s))) = \phi(r) = 1 + .5r \). So, \( 1 < \pi^1_{G_1}(g_1((r, s))) = 1 + .5r \leq 1.75 \) and hence, Case 3 applies. Here, let \( n = 4 \).

Case 5. Suppose \( (r, s) \in T \) such that \( .5 < r \leq 1 \). Then \( \pi^1_{G_1}(g_1((r, s))) = \phi(r) = r + .5 \). So, \( 1 < \pi^1_{G_1}(g_1((r, s))) = r + .5 \leq 1.5 \). Thus, Case 4 applies, so let \( n = 5 \).

Case 6. Suppose \( (r, s) \in T \) such that \( 0 < r \leq .5 \). Then \( \pi^1_{G_1}(g_1((r, s))) = \phi(r) = 2r \). Let \( m \) be an integer such that \( .5 < 2^m r \leq 1 \). Then, \( .5 < \pi^1_{G_1}(g_1((r, s))) \leq 2^m r \leq 1 \). Hence, let \( n = m + 5 \).

Lemma 6. Suppose that \( k \in \{1, 2\} \) and \( (r, s) \in T \cup L_k \). Then for every \( \epsilon > 0 \), there exists a negative integer \( n \) such that \( \pi^1_{G_1}(h^k_n(r, s)) < \epsilon \).

Proof. Proof follows from doing proof of Lemma 5 in reverse. \( \square \)

Lemmas 7 and 8 show how \( h_1 \) and \( h_2 \) expands the distances between points.

Lemma 7. Suppose \( (r, s), (x, y) \in T \cup L_1 \) such that \( r \neq x \). Then there exists a positive integer \( n \) such that \( d_{G_1}(h^1_n(r, s), h^1_n(x, y)) \geq .0625 \).

Proof. Assume \( r > x \).

Case 1. Suppose \( x = 0 \). We may assume that \( r < .0625 \). Let \( n \) be a positive integer such that \( 2^{n-1} r < .0625 \leq 2^n r \). Then \( d_{G_1}(h^1_n(r, s), h^1_n(0, y)) \geq |\pi^1_{G_1}(h^1_n(r, s)) - \pi^1_{G_1}(h^1_n(0, y))| = |(\phi^n(r) - \phi^n(0))| = 2^n r \geq .0625 \).
Case 2. Suppose $0 < x < r$. Then by Lemma 5, we may choose $n$ to be a positive integer such that $h^{n-1}_1(r, s) \in T \cup L_1$ and $h^n_1(r, s) \in P_1$.  

Case 2a. Suppose $h^n_0(x, y) \in P_1$. Let $r_n = \pi^{1}_{G_1}(h^n_0(r, s))$ and $x_n = \pi^{1}_{G_1}(h^n_0(x, y))$. Then $r_n$ and $x_n$ are distinct elements of $[2, 2.5]_P$. Since $f : [2, 2.5]_P \rightarrow [2.5, 3.5]_P$ is one-to-one, $f(r_n)$ and $f(x_n)$ are distinct elements of $[2.5, 3.5]_P$. Also, since $f : [2.5, 3.5]_P \rightarrow [0.5]_P \cup [2, 3]_P \cup [3.5, 4]_P$ is one-to-one, $f^2(r_n)$ and $f^2(x_n)$ are distinct elements of $[0.5]_P \cup [2, 3]_P \cup [3.5, 4]_P$.

It follows from Lemma 3 that there exists a $k \geq 0$ such that $d_P(f^k(r_n), f^k(r_n)) \geq 0.0625$. Hence $d_{G_1}(h^{n+k}(r, s), h^{n+k}(x, y)) = d_{P_1}((f^k(r_n), p_1), (f^k(x_n, p_1)) = d_P(f^k(r_n), f^k(r_n)) \geq 0.0625$.

Case 2b. Suppose $h^n_0(x, y) \not\in P_1$. We may assume that

$$d_{G_1}(\pi^{1}_{G_1}(h^n_0(r, s)), \pi^{1}_{G_1}(h^n_0(x, y))) \leq 0.0625.$$

Hence, $\pi^{1}_{G_1}(h^n_0(x, y)) \in [2.4375, 2.5]_{L_1}$ and $\pi^{1}_{G_1}(h^{n+1}_1(r, s)) \in [2, 2.0625]_P$. Then $x_{n+1} = \pi^{1}_{G_1}(h^{n+1}_1(x, y)) \in [2.4375, 2.5]_{L_1}$ and $r_{n+1} = \pi^{1}_{G_1}(h^{n+1}_1(r, s)) \in [2, 2.0625]_P$. Then the proof is similar to Case 2a. \hfill \Box

Lemma 8. Suppose $(r, s), (x, y) \in T \cup L_2$ such that $s \not= y$. Then there exists a positive integer $n$ such that $d_{G_2}(h^n_2(r, s), h^n_2(x, y)) \geq 0.0625$.

Proof. Proof is similar to proof of Lemma 7. \hfill \Box

Let $Y_1 = \lim\{G_1, h_1\}_{i=1}^{\infty}$, and $Y_2 = \lim\{G_2, h_2\}_{i=1}^{\infty}$. Let $k \in \{1, 2\}$. Each element $\tilde{w} \in Y_k$ is an infinite sequence of ordered pairs $(x_1, y_1), (x_2, y_2), ...$ where $h_k(x_i, y_i) = (x_{i-1}, y_{i-1})$. Let $H_1 : Y_1 \rightarrow Y_1$ and $H_2 : Y_2 \rightarrow Y_2$ be shift homeomorphisms induced from the inverse limit constructions.

Define projection maps $\{\pi^{1}_{Y_k}, \pi^{i, 1}_{Y_k}, \pi^{i, 2}_{Y_k}\}$ such that $\pi^{i}_{Y_k}(\tilde{w}) = (x_i, y_i)$, $\pi^{i, 1}_{Y_k}(\tilde{w}) = x_i$, and $\pi^{i, 2}_{Y_k}(\tilde{w}) = y_i$. If $\tilde{w}, \tilde{z} \in Y_k$, then $d_{Y_k}(\tilde{w}, \tilde{z}) = \sum_{i=1}^{\infty} 2^{-i}d_{G_k}(\pi^{i}_{Y_k}(\tilde{w}), \pi^{i}_{Y_k}(\tilde{z}))$. 


Lemma 9. Let \( k \in \{1, 2\} \). If \( \hat{w}, \hat{z} \in Y_k \) and \( \pi_{Y_k}^i(\hat{w}) \neq \pi_{Y_k}^i(\hat{z}) \) for some \( i \), then there exists an integer \( n \) such that \( d_{G_k}(H_k^n(\hat{w}), H_k^n(\hat{z})) \geq .0625 \).

Proof. Proof is for \( k = 1 \). Suppose that \( i \) is the smallest positive integer such that \( \pi_{Y_1}^{i_1}(\hat{w}) \neq \pi_{Y_1}^{i_1}(\hat{z}) \). Then for all \( m \geq i \), \( \pi_{Y_1}^{m_1}(\hat{w}) \neq \pi_{Y_1}^{m_1}(\hat{z}) \). Let \( w_1 = \pi_{Y_1}^{i_1}(\hat{w}), w_2 = \pi_{Y_1}^{i_2}(\hat{w}), z_1 = \pi_{Y_1}^{i_1}(\hat{z}), \) and \( z_2 = \pi_{Y_1}^{i_2}(\hat{z}) \). It will be shown that there exists an integer \( n \) such that \( d_{G_1}(h_1^n(w_1, w_2), h_1^n(z_1, z_2)) \geq .0625 \).

Case 1. Suppose that \( h_1^n(w_1, w_2), h_1^n(z_1, z_2) \in P_1 \) for every \( j \). Then \( h_1^n(w_1, w_2) = (f^n(w_1), p_1) \) and \( h_1^n(z_1, z_2) = (f^n(z_1), p_1) \). Hence, by Lemmas 2 and 3, there exists an integer \( n \) such that

\[
d_{G_1}(h_1^n(w_1, w_2), h_1^n(z_1, z_2)) = d_P(f^n(w_1), f^n(z_1)) \geq .0625.
\]

Case 2. Suppose that \( h_1^n(w_1, w_2) \in P_1 \) for every \( j \), but there exists an \( \alpha \) such that \( h_1^n(z_1, z_2) \notin P_1 \). Then, by Lemma 6, there exists an integer \( n \) such that \( \pi_{G_1}^i(h_1^n(z_1, z_2)) < 1 \). Then,

\[
d_{G_1}(h_1^n(w_1, w_2), h_1^n(z_1, z_2)) \geq 2.5 - 1 = 1.5.
\]

Case 3. Suppose that there exists an \( j \) such that \( h_1^j(w_1, w_2), h_1^j(z_1, z_2) \notin P_1 \). Then by Lemma 7, there exists an \( n \) such that \( d_{G_1}(h_1^n(w_1, w_2), h_1^n(z_1, z_2)) \geq .0625 \).

It follows that \( d_{Y_1}(H_1^{n+i}(\hat{w}), H_1^{n+i}(\hat{z})) = d_{Y_1}(H_1^n \circ H_1^i(\hat{w}), H_1^n \circ H_1^i(\hat{z})) \geq d_{G_1}(h_1^n(w_1, w_2), h_1^n(z_1, z_2)) \geq .0625 \). \( \square \)

Let \( X = Y_1 \cup Y_2/T \) such that if \( \hat{w} \in Y_1 \) and \( \hat{z} \in Y_2 \), then \( \hat{w} \) is identified to \( \hat{z} \) if and only if \( \pi_{Y_1}^1(\hat{w}), \pi_{Y_1}^1(\hat{z}) \in T, \pi_{Y_1}^1(\hat{w}) = \pi_{Y_1}^{1,1}(\hat{z}), \) and \( \pi_{Y_1}^{1,2}(\hat{w}) = \pi_{Y_1}^{1,2}(\hat{z}) \). The projection maps are defined as \( \pi_X(\hat{w}) = \pi_{Y_1}(\hat{w}) \) if \( \hat{w} \in Y_1 \) and \( \pi_X(\hat{w}) = \pi_{Y_2}(\hat{w}) \) if \( \hat{w} \in Y_2 \). Let

\[
d_X(\hat{w}, \hat{z}) = \begin{cases} 
  d_{Y_1}(\hat{w}, \hat{z}) & \hat{w}, \hat{z} \in Y_1 \\
  d_{Y_2}(\hat{w}, \hat{z}) & \hat{w}, \hat{z} \in Y_2 \\
  \inf(\{d_{Y_1}(\hat{w}, \hat{q}) + d_{Y_2}(\hat{q}, \hat{z}) | \hat{q} \in Y_1 \cap Y_2\}) & \hat{w} \notin Y_2, \hat{z} \notin Y_1
\end{cases}
\]
Let $F : X \to X$ be defined by

$$F(\hat{w}) = \begin{cases} 
H_1(\hat{w}) & 0 \leq \pi^{1,2}_X(\hat{w}) \leq 1.5 \\
H_1(\hat{w}) & \pi^{1,2}_X(\hat{w}) = p_1 \\
H_2^{-1}(\hat{w}) & 0 \leq \pi^{1,1}_X(\hat{w}) \leq 1.5 \\
H_2^{-1}(\hat{w}) & \pi^{1,1}_X(\hat{w}) = p_2
\end{cases}$$

Notice that if $0 \leq \pi^{1,2}_X(\hat{w}) \leq 1.5$ and $0 \leq \pi^{1,1}_X(\hat{w}) \leq 1.5$, then $H_1(\hat{w}) = H_2^{-1}(\hat{w})$. Hence, $F$ is a homeomorphism.

**Theorem 10.** $F : X \to X$ is an expansive homeomorphism.

**Proof.** Suppose $\hat{w}, \hat{z} \in X$ such that $\hat{w} \neq \hat{z}$. Then there exists an integer $i$ such that $\pi^i_X(\hat{w}) \neq \pi^i_X(\hat{z})$.

**Case 1.** Suppose $\pi^{i,1}_X(\hat{w}) \neq \pi^{i,1}_X(\hat{z})$. Then $\hat{w}, \hat{z} \in Y_1$. Hence, by Lemma 9, there exists an $n$ such that

$$d(F^n(\hat{w}), F^n(\hat{z})) = d_{Y_1}(H_1^n(\hat{w}), H_1^n(\hat{z})) \geq 0.0625.$$ 

**Case 2.** Suppose $\pi^{i,2}_X(\hat{w}) \neq \pi^{i,2}_X(\hat{z})$. Then $\hat{w}, \hat{z} \in Y_2$. Hence, by Lemma 9, there exists an $n$ such that

$$d(F^n(\hat{w}), F^n(\hat{z})) = d_{Y_2}(H_2^n(\hat{w}), H_2^n(\hat{z})) \geq 0.0625.$$

The fact that $X$ is planar follows from an application of the Anderson-Choquet Embedding Theorem [1] and although not difficult to show, it is tedious and will be left out.
REFERENCES


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