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A WEAK ALGEBRAIC STRUCTURE ON TOPOLOGICAL SPACES AND CARDINAL INVARIANTS

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Abstract. The notion of ω-diagonalizability, introduced in [6], is applied in this paper to the theory of cardinal invariants of topological spaces. A basic lemma stablishes a connection between ω-diagonalizability, the π-character, and the countability of the pseudocharacter in a Hausdorff space. This lemma permits one to prove that every ω-diagonalizable locally compact Hausdorff space of countable tightness is first countable, which answers a question asked in [4]. Applications to the study of power-homogeneous compacta of countable tightness are given. In particular, we show that every power-homogeneous locally compact monotonically normal space is first countable. This theorem implies a result of M. Bell in [8].

1. Introduction

Given an algebraic structure (a group structure, a ring structure, a vector space structure, and so on), a standard problem to consider is what kind of topologies can be introduced on this structure so that they fit it nicely (make the operations continuous, for example). It is much more rare that the inverse approach is adopted: given a topological space X, find out if it is possible to introduce

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some algebraic structure on this space in such a manner that the structure fits the topology of X well (again, makes the operations continuous, for example). Of course, such algebraic structures must be of a very general nature, if we want them to exist on rather general (for example, non-homogeneous) spaces. Curiously, it turns out that some such algebraic structures can indeed be defined on many spaces, and, what is more astonishing and important, they can provide effective techniques in situations which, apparently, cannot be treated by purely topological means. The general notion of diagonalizability, defined under this approach in [5] and [6], has been shown to have applications to extensions of continuous functions, to homogeneity problems, and to completions and compactifications (see [4], [5], [6]). In this paper, we continue to work with the notion of ω-diagonalizability, introduced in [6], and provide further applications of it to power homogeneity and to the theory of cardinal invariants.

Below, τ always stands for an infinite cardinal number. All spaces considered are assumed to be T_1. A set $A \subset X$ will be called *a G_τ-subset* of X, if there exists a family γ of open sets in X such that $|\gamma| \leq \tau$ and $A = \cap\gamma$. If $x \in X$ and $\{x\}$ is a G_τ-subset of X, we say that x is *a G_τ-point* in X. In this case, we also say that the *pseudocharacter of X at e* does not exceed τ, and write $\psi(e, X) \leq \tau$. Recall that the *character* of a space X at a point x does not exceed τ (notation: $\chi(x, X) \leq \tau$) if there exists a base $\mathcal{B}_x$ at x such that $|\mathcal{B}_x| \leq \tau$. A *compactum* is a compact Hausdorff space. In general, our terminology and notation follow [15].

2. Cardinal invariants and ω-diagonalizability

In this section, we give some sufficient conditions for a space to be ω-diagonalizable and consider how ω-diagonalizability influences relations beween cardinal invariants.

We say that the *$\pi\tau$-character of a space X at a point $e \in X$* is not greater than τ (and write $\pi\tau\chi(e, X) \leq \tau$) if there exists a family γ of non-empty G_τ-sets in X such that $|\gamma| \leq \tau$ and every open neighborhood of e contains at least one element of γ. Such a family γ is called a *$\pi\tau$-network* at e. If $\tau = \omega$, we use expressions $\pi\omega$-character and $\pi\omega$-network. In particular, if X has a countable π-base at e, then $\pi\omega\chi(e, X) \leq \omega$.

The following notion was introduced in [6]. A τ-*twister* at a point e of a space X is a binary operation on X, written as a product operation, satisfying the following conditions:

a) $ex = xe = x$, for each $x \in X$;

b) for every $y \in X$ and every G_τ-subset V in X containing y, there is a G_τ-subset P of X such that $e \in P$ and $xy \in V$, for each $x \in P$ (that is, $Py \subset V$) (this is the G_τ-continuity at e on the right); and

c) if $e \in \overline{B}$, for some $B \subset X$, then, for every $x \in X$, $x \in \overline{xB}$ (this is the continuity at e on the left).

Proposition 2.1. [6] *If Z is a retract of X and $e \in Z$, and there is a τ-twister at e on X, then there is a τ-twister on Z at e.*

Proof: Fix a retraction r of X onto Z and a τ-twister on X at e, and define a binary operation ϕ on Z by the rule: $\phi(z, h) = r(zh)$. Clearly, the operation ϕ is a τ-twister on Z at e. □

If a space X has a τ-twister at a point $e \in X$, we will say that X is τ-*diagonalizable at* e. A space is called τ-*diagonalizable* if it is τ-diagonalizable at every point. It follows from Proposition 2.1 that a retract of a τ-diagonalizable space is τ-diagonalizable.

We also need the next easy-to-prove statement from [6]:

Proposition 2.2. *If e is a G_τ-point in a space X, then there exists a τ-twister on X at e.*

Proof: Put $ey = y$ for every $y \in X$, and put $xy = x$ for every x and y in X such that $x \neq e$. This operation is obviously a τ-twister on X. □

Theorem 2.3. *Let X be a Hausdorff space. Then $\psi(e, X) \leq \tau$ if and only if $\pi\tau\chi(e, X) \leq \tau$ and X is τ-diagonalizable at e.*

Proof: If $\psi(e, X) \leq \tau$, then X is τ-diagonalizable at e by Proposition 2.2, and $\gamma = \{e\}$ is a $\pi\tau$-network at e. Therefore, $\pi\tau\chi(e, X) \leq \tau$.

Now assume that X is τ-diagonalizable at e and $\pi\tau\chi(e, X) \leq \tau$. Fix a τ-twister at e and a $\pi\tau$-network γ at e. Take any $V \in \gamma$ and fix $y_V \in V$. There exists a G_τ-set P_V such that $e \in P_V$ and $P_V y_V \subset V$. Put $Q = \cap\{P_V : V \in \gamma\}$. Clearly, Q is a G_τ-set and $e \in Q$.

Claim: $Q = \{e\}$. Assume the contrary. Then we can fix $x \in Q$ such that $x \neq e$. Since X is Hausdorff, there exist open sets U and W such that $x \in U$, $e \in W$, and $U \cap W = \varnothing$. Since $xe = x \in U$ and the multiplication on the left is continuous at e, we can also assume that $xW \subset U$.

Since γ is a $\pi\tau$-network at e, there exists $V \in \gamma$ such that $V \subset W$. Then for the point y_V, we have $y_V \in W$, $xy_V \in P_V y_V \subset V \subset W$ and $xy_V \in xV \subset xW \subset U$. Hence, $xy_V \in W \cap U$ and $W \cap U \neq \varnothing$, a contradiction. It follows that $Q = \{e\}$. □

Corollary 2.4. *Suppose that X is a Hausdorff space ω-diagonalizable at a point e. Suppose further that there exists a countable set A of G_δ-points in X such that $e \in \overline{A}$. Then e is also a G_δ-point in X.*

Corollary 2.5. *The space $\beta\omega$ is not ω-diagonalizable at any point e of $\beta\omega \setminus \omega$.*

Now we can identify many other examples of non-ω-diagonalizable spaces. For example, the Alexandroff compactification of an uncountable discrete space is not ω-diagonalizable at the non-isolated point, by Corollary 2.4.

Recall that a space X is of *point-countable type* [1] if every point of X is contained in a compact subspace $F \subset X$ such that F has a countable base of neighborhoods in X. The class of spaces of point-countable type contains all locally compact Hausdorff spaces, all Čech-complete spaces, and all p-spaces.

Since the character and the pseudocharacter coincide in Hausdorff spaces of point-countable type ([1], [15]), the next statement follows from Theorem 2.3.

Proposition 2.6. *Suppose that X is a Hausdorff space of point-countable type and $e \in X$. Then the following conditions are equivalent:*

a) X has a base of cardinality $\leq \tau$ at e; and

b) X is τ-diagonalizable at e and has a π-base at e of cardinality $\leq \tau$.

We also need the following statement:

Proposition 2.7. *If X is a Hausdorff space of point-countable type and the tightness of X is countable, then the $\pi\omega$-character of X is also countable.*

Proof: Take any $x \in X$, and fix a compact subspace F of X such that $x \in F$ and F is a G_δ-subset of X. Since $t(F) \leq \omega$ and F is compact, there exists a countable π-base η of the space F at x (by a theorem of B. E. Šapirovskiĭ [25]). Every $P \in \eta$ is a G_δ-subset in X, since F is a G_δ-subset of X. Therefore, η is a $\pi\tau$-network of X at x. Hence, $\pi\omega\chi(x, X) \leq |\eta| \leq \omega$. □

Theorem 2.8. *If X is an ω-diagonalizable Hausdorff space of point-countable type, then the tightness of X is countable if and only if X is first countable.*

Proof: Indeed, if the tightness of X is countable, then the $\pi\omega$-character of X is also countable, by Proposition 2.7, and it remains to apply Theorem 2.3. □

Theorem 2.8 answers a question from [6]. This theorem has a pointwise version which we can prove only under (CH) or something similar to it.

Theorem 2.9. *Under (CH), if X is a Hausdorff space of point-countable type and is ω-diagonalizable at a point $x \in X$, then the tightness of X at x is countable if and only if X is first countable at x.*

Proof: Indeed, if the tightness of X is countable at x, then, under (CH), the $\pi\omega$-character of X at x is also countable, by a result of Šapirovskiĭ [25], and it remains to apply Proposition 2.6. □

Under Generalized Continuum Hypothesis, a similar statement holds for any cardinal number τ.

The next result shows that hereditary normality and ω-diagonalizability rarely go together in compacta. First, a definition. A compact Hausdorff space F will be called *Tychonoff small* if F cannot be mapped continuously onto the Tychonoff cube I^{ω_1}. The *Tychonoff number* of a space X will be said to be countable if every compact subspace of X is Tychonoff small (notation: $Tych(X) = \omega$). Clearly, every compactum of countable tightness is Tychonoff small. Also, every hereditarily normal compactum is Tychonoff small, since I^{ω_1} is not hereditarily normal.

Theorem 2.10. *If X is an ω-diagonalizable Hausdorff space of point-countable type, and the Tychonoff number of X is countable, then X is first countable at a dense set of points.*

Proof: Take any $x \in X$ and any open neighborhood Ox of x. Since X is a space of point-countable type, we can fix a compact subspace F of X such that $x \in F \subset Ox$ and F has a countable base of neighborhoods in X. The space F is compact and cannot be mapped continuously onto the Tychonoff cube I^{ω_1}. By a theorem of Šapirovskiĭ [26], it follows that the set H of all points $y \in F$ at which the space F has a countable π-base is dense in F. Fix any $y \in H$. Clearly, $\pi\omega\chi(y, X) \leq \omega$. Since X is ω-diagonalizable at y, it follows from Theorem 2.3 that y is a G_δ-point in X. Under the restrictions on X, this implies that X is first countable at y. □

Corollary 2.11. *Every hereditarily normal ω-diagonalizable space X of point-countable type is first countable at a dense set of points.*

Example 2.12. The ordinal space $\omega_1 + 1$ is a hereditarily normal ω-diagonalizable compactum [4]. This space is first countable at a dense set of points but not at all points. Thus, the conclusion in Corollary 2.11 cannot be strengthened in the obvious way.

The approach we follow can be used to obtain some information on the structure of compacta in semitopological groups. Recall that a *semitopological group* is a group with a topology such that all left and all right translations in the group are homeomorphisms. Clearly, every semitopological group is ω-diagonalizable at the neutral element by the product operation given in the group. Since every semitopological group is a homogeneous space, it is ω-diagonalizable at every point. The following well known fact shows how strong the influence of an algebraic structure can be on the topology of a space: If G is a topological group and the space G contains a non-empty compact subspace F with a countable base of neighborhoods in G, then G is a paracompact p-space (see [22]) and F is a dyadic compactum. For semitopological groups, we have a weaker, but still curious, result:

Theorem 2.13. *Suppose that G is a Hausdorff semitopological group and that F is a non-empty compact subspace of G with a countable base of neighborhoods in G. Suppose further that F is Tychonoff small. Then the space G is first countable, the diagonal in $G \times G$ is a G_δ-set, and F is metrizable.*

Proof: Indeed, the $\pi\omega$-character of F at some point is countable, since F is Tychonoff small [25]. Since F is a G_δ-set in G, it follows that the $\pi\omega$-character of G is countable at every point of F. Hence, by homogeneity of G, the $\pi\omega$-character of G is countable at every point of G. Since G is ω-diagonalizable, it follows from Theorem 2.3 that each point of G is a G_δ-point. Clearly, G is a Hausdorff space of point-countable type. Therefore, G is first countable. It was shown by Y.-Q. Chen that every first countable Hausdorff semitopological group G has a G_δ-diagonal [10]. It remains to refer to the well known fact that every compact Hausdorff space with a G_δ-diagonal is metrizable [15]. □

Corollary 2.14. *Suppose that G is a Hausdorff semitopological group of point-countable type such that the tightness of G is countable. Then the space G is first countable, and the diagonal in $G \times G$ is a G_δ-set.*

Corollary 2.15. *Suppose that G is a Hausdorff semitopological group such that G is a paracompact p-space and the tightness of G is countable. Then the space G is metrizable.*

Proof: Every paracompact p-space with a G_δ-diagonal is metrizable [9], and every p-space is a space of point-countable type [1]. Hence, the conclusion follows from Corollary 2.14. □

Remark 2.16. *Compactness in Theorem 2.13 cannot be replaced by countable compactness (consider the Σ-product of uncountably many copies of the discrete group $D = \{0, 1\}$).*

The natural question of whether $\beta\omega \setminus \omega$ is ω-diagonalizable at some point turns out to be rather delicate. In this direction, we have:

Proposition 2.17. *Suppose that z is a point in $X = \beta\omega \setminus \omega$ such that $z \in \overline{A} \setminus A$, for some countable discrete subspace A of X. Then X is not ω-diagonalizable at z.*

Proof: Assume that X is ω-diagonalizable at z. The subspace $Z = \overline{A}$ is a retract of $\beta\omega$ [19]. Therefore, Z is a retract of X. Since X is ω-diagonalizable at z, it follows from Proposition 2.1 that Z is ω-diagonalizable at z. Since Z is homeomorphic to $\beta\omega$, and z is not a G_δ-point in Z, it follows that Z is not ω-diagonalizable at z (see Corollary 2.4), a contradiction. □

There are many points in $\beta\omega \setminus \omega$, such as in Proposition 2.17. Hence, $\beta\omega \setminus \omega$ at some points is not ω-diagonalizable. However, it was shown in [5] that, consistently, $\beta\omega \setminus \omega$ is ω-diagonalizable at some point. Below, we prove a slightly more general statement than in [5]. A point x of a space X is called *a chain-point* [6] if there exists a family γ of open subsets of X satisfying the following conditions:

a) $\cap\gamma = \cap\{\overline{V} : V \in \gamma\} = \{x\}$; and

b) γ is a chain; that is, for any $V, U \in \gamma$, either $V \subset U$ or $U \subset V$.

Any such family γ will be called *a strong chain* at x.

Now we need a slightly stronger version of ω-diagonalizability. A *point-continuous twister* at a point e of a space X is a binary operation on X satisfying the following conditions:

a) $ex = xe = x$, for each $x \in X$;

b) The multiplication is jointly continuous at (x, y) whenever $x = e$ or $y = e$.

A space X will be called *point-continuously diagonalizable* at $e \in X$ if there exists a point-continuous twister on X at e. A space X is said to be *point-continuously diagonalizable* if it is point-continuously diagonalizable at every point of X. We need the following statement from [6]. For the sake of completeness, we present the simple proof of it as well.

Proposition 2.18. [6] *Any space X is point-continuously diagonalizable at any chain-point in X.*

Proof: Let e be a chain-point in X and γ a strong chain at e. Take any $x, y \in X$. Put $xy = y$ if there exists $V \in \gamma$ such that $x \in V$ and $y \notin \overline{V}$. Otherwise, put $xy = x$. In particular, it follows that $ey = y$, for each $y \in X$, and $xe = x$, for each $x \in X$. It cannot occur that, for some $V, U \in \gamma$ and $x, y \in X$, $x \in V$, $y \notin V$, $y \in U$, and $x \notin U$, since γ is a chain. Therefore, the definition of multiplication is correct. Let us check that the binary operation so defined is a point-continuous twister on X at e.

Case 1: Assume that $a \neq e$. Then there exists $V \in \gamma$ such that $a \in Oa = X \setminus \overline{V}$. Then $xy = y \in Oa$, for any $x \in V$ and any $y \in Oa$. Thus, the multiplication is jointly continuous at (e, a). It is also clear that $yx = y \in Oa$, for each $y \in Oa$ and each $x \in V$. Therefore, the multiplication is jointly continuous at (a, e) as well.

Case 2: The multiplication at (e, e) is continuous, since whenever W is an open neighborhood of e and x, y are any elements of W, the product xy is either x or y and, therefore, belongs to W.

Hence, X is point-continuously diagonalizable at e. □

Clearly, point-continuous diagonalizability implies ω-diagonalizability. We also have:

Proposition 2.19. *Every regular space X is point-continuously diagonalizable at any G_δ-point e in X.*

Proof: Indeed, in a regular space X every G_δ-point is a chain-point. It remains to apply Proposition 2.18. □

Recall that a point $x \in X$ is *a P-point* in X if, for every countable family γ of open neighborhoods of x, the intersection of γ contains an open neighborhood of x.

Theorem 2.20. *If $\omega_1 = 2^\omega$, then $\beta\omega \setminus \omega$ is point-continuously diagonalizable at any P-point in $\beta\omega \setminus \omega$. Therefore, under (CH), $\beta\omega \setminus \omega$ is point-continuously diagonalizable (and hence, ω-diagonalizable) at some point.*

Proof: Under (CH), there exists a P-point in $\beta\omega \setminus \omega$ [24]. Clearly, (CH) implies that every P-point in $\beta\omega \setminus \omega$ is a chain-point, since the weight of $\beta\omega \setminus \omega$ does not exceed ω_1 in this case. It follows from Proposition 2.18 that, under (CH), $\beta\omega \setminus \omega$ is point-continuously diagonalizable at any P-point. □

Problem 2.21. *Is it true in ZFC that $\beta\omega \setminus \omega$ is ω-diagonalizable (point-continuously diagonalizable) at some point?*

Here is a curious generalization of Theorem 2.20.

Proposition 2.22. *Suppose that X is a space admitting a one-to-one continuous mapping f onto $\beta\omega \setminus \omega$. Then, under (CH), X is point-continuously diagonalizable at some point.*

Proof: The proof of Theorem 2.20 shows that, under (CH), there exists a chain-point y in $\beta\omega \setminus \omega$. The point $x \in X$, which is a preimage of y under f, is obviously a chain-point in X, since the preimage of a strong chain at y in $\beta\omega \setminus \omega$ under f is a strong chain in X at x. Therefore, X is point-continuously diagonalizable at x, by Proposition 2.18. □

We also need the following simple fact [6]:

Proposition 2.23. *Suppose that Y is an open subspace of a space X, $e \in Y$, and Y is τ-diagonalizable at e. Then X is also τ-diagonalizable at e.*

Proof: Fix a τ-twister on Y at e. Take any $x, y \in X$. If both x, y are in Y, xy and yx are already defined and we stick to these definitions. Suppose that $x \notin Y$ and $y \in Y$. Then we put $xy = x$ and $yx = x$. If $x \notin Y$ and $y \notin Y$, then we put $xy = x$ and $yx = y$. Since Y is open in X and $e \in Y$, it is clear that the binary operation on X so defined is a τ-twister on X. Hence, X is τ-diagonalizable at e. □

3. ω-DIAGONALIZABILITY AND POWER HOMOGENEITY

Now we are going to apply the techniques developed above and in [6], based on the notion of τ-diagonalizability, to study power homogeneity. Recall, that a space X is said to be *power-homogeneous* if some power X^λ of X is homogeneous. The notion was introduced by E. van Douwen in [12], who, it seems, was motivated by a question of M. Bell. Recently some original results on power homogeneity were obtained in [18], [8], [14], [19], [21], [5], [6], and [28]. One of basic questions on power homogeneity, which remains unsolved, can be formulated as follows:

Problem 3.1. *Identify a power-homogeneous compact Hausdorff space X such that the Souslin number $c(X)$ of X is greater than 2^ω, or prove that such X doesn't exist.*

This question is related in an obvious way to the famous problem, posed by van Douwen [12], whether the cellularity can exceed the cardinality of the continuum in homogeneous compacta. Some versions of the next statement were instrumental in [5] and [6].

Proposition 3.2. *If a power-homogeneous space X is τ-diagonalizable at some point, then X is τ-diagonalizable at every point.*

Proof: Take a cardinal number $\lambda > 0$ such that X^λ is homogeneous. The space X^λ is τ-diagonalizable at some point. (Define a τ-twister on X^λ as a result of coordinatewise application of the τ-twister given on X.) Since X^λ is homogeneous, it follows that X^λ is τ-diagonalizable at every point. Therefore, by Proposition 2.1, X is τ-diagonalizable at every point, since X is a retract of X^λ. □

Now we will prove the following partial generalization of van Douwen's theorem that $\beta\omega\setminus\omega$ is not power-homogeneous [12]. Notice, that we use (CH), while van Douwen proved his result in ZFC.

Theorem 3.3. *Assume (CH). Suppose that $\beta\omega\setminus\omega$ is an open subspace of a Hausdorff space X. Then X is not power-homogeneous.*

Proof: By Proposition 2.23 and Theorem 2.20, the space X is ω-diagonalizable at some point. If X is power-homogeneous, then X is ω-diagonalizable at every point. Since X is Hausdorff, the compact subspace $\beta\omega\setminus\omega$ is closed in X. Since $\beta\omega\setminus\omega$ is open in X, it follows that $\beta\omega\setminus\omega$ is a retract of X. Hence, $\beta\omega\setminus\omega$ is ω-diagonalizable at every point, a contradiction with Proposition 2.17. □

Problem 3.4. *Can one drop (CH) in the last theorem?*

Problem 3.5. *Is there a compact Hausdorff space Y such that the space $(\beta\omega\setminus\omega)\times Y$ is homogeneous?*

In power-homogeneous compacta X, the relationship between the tightness $t(X)$ and the character $\chi(X)$ simplifies greatly. Indeed, we have:

Theorem 3.6. *For every power-homogeneous Hausdorff space of point-countable type, $\chi(X)\le 2^{t(X)}$.*

Proof: Put $\tau = t(X)$. By the fundamental lemma on tightness in compacta [2], there exist sets $A\subset X$ and F such that F is a non-empty compact G_τ-subset of X, $|A|\le\tau$, and $F\subset\overline{A}$. Then the weight of F does not exceed 2^τ. It follows that $\psi(x,X)\le 2^\tau$, for every $x\in F$. Hence, X is 2^τ-diagonalizable at some point. Since X is power-homogeneous, it follows that X is 2^τ-diagonalizable at every point. However, $\pi\tau\chi(X)\le\tau$, since X is a Hausdorff space of point-countable type and $t(X)\le\tau$ (see Proposition 2.7). From Theorem 2.3, it follows that $\psi(X)\le 2^\tau$. Again using the fact that X is a Hausdorff space of point-countable type, we conclude that $\chi(X)\le 2^{t(X)}$. □

Corollary 3.7. *If X is a power-homogeneous compact Hausdorff space of countable tightness, then $|X|\le 2^c$, where $c=2^\omega$.*

An important subclass of the class of spaces of countable tightness constitutes the class of sequential spaces. If, in Corollary 3.7, we replace the restriction on the tightness of X by the assumption that X is sequential, we can considerably strengthen the conclusion.

Corollary 3.8. *If X is a power-homogeneous compact Hausdorff sequential space, then $|X| \leq 2^{\omega}$.*

Proof: Since X is sequential, the tightness of X is countable. Therefore, by Theorem 3.6, $\chi(X) \leq 2^{\omega}$. It follows that $|X| \leq 2^{\omega}$, since this is so for every sequential compact Hausdorff space the character of which does not exceed 2^{ω} (see [7]). □

Problem 3.9. *Suppose that X is a power-homogeneous compact Hausdorff space. Is it then true that $|X| \leq 2^{t(X)}$?*

Below we will need the next key result from [6]:

Theorem 3.10. *Suppose that X is a power-homogeneous Hausdorff space of point-countable type. Then, for any cardinal number τ, the set of all $x \in X$ such that $\chi(x, X) \leq \tau$ is closed in X.*

A. Dow [13] showed that, under Proper Forcing Axiom (PFA), every compact Hausdorff space of countable tightness is first countable at a dense set of points. Therefore, under (PFA), every power-homogeneous compact Hausdorff space X of countable tightness is first countable and satisfies $|X| \leq 2^{\omega}$; that is, the answer to the last question is positive in this case. In fact, we can formulate a considerably more general result which easily follows from Theorem 3.10:

Theorem 3.11. *Suppose that (PFA) holds and X is a power-homogeneous Hausdorff space of point-countable type such that the tightness of X is countable. Then X is first countable.*

Dow and E. Pearl have proved that, for any zero-dimensional first countable Hausdorff space X, the space X^{ω} is homogeneous [14]. Therefore, by Theorem 3.11, for any zero-dimensional Hausdorff space X of point-countable type such that $t(X) \leq \omega$, the following conditions are pairwise equivalent under (PFA): 1) X is power-homogeneous; 2) X^{ω} is homogeneous; 3) X is first countable.

Problem 3.12. *Is it consistent with ZFC that, for arbitrary compact Hausdorff space X, the character of X at some point does not exceed the tightness of X?*

Theorem 3.13. *Suppose that X is a power-homogeneous space of point-countable type such that all compact subspaces of X are Tychonoff small. Then $\chi(a, X) = \chi(b, X)$, for any $a, b \in X$. In particular, if X is also first countable at least at one point, then X is first countable.*

Proof: Put $\tau = min\{\chi(x, X) : x \in X\}$. Clearly, we can assume that $\chi(a, X) = \tau$. Then X is τ-diagonalizable at a. Since X is power-homogeneous, X is τ-diagonalizable at every other point of X as well, by Proposition 3.2. Arguing as in the proof of Theorem 2.10, we come to the conclusion that the set Y of all $x \in X$ such that $\pi\tau\chi(x, X) \leq \tau$ is dense in X. It follows from Theorem 2.3 that $\psi(y, X) \leq \tau$, for every $y \in Y$, since X is τ-diagonalizable at y. We also have $\psi(x, X) = \chi(x, X)$, for each $x \in X$, since X is a Hausdorff space of point-countable type. Therefore, $\chi(y, X) \leq \tau$, for every $y \in Y$. Since Y is dense in X, it follows from Theorem 3.10 that $\chi(x, X) \leq \tau$, for every $x \in X$. Hence, by the choice of τ, $\chi(b, X) = \tau = \chi(a, X)$, and the argument is complete. □

Corollary 3.14. *Suppose that X is a hereditarily normal power-homogeneous space of point-countable type. Then $\chi(a, X) = \chi(b, X)$ for any $a, b \in X$. In particular, if X is also first countable at least at one point, then X is first countable.*

Corollary 3.15. *Suppose that X is a power-homogeneous Hausdorff space of point-countable type such that all compact subspaces of X are countably tight. Then $\chi(x, X) = \chi(y, X)$ for any x, y in X.*

Problem 3.16. [5] *Is it true in ZFC that every power-homogeneous compact Hausdorff space of countable tightness is first countable?*

Problem 3.17. *Is every power-homogeneous hereditarily normal compact Hausdorff space X first countable?*

The answers to the last two questions are not known even when the space X is homogeneous. For some recent results on homogeneous hereditarily normal compacta and further questions about them, see [21]. In an important special case, a positive answer to Problem 3.17 can be derived from some deep results of Bell and M. E. Rudin. Bell [8] has shown that if a compactum X is a continuous image of some linearly ordered compact space, and X is power-homogeneous, then X must be first countable. On the other hand, M. E. Rudin [23] has shown that a compactum X can be represented as a continuous image of some linearly ordered compact space if and only if X is a monotonically ordered compactum. It follows that every power-homogeneous monotonically normal compactum is first countable. Since the arguments in [8] and especially in [23] are far from easy, we present here an elementary, direct, and complete, proof of the following more general results (which do not seem to reduce to the compact case).

Theorem 3.18. *Suppose that X is a locally compact monotonically normal Hausdorff space, and that Y is a Hausdorff space ω-diagonalizable at least at one point. Suppose also that $X \times Y$ is power-homogeneous. Then X is first countable at a dense set of points.*

Proof: Fix a non-empty open set U in X such that $\overline{U}$ is compact. Since $\overline{U}$ is also monotonically normal, Theorem 3.12 (iii) in [16] implies that there is a chain-point e in $\overline{U}$ such that $e \in U$. Then, clearly, e is a chain-point in X. It follows that X is ω-diagonalizable at e. Since Y is also ω-diagonalizable at some point y, the space $X \times Y$ is ω-diagonalizable at (e, y). Since $X \times Y$ is power-homogeneous, it follows that $X \times Y$ is ω-diagonalizable at all points. Therefore, by Proposition 2.1, the space X is ω-diagonalizable at every point. It remains to apply Theorem 2.10. □

Notice, that the above proof of Theorem 3.18 also shows that the following statement is true:

Proposition 3.19. *Every non-empty monotonically normal locally compact Haudorff space is ω-diagonalizable at some point.*

Now, to establish our main result on homogeneity of products of monotonically normal locally compact spaces, we need the following obvious corollary of Theorem 13 in [6]. (We could also refer to Theorem 6.20 in [5].)

Theorem 3.20. *Suppose that X is the product of a family $\gamma = \{X_\alpha : \alpha \in A\}$ of non-empty Hausdorff spaces X_α of point-countable type each of which is first countable at least at one point and that X is homogeneous; then, for each $\alpha \in A$, the set of all points at which X_α is first countable is closed in X_α.*

Theorem 3.21. *If X is the product of a family $\gamma = \{X_\alpha : \alpha \in A\}$ of non-empty locally compact monotonically normal spaces X_α, and X is homogeneous, then every $X_\alpha \in \gamma$ is first countable.*

Proof: Fix $\alpha \in A$ and put $B = A \setminus \{\alpha\}$, $Y = X_\alpha$, and $Z = \Pi\{X_\beta : \beta \in B\}$. Then $X = Y \times Z$. It follows from Proposition 3.19 that the space Z is ω-diagonalizable at some point. Therefore, by Theorem 3.18, the space $X_\alpha = Y$ is first countable at a dense set of points. Since this is true for each $\alpha \in A$, it follows from Theorem 3.20 that each X_α is first countable. □

Corollary 3.22. *If a locally compact monotonically normal space X is power-homogeneous, then X is first countable.*

Problem 3.23. *Suppose that X is a monotonically normal power-homogeneous space X of point-countable type. Is then X first countable?*

In connection with Theorem 3.18, the following question seems to be natural:

Problem 3.24. *Is there a compact Hausdorff space Y such that the product $(\omega_1 + 1) \times Y$ is homogeneous?*

Note, that if Y is any Hausdorff space (not necessarily compact) first countable at least at one point, then the space $(\omega_1 + 1) \times Y$ is not homogeneous. This easily follows from the results in this article. On the other hand, there exists a Tychonoff space Y such that $(\omega_1 + 1) \times Y$ is homogeneous, according to a result of V. V. Uspenskiĭ in [27].

Among monotonically normal spaces are all linearly ordered spaces. It was shown in [6] that every power-homogeneous linearly ordered space of point-countable type is, indeed, first countable. This result gives some hope that Problem 3.23 might get a positive answer.

The next result provides yet another condition, in terms of cardinal invariants, for a Hausdorff space of point-countable type to be non-power-homogeneous.

Theorem 3.25. *Suppose that X is a Hausdorff space of point-countable type and τ is an isolated infinite cardinal number such that $\tau \leq \chi(X)$ and $|X| < 2^\tau$. Then X is not power-homogeneous.*

Proof: Assume that X is power-homogeneous. Since τ is an isolated infinite cardinal, it follows from Theorem 3.10 that the set Y of all points $x \in X$ at which the character of X is smaller than τ is closed in X. Hence, the set $Z = X \setminus Y$ is open. Since $\tau \leq \chi(X)$ and τ is isolated, the set Z is not empty. Since X is a Hausdorff space of point-countable type, there exists a non-empty compact subspace F of X with a countable base of neighborhoods in X such that $F \subset Z$. It follows that $\chi(z, F) \geq \tau$, for every $z \in F$. The well known theorem of Čech and Pospišil [15] now implies that $|F| \geq 2^{\tau}$. Hence, $|X| \geq 2^{\tau}$, a contradiction. □

Below $H(X)$ is the space of non-empty closed subsets of a space X, in the Vietoris topology [15]. Recall that a *Corson compactum* is a compact subspace of the Σ-product of separable metrizable spaces (see [3], [5]). It easily follows from this definition (and is well known) that each Corson compactum X is *monolithic*; that is, the weight of the closure of an arbitrary subset A of X does not exceed the cardinality of A. The tightness of any Corson compactum is countable, since the tightness of the Σ-product of any family of separable metrizable spaces is countable (see [15, 3.10.D]).

Theorem 3.26. *Suppose that X is a Corson compactum such that $H(X)$ is power-homogeneous. Then X is metrizable.*

Proof: If X is a Corson compactum, then there exists a dense subspace Y of X such that X is first countable at every point of Y. Indeed, Theorem 1 in section 4 of [3] says that every monolithic compactum of countable tightness is first countable at a dense set of points. The set Z of all finite subsets of Y is a dense subspace of $H(X)$, and $H(X)$ is, obviously, first countable at each $F \in Z$. However, by Theorem 3.10, the space $H(X)$ is character closed, since it is compact and power-homogeneous. It follows that the space $H(X)$ is first countable. By a result of M. Čoban in [11], this implies that X is separable. Hence, X is metrizable, since X is monolithic. □

Clearly, the above theorem holds for all monolithic compacta of countable tightness.

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References

[1] A. V. Arhangel'skii, *On a class of spaces containing all metric spaces and all locally bicompact spaces*, Soviet. Math. Dokl. **4** (1963), 751–754.

[2] A. V. Arhangel'skii, *Souslin number and cardinality. The characters of points in sequential bicompacta*, Soviet Math. Dokl. **11** (1970), 255–258.

[3] A. V. Arhangel'skii, *On some spaces that arise in functional analysis*, Russian Math. Surveys **31** (1976), no. 5, 14–30.

[4] A. V. Arhangel'skii, *An operation on topological spaces*, Appl. Gen. Topol. **1** (2000), no. 1, 13–28.

[5] A. V. Arhangel'skii, *Diagonal structures on topological spaces*, Topology Appl. **125** (2002), 419–446.

[6] A. V. Arhangel'skii, *Homogeneity of powers of spaces and the character.* Submitted, August 2003.

[7] A. V. Arhangel'skii and R. Z. Buzyakova, *On some properties of linearly Lindelöf spaces*, Topology Proc. **23** (1998), Summer, 1–11.

[8] M. G. Bell, *Nonhomogeneity of powers of cor images.* Rocky Mountain J. Math. **22** (1992), 805–812.

[9] C. R. Borges, *On metrizability of topological spaces*, Canad. J. Math. **20** (1968), 795–804.

[10] Y.-Q. Chen, *Note on two questions of Arhangel'skii*, Questions Answers Gen. Topology **17** (1999), 91–94.

[11] M. Čoban, *Note sur topologie exponentielle* (French), Fund. Math. **71** (1971), 27–41.

[12] E. K. van Douwen, *Nonhomogeneity of products of preimages and π-weight.* Proc. Amer. Math. Soc. **69** (1978), no. 1, 183–192.

[13] A. Dow, *An introduction to applications of elementary submodels to topology*, Topology Proc. **13** (1988), 17–72.

[14] A. Dow and E. Pearl, *Homogeneity in powers of zero-dimensional first-countable spaces*, Proc. Amer. Math. Soc. **125** (1997), 2503–2510.

[15] R. Engelking, *General Topology.* Warsaw: PWN, 1977.

[16] I. Juhász, *Cardinal functions*, in Recent Progress in General Topology, Ed. M. Hušek and J. van Mill. Amsterdam: Elsevier Science Publishers, 1992. 417–441.

[17] K. Kunen, *Weak P-points in N^**, in Topology, Vol. II. Colloq. Math. Soc. Bolyai János, 23. Amsterdam–New York: North Holland, 1980. 741–749.

[18] K. Kunen, *Large compact homogeneous spaces*, in Handbook of Set-Theoretic Topology, Ed. K. Kunen and J. Vaughan. Amsterdam: North-Holland, 1984. 261–270.

[19] J. van Mill, *An introduction to $\beta\omega$*, in Handbook of Set-Theoretic Topology, Ed. K. Kunen and J. Vaughan. Amsterdam: North-Holland, 1984. 503–567.

[20] J. van Mill, *On the character and π-weight of homogeneous compacta*, Israel J. Math. **133** (2003), 321–338.

[21] J. van Mill, *On the cardinality of power-homogeneous compacta*. preprint, 2002.

[22] W. Roelcke and S. Dierolf, *Uniform Structures on Topological Groups and their Quotients*. New York: McGraw-Hill, 1981.

[23] M. E. Rudin, *Nikiel's conjecture*, Topology Appl. **116** (2001), no. 3, 305–331.

[24] W. Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J. **23** (1956), 409–419.

[25] B. E. Šapirovskiĭ, *The π-character and the π-weight of compact Hausdorff spaces*, Soviet Math. Dokl. **16** (1975), 999–1004.

[26] B. E. Šapirovskiĭ, *Maps onto Tychonoff cubes*, Russian Math. Surveys **35** (1980), no. 3, 145–156.

[27] V. V. Uspenskiĭ, *For any X, the product $X \times Y$ is homogeneous for some Y*, Proc. Amer. Math. Soc. **87** (1983), no. 1, 187–188.

[28] R. de la Vega and K. Kunen, *A compact homogeneous S-space*, Topology Appl. **136** (2004), 123–127.

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