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SYMMETRIC PRODUCTS AS CONES AND PRODUCTS

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Abstract. In this paper we prove that:
(a) if X is a finite graph, then the second symmetric product $F_2(X) = \{\{p, q\} \subset X : p, q \in X\}$ of X is the cone over some continuum Y, if and only if X is a simple n-od or an arc, and
(b) if X is a finite graph, then $F_2(X)$ is a product of two nondegenerate continua, if and only if X is an arc.

1. Introduction

A *continuum* is a compact, connected metric space. Let $\mathbb{N}$ denote the set of positive integers. Let $n \in \mathbb{N}$. Consider the following hyperspaces of a continuum X:

$$C(X) = \{A \subset X : A \text{ is a closed, nonempty connected subset of } X\},$$

$$F_n(X) = \{A \subset X : A \text{ is a nonempty subset of } X \text{ with at most } n \text{ elements}\}.$$

Both hyperspaces are considered with the Hausdorff metric.

The problem of determing those finite dimensional continua X for which $C(X)$ is homeomorphic to cone(X) has been extensively studied. (See [1], [4], [8], [9], [10], [11], [12], [14], [15], [17], [20], [21], [22] and [23].) A detailed discussion about this topic can be found in [11, sections 7 and 80]. The case that X is hereditarily decomposable has been completely solved by S. B. Nadler, Jr. in [17].

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He showed that there exist exactly eight such continua, pictured on page 63 of [11]. In the case that X contains an indecomposable subcontinuum Y, it is known that Y is unique [11, Theorem 80.12], $X - Y$ is arcwise connected [8], and Y has the cone = hyperspace property. A characterization in terms of selections of finite dimensional continua Y with the cone = hyperspace has recently appeared in [9].

Continua X for which there exists a finite dimensional continuum Z such that $C(X)$ is homeomorphic to cone(Z) have been completely described. Using a previous result by S. Macías [15], A. Illanes and M. de J. López [10] give a complete list of that continua X, in the case that X is hereditarily decomposable, and López [14] gives a precise description of that continua X, in the case that X is not hereditarily decomposable.

With respect to products, Illanes [11, Theorem 79.2] has shown that a continuum X has the properties that $C(X)$ is finite dimensional and it is homeomorphic to the product of two nondegenerate continua if and only if X is an arc or a simple closed curve.

The hyperspace $F_n(X)$ is the so called $n^{th}-$ *symmetric product of* X. Symmetric products were introduced by K. Borsuk and S. Ulam in [2]. They proved that, for $I = [0, 1]$ and $n = 1, 2, 3$, $F_n(I)$ is homeomorphic to I^n. It is known that $F_2(S^1)$ is homeomorphic to Möebius strip (see [2, p. 877] or [11, 1.26]). Given an n-od T_n (see the definition below in conventions), it was proved in [3, Lemma 1] that $F_2(T_n)$ is homeomorphic to the cone over a continuum Z.

In this paper we consider the problem of determining the finite graphs X such that $F_2(X)$ is homeomorphic to a cone or to the product of two nondegenerate continua. We prove that:

Theorem 3.11. If X is a finite graph, then $F_2(X)$ is homeomorphic to the product of two nondegenerate continua if and only if X is an arc.

Theorem 3.14. If X is a finite graph, then $F_2(X)$ is homeomorphic to the cone over a continuum Z if and only if X is a simple n-od or an arc.

2. Conventions

A *finite (connected) graph* is a continuum which is a finite union of arcs (called *edges*) such that every two of them meet at a subset of their end points. If X is a finite graph, in X are defined *edges* and *vertices*. The vertices of X are the end points of the edges of X. A finite graph which is different from a simple closed curve is called an *acircular graph*.

We are interested in distinguishing the ramification points of the graph X from the rest of the points, so we assume the each vertex of an acircular graph X is either an end point of X or a ramification point of X. With this restriction, the two end points of an edge of X may coincide and such an edge is a simple closed curve. These edges will be called *loops*. Thus, the edges of X are arcs or simple closed curves, and in X there are only three kinds of edges: loops, edges that contain some end point, and edges joining ramification points.

We assume that the metric d in X is the metric of arc length and each edge of X has length equal to one. The set of ramification points of X is denoted by $R(X)$. A *simple n-od* T_n is a finite graph which is the union of n arcs $J_1, ..., J_n$ such that there exists a point $p \in T_n$ with the property $J_i \cap J_j = \{p\}$, if $i \neq j$, and p is an end point of each one of the arcs J_i. The point p is called the *core of* T_n. A simple 3-od is called a *simple triod*. Given $n, m \in \mathbb{N}$, where $n, m \geq 3$, let $K_{n,m}$ denote the *complete bipartite graph*; that is, $K_{n,m}$ contains $n + m$ vertices $\{v_1, ..., v_n, w_1, ..., w_m\}$ and $n \cdot m$ edges $v_i w_j$, where $v_i w_j$ joins v_i and w_j, for every $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$.

Given a continuum Z and a subset A of Z, $\mathrm{bd}_Z(A)$, $\mathrm{cl}_Z(A)$, and $\mathrm{int}_Z(A)$ denote the respective boundary, closure, and interior of A in Z. Let Z be a continuum and $p \in Z$. Let β be a cardinal number. We say that *p is of order less than or equal to β in Z*, written $\mathrm{ord}(p, Z) \leq \beta$, provided that for each open subset U of Z such that $p \in U$, there exists an open subset V of Z such that $p \in V \subset U$ and $|\mathrm{bd}_Z(V)| \leq \beta$. We say the *$p$ is of order β*, written $\mathrm{ord}(p, Z) = \beta$, provided that $\mathrm{ord}(p, Z) \leq \beta$ and $\mathrm{ord}(p, Z) \not\leq \alpha$ for any cardinal number $\alpha < \beta$. A point $p \in Z$ is called an *end point of Z* provided that $\mathrm{ord}(p, Z) = 1$. A point $p \in Z$ is called a *ramification point of Z* provided that $\mathrm{ord}(p, Z) \geq 3$. If A is a subset of Z, $p \in Z$, and $\varepsilon > 0$, let $B_Z(\varepsilon, p) = \{q \in Z : d_Z(p, q) < \varepsilon\}$ and $N_Z(\varepsilon, A) = \{q \in Z :$ there exists $p \in A$ such that $d_Z(p, q) < \varepsilon\}$. A *Peano continuum* is a locally connected continuum. Given open subsets $U_1, ..., U_m$ of a continuum X, let $\langle U_1, ..., U_m \rangle_n = \{A \in F_n(X) : A \subset U_1 \cup ... \cup U_m$

and $A \cap U_i \neq \emptyset\}$. It is known that the sets of the form $\langle U_1, ..., U_m \rangle_n$ form a basis for the topology of $F_n(X)$.

3. Symmetric products as cones or products

Lemma 3.1. *Let T_n and T_m denote a simple n-od and a simple m-od, respectively. If $2 \leq m < n$, then $T_n \times [0,1]$ cannot be embedded in $T_m \times [0,1]$.*

Proof: Let $X = T_n \times [0,1]$ and $Y = T_m \times [0,1]$; let $T_n = pa_1 \cup ... \cup pa_n$ and $T_m = qb_1 \cup ... \cup qb_m$, where the sets pa_i and qb_j are arcs, $pa_i \cap pa_j = \{p\}$, if $i \neq j$ and $qb_k \cap qb_r = \{q\}$, if $k \neq r$. Let $C_X = \{p\} \times [0,1]$, $C'_X = \{p\} \times (0,1)$, $C_Y = \{q\} \times [0,1]$, and $C'_Y = \{q\} \times (0,1)$. Suppose that there exists an embedding $h : X \to Y$.

Claim. $h(C_X) \subset C_Y$.

Suppose to the contrary that there exists a point $(p,t) \in C_X$ such that $h(p,t) = (y,s)$ and $y \neq q$. We may assume that $y \in qb_1 - \{q\}$. Then there exists a simple n-od $T \subset T_n$ and an arc $J \subset [0,1]$ such that $(p,t) \in T \times J$ and $h(T \times J) \subset (qb_1 - \{q\}) \times [0,1]$. This implies that the product $T \times J$ can be embedded in $[0,1]^2$. By using the Theorem on the Invariance of the Domain [6, Theorem VI 9, p. 95], it can be shown that this is impossible. This contradiction completes the proof of the claim.

Fix a point $x_0 \in C'_X$. Since $h(C'_X)$ is homeomorphic to $\mathbb{R}$ and it is contained in the arc C_Y, we have that $C_Y - h(C'_X)$ is compact and $x_0 \notin h^{-1}(C_Y - h(C'_X))$. Thus, there exists a simple n-od $T \subset T_n$ and an arc $J \subset (0,1)$ such that $x_0 \in T \times J \subset X - h^{-1}(C_Y - h(C'_X))$. Let $D = \{p\} \times J$. Given a point $x \in (T \times J) - D$, $h(x) \notin C_Y - h(C'_X)$. If $h(x) \in C_Y$, then $h(x) \in h(C'_X)$. Thus, $x \in C'_X \cap (T \times J) = D$. This contradicts the choice of x and proves that $h(x) \notin C_Y$. We have shown that $h((T \times J) - D) \subset Y - C_Y$. Suppose that $T = pc_1 \cup ... \cup pc_n$, where each $c_i \in pa_i - \{p\}$. Since the components of $h((T \times J) - D)$ are the sets $h((pc_1 - \{p\}) \times J), ..., h((pc_n - \{p\}) \times J)$ and the components of $Y - C_Y$ are the separated sets $(qb_1 - \{q\}) \times [0,1], ..., (qb_m - \{q\}) \times [0,1]$ and $m < n$, we may assume that $h((pc_1 - \{p\}) \times J) \cup h((pc_2 - \{p\}) \times J) \subset (qb_1 - \{q\}) \times [0,1]$ and $h((pc_3 - \{p\}) \times J) \subset ((qb_1 - \{q\}) \cup (qb_2 - \{q\})) \times [0,1]$. Thus,

$$h(((pc_1 - \{p\}) \cup (pc_2 - \{p\}) \cup (pc_3 - \{p\})) \times J) \subset (qb_1 \cup qb_2) \times [0,1].$$

Therefore, $h((pc_1 \cup pc_2 \cup pc_3) \times J) \subset (qb_1 \cup qb_2) \times [0,1]$. This is impossible since, by using the Theorem on the Invariance of the Domain [6, Theorem VI 9, p. 95], it can be shown that the product of a simple triod and an arc cannot be embedded in the plane. The proof of the lemma is complete. □

Lemma 3.2. *Let T_n and T_m denote a simple n-od and a simple m-od, respectively, where $n, m \in \mathbb{N}$ and $n, m \geq 3$. Then $T_n \times T_m$ is homeomorphic to the cone over the complete bipartite graph $K_{n,m}$.*

Proof: Suppose that $T_n = \bigcup_{i=1}^{n} \theta e_i$ and $T_m = \bigcup_{j=1}^{m} \theta' e'_j$, where $\{e_1, ..., e_n\}$ and $\{e'_1, ..., e'_m\}$ are the respective canonical basis of the linear spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ and $\{e_1, ..., e_n\} \cap \{e'_1, ..., e'_m\} = \emptyset$; moreover, θ and θ' are the respective origins of $\mathbb{R}^n$ and $\mathbb{R}^m$, and θe_i and $\theta' e'_j$ are the convex segments joining θ and e_i and θ' and e'_j, respectively.

Let $Z = \{(x, y) \in T_n \times T_m : x \in \{e_1, ..., e_n\} \text{ or } y \in \{e'_1, ..., e'_m\}\}$.

CLAIM. Z is a complete bipartite graph $K_{n,m}$.

In order to prove this claim, we need to define the vertices and the edges of Z. Let $\mathcal{U} = \{(e_i, \theta') : i \in \{1, ..., n\}\}$ and $\mathcal{V} = \{(\theta, e'_j) : j \in \{1, ..., m\}\}$.

For each $i \in \{1, ..., n\}$ and each $j \in \{1, ..., m\}$, let

$$\mathcal{A}_{i,j} = (\{e_i\} \times \theta' e'_j) \cup (\theta e_i \times \{e'_j\}).$$

Clearly, $\mathcal{A}_{i,j}$ is an arc that joins (e_i, θ') and (θ, e'_j). Now suppose that there exists a point $(x, y) \in \mathcal{A}_{i,j} \cap \mathcal{A}_{k,r}$ and $(i, j) \neq (k, r)$. It is easy to show that (x, y) is an end point of $\mathcal{A}_{i,j}$ and of $\mathcal{A}_{k,r}$.

We have shown that the set of vertices $\mathcal{U} \cup \mathcal{V}$ and the set of edges $\mathcal{A}_{i,j}$ form a complete bipartite graph $K_{n,m}$.

Since $Z = \bigcup\{\mathcal{A}_{i,j} : i \in \{1, ..., n\} \text{ and } j \in \{1, ..., m\}\}$, the claim is proved.

Now, let $F : \text{cone}(Z) \to T_n \times T_m$ be given by $F((x, y), t) = ((1 - t)x, (1 - t)y)$. Clearly, F is continuous. In order to show that F is one-to-one, suppose that $F((x, y), t) = F((u, v), s)$. Then $(1 - t)x = (1 - s)u$ and $(1 - t)y = (1 - s)v$. By the definition of Z, we may assume that $x = e_i$ for some $i \in \{1, ..., n\}$. We consider four cases:

Case 1: $u = e_k$ for some $k \in \{1, ..., n\}$ and $t \neq 1$. Since $(1-t)e_i = (1-s)e_k$, it follows that $t = s$ and $e_i = e_k$. Since $(1-t)y = (1-s)v$, we conclude that $y = v$.

Case 2: $u = e_k$ for some $k \in \{1, ..., n\}$ and $t = 1$. Since $(1-t)e_i = (1-s)e_k$, it follows that $s = 1$. Thus, $((x, y), t)$ and $((u, v), s)$ are the same point in cone(Z).

Case 3: $v = e'_r$ for some $r \in \{1, ..., m\}$ and $t \neq 1$. Since $(1-t)e_i = (1-s)u$, $|(1-t)e_i| = |(1-s)u|$. This implies that $1-t \leq 1-s$. On the other hand, $|(1-t)y| = |(1-s)e'_r|$ implies that $1-t \geq 1-s$. Thus, $s = t$. Hence, $u = e_i = x$ and $y = e'_r = v$.

Case 4: $v = e'_r$ for some $r \in \{1, ..., m\}$ and $t = 1$. Thus, $\theta = (1-t)y = (1-s)e'_r$. This implies that $s = 1$. Thus, $((x, y), t)$ and $((u, v), s)$ are the same point in cone(Z).

Now we show that F is an onto map. Let $(x, y) \in T_n \times T_m$. Then $x = ae_i$ and $y = be'_j$ for some $a, b \in [0, 1]$, $i \in \{1, ..., n\}$, and $j \in \{1, ..., m\}$. We may assume that $a \leq b$. If $b = 0$, then $(x, y) = (x, \theta) = F((e_i, \theta), (1-a))$. If $b \neq 0$, then $(x, y) = F(((a/b)e_i, e'_j), 1-b)$. This completes the proof that F is onto. Therefore, F is a homeomorphism and the proof of the lemma is complete. □

Given topological spaces X and Y and points $p \in X$ and $q \in Y$, we write $(X, p) \approx (Y, q)$ if there exists a homeomorphism $f : X \to Y$ such that $f(p) = q$. Given an n-od T_n with core z and end points $z_1, ..., z_n$, let $Z_n = \{A \in F_2(T_n) : z_i \in A$ for some $i \in \{1, ..., n\}\}$. It was proved in [3, Lemma 1] that $F_2(T_n)$ is homeomorphic to cone(Z_n).

Lemma 3.3. *Let X be a finite graph and $p, q \in X$. Then the element $A = \{p, q\}$ has a basis of neighborhoods $\mathcal{B}$ in $F_2(X)$ with the property that for each $\mathcal{U} \in \mathcal{B}$:*

(a) if ord$(p, X) = 1$ or ord$(p, X) = 2$ and ord$(q, X) = 1$, then $(\mathcal{U}, A) \approx ([0, 1]^2, (0, 0))$;

(b) if ord$(p, X) = 2$, $p \neq q$ and ord$(q, X) = 2$, then $(\mathcal{U}, A) \approx ([0, 1]^2, (\frac{1}{2}, \frac{1}{2}))$;

(c) if ord$(p, X) = 2$ and $p = q$, then $(\mathcal{U}, A) \approx ([0, 1]^2, (0, 0))$;

(d) if ord$(p, X) = 1$ and ord$(q, X) = m \geq 3$, then $(\mathcal{U}, A) \approx ([0, 1] \times T_m, (0, z))$, where T_m is a simple m-od and z is the core of T_m;

(e) if ord$(p, X) = 2$ and ord$(q, X) = m \geq 3$, then $(\mathcal{U}, A) \approx ([0, 1] \times T_m, (\frac{1}{2}, z))$, where T_m is a simple m-od and z is the core of T_m;

(f) if ord$(p, X) = n \geq 3$, $p \neq q$ and ord$(q, X) = m \geq 3$, then $(\mathcal{U}, A) \approx (cone(K_{n,m}), v)$, where v is the vertex of cone$(K_{n,m})$;

(g) if ord$(p, X) = n \geq 3$, and $p = q$, then $(\mathcal{U}, A) \approx (cone(Z_n), v)$, where v is the vertex of $cone(Z_n)$, where Z_n is as described above.

Proof: Given points $p, q \in X$ such that $p \neq q$, there exist bases of closed connected neighborhoods K and L of p and q, respectively, in X such that K satisfies the following: (i) if ord$(p, X) = 1$, then K is an arc and p is an end point of K; (ii) if ord$(p, X) = 2$, then K is an arc and p is not an end point of K; (iii) if ord$(p, X) = n \geq 3$, then K is an n-od and p is the core of K. The set L satisfies analogous properties, depending on ord(q, X). Notice that $K \times L$ is homeomorphic to the neighborhood $\langle K, L\rangle_2$ of $A = \{p, q\}$ in $F_2(X)$ (with the homeomorphism that sends the pair (x, y) into the set $\{x, y\}$).

In the case that $p = q$, we can select a basis of neighborhoods K of p in X as in the previous paragraph and, in this case, a $\langle K\rangle_2 = F_2(K)$ is a neighborhood of $A = \{p\}$.

With the observations contained in the previous paragraphs and Lemma 3.2, the proof of this lemma is easy. □

The following lemma provides a description of the neighborhoods of a point in the product of two finite graphs. The proof is immediate ((e) follows from Lemma 3.2).

Lemma 3.4. *Let Y and Z be finite graphs $y \in Y$ and $z \in Z$. Then the element $A = (y, z)$ has a basis of neighborhoods $\mathcal{B}$ with the property that for each $\mathcal{U} \in \mathcal{B}$:*

(a) if ord$(y, Y) = 1$ or ord$(y, Y) = 2$ and ord$(z, Z) = 1$, then $(\mathcal{U}, A) \approx ([0, 1]^2, (0, 0))$;

(b) if ord$(y, Y) = 2$ and ord$(z, Z) = 2$, then $(\mathcal{U}, A) \approx ([0, 1]^2, (\frac{1}{2}, \frac{1}{2}))$;

(c) if ord$(y, Y) = 1$ and ord$(z, Z) = m \geq 3$, then $(\mathcal{U}, A) \approx ([0, 1] \times T_m, (0, z))$, where T_m is a simple m-od and z is the core of T_m;

(d) if ord$(y, Y) = 2$ and ord$(z, Z) = m \geq 3$, then $(\mathcal{U}, A) \approx ([0, 1] \times T_m, (\frac{1}{2}, z))$, where T_m is a simple m-od and z is the core of T_m;

(e) if ord$(p, X) = n \geq 3$ and ord$(q, X) = m \geq 3$, then $(\mathcal{U}, A) \approx (cone(K_{n,m}), v)$, where v is the vertex of $cone(K_{n,m})$.

Lemma 3.5. *Let T_3 and T_n denote a simple 3-od and a simple n-od, respectively, with $n \in \mathbb{N}$ and $n \geq 3$. Then $F_2(T_3)$ cannot be embedded in $T_n \times [0, 1]$.*

Proof: Suppose, to the contrary, that there exists an embedding $h : F_2(T_3) \to T_n \times [0, 1]$. Let p be the core of T_3 and $T = \{\{p, x\} \in$

$F_2(T_3) : x \in T_3\}$. Notice that T is homeomorphic to T_3. Let z be the core of T_n. Given $x \in T_3 - \{p\}$, there exists an arc J in T_3 such that J is a neighborhood of x in T_3 and $p \notin J$. Thus, there exists a simple subtriod S of T_3 such that p is the core of S and $S \cap J = \emptyset$. Then each neighborhood of $\{p, x\}$ in $F_2(T_3)$ which is contained in $\langle S, J\rangle_3$ contains a homeomorphic copy of the product $S \times J$. Since this space cannot be embedded in the plane, and the components of $(T_n \times [0,1]) - (\{z\} \times [0,1])$ are planable spaces, we conclude that $h(\{p, x\}) \in \{z\} \times [0,1]$ for each $x \neq p$. By continuity, $h(T) \subset \{z\} \times [0,1]$. This is impossible since an arc cannot contain a simple triod. This contradiction proves the lemma. □

Lemma 3.6. *Let $n, m, r \in \mathbb{N}$ be such that $n \geq 3$ and $r, m \geq 2$. Let T_n be a simple n-od with core p. Then there is not a continuum Z, a point $z \in Z$ and closed neighborhoods A and B of z in Z such that $A \subset B$, $(B, z) \approx$ (cone($K_{r,m}$),vertex of cone($K_{r,m}$)) and $(A, z) \approx (F_2(T_n), \{p\})$.*

Proof: Suppose, to the contrary, that there exists a continuum Z, a point $z \in Z$, and closed neighborhoods A and B of z in Z such that $A \subset B$, $(B, z) \approx$ (cone($K_{r,m}$),vertex of cone($K_{r,m}$)) and $(A, z) \approx (F_2(T_n), \{p\})$. Let $h : B \to \text{cone}(K_{r,m})$ and $g : F_2(T_n) \to A$ be homeomorphisms such that $h(z) = v$ and $g(\{p\}) = z$, where v is the vertex of cone($K_{r,m}$). Let W be an open subset of Z such that $z \in W \subset A - h^{-1}(K_{r,m} \times \{0\})$. Let $q \in T_n - \{p\}$ be such that $g(\{q\}) \in W$. Then there exists an arc $J \subset T_n - \{p\}$ such that J has end points u and v, $J - \{u, v\}$ is an open subset of T_n, $q \in J - \{u, v\}$, and $g(F_2(J)) \subset W$. Thus, $g(F_2(J - \{u, v\}))$ is an open subset of A. So, there exists an open subset U of Z such that $g(F_2(J - \{u, v\})) = A \cap U = A \cap U \cap W = U \cap W$. Hence, $g(F_2(J - \{u, v\}))$ is an open subset of Z that contains the point $g(\{q\})$. Since $(F_2(J - \{u, v\}), \{q\}) \approx ([0,1)^2, (0,0))$, $(h(g(F_2(J - \{u, v\}))), h(g(\{q\}))) \approx ([0,1)^2, (0,0))$.

On the other hand, the set $V = h(g(F_2(J - \{u, v\})))$ is an open subset of $\text{cone}(K_{r,m}) - (K_{r,m} \times \{0\})$ that contains the point $y = h(g(\{q\}))$ and satisfies $(V, y) \approx ([0,1)^2, (0,0))$. However, it is easy to show that $K_{r,m}$ is a finite graph without end points. Thus, it is impossible that the point $y \in \text{cone}(K_{r,m}) - (K_{r,m} \times \{0\})$ has a neighborhood V such that $(V, y) \approx ([0,1)^2, (0,0))$. This contradiction proves the lemma. □

Lemma 3.7. *Let X be a finite graph and Y be a Peano continuum. If $Y \times [0,1]$ can be embedded in $F_2(X)$, then ord(y, Y) is finite for every $y \in Y$.*

Proof: Suppose, to the contrary, that there exists a point $y \in Y$ such that $\text{ord}(y, Y)$ is infinite and suppose also that there exists an embedding $h : Y \times [0,1] \to F_2(X)$. Let $\mathcal{R} = \{A \in F_2(X) : A \subset R(X)\}$. Since $\mathcal{R}$ is finite, $h^{-1}(\mathcal{R})$ is finite. Thus, there exists a subarc J of $[0,1]$ such that $h(\{y\} \times J) \subset F_2(X) - \mathcal{R}$. Let $m = \max\{\text{ord}(x, X) : x \in X\}$. Since $\text{ord}(y, Y)$ is infinite, there exists a simple $(m+1)$-od T_{m+1} in Y such that y is the core of T_{m+1} (See [13, Ch. VI, § 51, Example 8, p. 277].) Fix $t \in J$. Since $h(y,t) \in F_2(X) - \mathcal{R}$, $h(y,t)$ is an element of $F_2(X)$ of one of the forms described in (a), (b), (c), (d) or (e) in Lemma 3.3. Thus, $h(y,t)$ has a neighborhood U in $F_2(X)$ of the form $[0,1] \times T$, where T is either an arc or an r-od for some $r \leq m$. By continuity, there exists a subarc L of J and a simple $(m+1)$-od S_{m+1}, contained in T_{m+1} such that $h(L \times S_{m+1}) \subset U$. Thus, it is possible to embed the product $L \times S_{m+1}$ into $[0,1] \times T$. This contradicts Lemma 3.1 and completes the proof of the lemma. □

Lemma 3.8. *For $m \geq 3$, let T_m be a simple m-od and Y a Peano continuum. If $Y \times [0,1]$ can be embedded in $T_m \times [0,1]$, then the number of points $y \in Y$ such that ord$(y, Y) > 2$ is finite.*

Proof: Let p be the core of T_m. Suppose, to the contrary, that the number of points $y \in Y$ such that $\text{ord}(y, Y) > 2$ is infinite and there exists an embedding $h : Y \times [0,1] \to T_m \times [0,1]$. Choose a sequence of different points $y_1, y_2, \ldots$ such that $\text{ord}(y_n, Y) > 2$ for each $n \in \mathbb{N}$ and $\lim y_n = y_0$ for some $y_0 \in Y$.

Given $n \in \mathbb{N}$ and $t \in [0,1]$, each neighborhood of $h(y_n, t)$ contains a topological copy of the set $T_3 \times [0,1]$, where T_3 is a simple triod (see [13, Ch. VI, § 51, Example 8, p. 277]). Since $(Y \times [0,1]) - (\{p\} \times [0,1])$ is a finite union of open sets homeomorphic to the space $(0,1] \times [0,1]$, then $h(y_n, t) \in \{p\} \times [0,1]$. We have shown that $h(\{y_n\} \times [0,1]) \subset \{p\} \times [0,1]$ for each $n \in \mathbb{N}$. By continuity, $h(\{y_0\} \times [0,1]) \subset \{p\} \times [0,1]$. Thus, the sets $h(\{y_1\} \times [0,1]), h(\{y_2\} \times [0,1]), \ldots$ are pairwise nondegenerate disjoint continua, contained in the arc $\{p\} \times [0,1]$ and they tend to $h(\{y_0\} \times [0,1])$. Since this situation is impossible for an arc, the lemma is proved. □

Lemma 3.9. *Let X be a finite graph and Y a Peano continuum. If $Y \times [0,1]$ can be embedded in $F_2(X)$, then the number of points $y \in Y$ such that $ord(y,Y) > 2$ is finite.*

Proof: Suppose, to the contrary, that the number of points $y \in Y$ such that $\text{ord}(y,Y) > 2$ is infinite and there exists an embedding $h : Y \times [0,1] \to F_2(X)$. Choose a sequence of different points $y_1, y_2, ...$ such that $\text{ord}(y_n, Y) > 2$ and $\lim y_n = y_0$ for some $y_0 \in Y$.

Given $t \in [0,1]$, each neighborhood of $h(y_0,t)$ contains points of the form $h(y_n,t)$, and then these neighborhoods contain topological copies of the set $T_3 \times [0,1]$, where T_3 is a simple triod. (See [13, Ch. VI, § 51, Example 8, p. 277].) Thus, $h(y_0,t)$ is not of any of the forms described in (a), (b), and (c) of Lemma 3.3.

Let $\mathcal{A} = \{A \in F_2(X) : A \subset R(X)\}$. Since $\mathcal{A}$ is finite, there exists an arc $J \subset [0,1]$ such that $h(\{y_0\} \times J) \cap \mathcal{A} = \emptyset$. Therefore, $h(y_0,t)$ is not of any of the forms described in (f) and (g) of Lemma 3.3 for all $t \in [0,1]$.

Fix $t_0 \in J$. By the previous paragraphs, $h(y_0,t_0)$ is of one of the forms (d) or (e) of Lemma 3.3. In both cases, $h(y_0,t_0)$ has a neighborhood $\mathcal{U}$ of the form $T_m \times [0,1]$ for some simple m-od T_m. Let Y_1 be a closed connected, locally connected neighborhood of y_0 in Y and L a subarc of J such that $h(Y_1 \times L) \subset \mathcal{U}$. Since Y_1 contains infinitely many points y such that $\text{ord}(y,Y_1) > 2$, we obtain a contradiction to Lemma 3.8. This finishes the proof of this lemma. □

Lemma 3.10. *Let X be a finite graph. If $F_2(X)$ is homeomorphic to the product of two nondegenerate continua Y and Z, then Y and Z are finite graphs.*

Proof: Since X is locally connected, $F_2(X)$ is locally connected. (See [2, property (a), p. 877].) Thus, Y and Z are locally connected. Fix an arc J in Z. Then $Y \times J$ can be embedded in $F_2(X)$. By lemmas 3.7 and 3.9, $\text{ord}(y,Y)$ is finite for every $y \in Y$ and the number of points $y \in Y$ such that $\text{ord}(y,Y) > 2$ is finite. By [19, Theorem 9.10], Y is a finite graph. Similarly, Z is a finite graph. □

Theorem 3.11. *Let X be a finite graph. Then $F_2(X)$ is homeomorphic to the product of two nondegenerate continua Y and Z if and only if X is an arc.*

Proof: Since $F_2([0,1])$ is homeomorphic to $[0,1]^2$, the sufficiency is immediate.

Now, we prove the necessity.

First, we show that X does not contain ramification points. Suppose, to the contrary, that X contains a ramification point p. Let $h : F_2(X) \to Y \times Z$ be a homeomorphism. By Lemma 3.10, Y and Z are finite graphs.

Since the point p has a basis of neighborhoods in X of the form T_m, where T_m is a simple m-od, p is the core of T_m and $m = \mathrm{ord}(p, X)$, we have that $\{p\}$ has a basis of neighborhoods in $F_2(X)$ of the form $F_2(T_m)$. By Lemma 3.5, the sets of the form $F_2(T_m)$ cannot be embedded in sets of the form $T_n \times [0,1]$, where T_n is a simple n-od. By Lemma 3.4, $h(p)$ is of the form $h(p) = (y, z)$ for some $y \in Y$ and $z \in Z$ such that $\mathrm{ord}(y, Y) = r \geq 3$ and $\mathrm{ord}(z, Z) = s \geq 3$. By Lemma 3.4, (y, z) has a basis of neighborhoods $\mathcal{B}$ in $Y \times Z$ with the property that, for each $\mathcal{U} \in \mathcal{B}$, $(\mathcal{U}, (y, z)) \approx (\mathrm{cone}(K_{r,s}), v)$, where v is the vertex of $\mathrm{cone}(K_{r,s})$. Since h is a homeomorphism, (y, z) also has a basis of neighborhoods $\mathcal{B}_0$ in $Y \times Z$ with the property that, for each $\mathcal{V} \in \mathcal{B}_0$, $(\mathcal{V}, (y, z)) \approx (F_2(T_m), \{p\})$. According to Lemma 3.6, this is absurd. Therefore, X does not contain ramification points. Thus X is an arc or a simple closed curve. If X is a simple closed curve, then $F_2(X)$ is homeomorphic to the Möbius strip. Thus, $F_2(X)$ is not homeomorphic to the product of two nondegenerate continua. This proves that X is not a simple closed curve. Therefore, X is an arc. □

Lemma 3.12. *Let X be a finite graph. If $F_2(X)$ is homeomorphic to the cone over a continuum Y, then Y is a finite graph.*

Proof: Since X is locally connected, $F_2(X)$ is also locally connected [2, property (a), p. 877]), so Y is locally connected. Since $Y \times [0, \frac{1}{2}]$ can be embedded in $F_2(X)$, by lemmas 3.7 and 3.9, $\mathrm{ord}(y, Y)$ is finite for every $y \in Y$ and the number of points $y \in Y$ such that $\mathrm{ord}(y, Y) > 2$ is finite. By [19, Theorem 9.10], Y is a finite graph. □

Lemma 3.13. *Let X be a finite graph. If $F_2(X)$ is homeomorphic to the cone over a continuum Y, then X is unicoherent.*

Proof: Suppose, to the contrary, that X is not unicoherent. Since X is a locally connected continuum, this implies, by [7, Theorem 1.6], that $F_2(X)$ is not unicoherent. By [5, theorems 2 and 3] and [16], there exists a continuous function $f : F_2(X) \to S^1$, where S^1 is the unit circle in the plane, such that f is not homotopic to a constant map. Since cone(Y) is contractible, $F_2(X)$ is contractible and every map from $F_2(X)$ to S^1 is homotopic to a constant map. This contradiction completes the proof of the lemma. □

Theorem 3.14. *Let X be a finite graph. Then $F_2(X)$ is homeomorphic to the cone over a continuum Y if and only if X is a simple n-od or an arc.*

Proof: (Necessity). By Lemma 3.12, Y is a finite graph. Let $h : F_2(X) \to \text{cone}(Y)$. First, we show that if p is a ramification point of X and v is the vertex of cone(Y), then $h(\{p\}) = v$. Suppose, to the contrary, that $h(\{p\}) = (y,t)$ for some $t \in [0,1)$. Since Y is a finite graph, (y,t) has a basis of neighborhoods $\mathcal{B}$ in cone(Y) such that, for each $U \in \mathcal{B}$, U is of the form $[0,1] \times T$, where T is an arc of a simple r-od. On the other hand, $\{p\}$ has a basis of neighborhoods of the form $F_2(T_m)$, where $m = \text{ord}(p,X)$ and T_m is a simple m-od. By Lemma 3.5, the basic neighborhoods of $\{p\}$ cannot be embedded in the basic neighborhoods of (y,t). This contradiction proves that $h(\{p\}) = v$.

Since h is one-to-one, we conclude that X has at most one ramification point. Since X is unicoherent (Lemma 3.13), we conclude that X is an arc or a simple n-od.

(Sufficiency). Is immediate from [3, Lemma 1]. □

Question 3.15. Is $[0,1]$ the only finite graph such that $F_3(X)$ is a product of two nondegenerate continua? By [2, Theorem 6], $F_3([0,1])$ is homeomorphic to $[0,1]^3$.

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