A REFLECTION THEOREM FOR I-WEIGHT

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ABSTRACT. We will show that, under GCH, for every cardinal \( \kappa > \omega \) and an arbitrary compact Hausdorff space \( X \), if we have \( iw(Y) < \kappa \) whenever \( Y \subset X \) and \( |Y| \leq \kappa \), then \( iw(X) < \kappa \).

1. Introduction

The following remarkable theorem is due to A. Hajnal and I. Juhász [5]: The weight function \( w \) reflects every infinite cardinal \( \kappa \) (that is, if \( w(X) \geq \kappa \), then \( w(Y) \geq \kappa \) for some subspace \( Y \) of cardinality \( \leq \kappa \)). The main aim of this paper is to prove an analogous statement for the i-weight: Assume GCH (Generalized Continuum Hypothesis); for the class of compact Hausdorff spaces, \( iw \) strongly reflects all infinite cardinals. The study of reflection and the increasing union property was initiated by M. G. Tkacenko in [9] and continued by Juhász in [8]. R. E. Hodel and J. E. Vaughan in [7] made a systematic study of reflection theorems for cardinal functions.

Let \( w \), \( \chi \) and \( \psi \) denote the following standard cardinal functions: weight, character, and pseudo-character, respectively. If \( \phi \) is a cardinal function, then \( h\phi \) is the hereditary version of \( \phi \); i.e., \( h\phi(X) = \sup \{ \phi(Y) : Y \subseteq X \} \). As well, it is known that \( \phi \) is monotone if and only if \( \phi = h\phi \). (For definitions, see [6].)

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A map $f : X \to Y$ is called a map onto if $f(X) = Y$. A condensation is a bijective continuous map onto.

For a topological space $X$, $iw(X)$ denotes the minimal weight of all (Tychonoff) spaces onto which $X$ can be condensed. The cardinal invariant $iw(X)$ is called the $i$–weight of $X$. For example, a space $X$ has $i$–weight $\omega$ iff $X$ has a weaker separable metric topology.

The next theorem summarizes some properties of the cardinal invariant $iw$. (See [1] and [2].)

**Theorem 1.1.** (a) The function $iw$ is monotone. (b) $\psi(X) \leq iw(X)$. (c) For each Tychonoff space $X$, $iw(X) \leq w(X)$. If, in addition, $X$ be a compact Hausdorff space, then $iw(X) = w(X)$.

Finally, a compact Hausdorff space $X$ is called a dyadic space if $X$ is a continuous image of the Cantor cube $D^\kappa$ for some cardinal number $\kappa$.

2. Main Results

For the sake of the reader’s comfort the formulations of the necessary results of Reflection Theory are given here. (See [7].)

**Definition 2.1.** Let $\phi$ be a cardinal function and $\kappa \geq \omega$ a cardinal number.

(a) $\phi$ reflects $\kappa$ means: if $\phi(X) \geq \kappa$, then there exists $Y \subseteq X$ with $|Y| \leq \kappa$ and $\phi(Y) \geq \kappa$.

(b) $\phi$ strongly reflects $\kappa$ means: if $\phi(X) \geq \kappa$, then there exists $Y \subseteq X$ such that:

1. $|Y| \leq \kappa$;
2. if $Y \subseteq Z \subseteq X$, then $\phi(Z) \geq \kappa$.

Note. For some cardinal functions, it is necessary to restrict the class of spaces under consideration in order to obtain a reflection theorem. The appropriate definition in this case is as follows: $\phi$ reflects $\kappa$ for the class $\mathcal{C}$ if given $X \in \mathcal{C}$ with $\phi(X) \geq \kappa$, there exist $Y \subseteq X$ with $|Y| \leq \kappa$ and $\phi(X) \geq \kappa$.

The next two lemmas play a very important role in streamlining reflection proofs. (See [7].)

**Lemma 2.2.** If $\phi$ reflects $\kappa^+$, then $h\phi$ strongly reflects $\kappa^+$. 
Lemma 2.3. If $\phi$ strongly reflects all successor cardinals, then $\phi$ strongly reflects all infinite cardinals. In particular, if $\phi$ is monotone and reflects all successor cardinals, then $\phi$ strongly reflects all infinite cardinals.

To present our main result, we need the next theorem. (See [7].)

Theorem 2.4. Assume GCH: For the class of compact Hausdorff spaces, $\psi$ strongly reflects all infinite cardinals.

Theorem 2.5. Assume GCH: For the class of compact Hausdorff spaces, $iw$ strongly reflects all infinite cardinals.

Proof: Since $iw$ is monotone, it suffices to prove that $iw$ reflects every successor cardinal $\kappa^+$. (See lemmas 2.2, 2.3.)

Let $X$ be a compact Hausdorff space such that $iw(X) \geq \kappa^+$. We will show that there is a subset $Z$ of $X$ with $|Z| \leq \kappa^+$ and $iw(Z) \geq \kappa^+$.

Since $iw(X) = w(X)$, then $w(X) \geq \kappa^+$; hence, by the Hajnal-Juhász reflection theorem, there is a subspace $Y$ of $X$ with $|Y| \leq \kappa^+$ and $w(Y) \geq \kappa^+$. Let $Y_0 = cl_X(Y)$, then $Y_0$ is a compact Hausdorff space.

We now consider two cases:

(a) $\psi(Y_0) \geq \kappa^+$. By Theorem 2.4, there is a subspace $Z$ of $Y_0$ such that $|Z| \leq \kappa^+$ and $\psi(Z) \geq \kappa^+$. It follows that $iw(Z) \geq \kappa^+$. (See Theorem 1.1 (a).)

(b) $\psi(Y_0) < \kappa^+$; i.e., $\psi(Y_0) \leq \kappa$. Since $Y_0$ is a compact Hausdorff space, $\psi(Y_0) = \chi(Y_0)$, and $|Y_0| \leq 2^\chi(Y_0)$. Hence, $|Y_0| \leq 2^\kappa = \kappa^+$, (by GCH). On the other hand, it is clear that at $iw(Y_0) = w(Y_0) \geq w(Y) \geq \kappa^+$. Thus, $Z = Y_0$ witnesses the fact that $iw$ reflects $\kappa^+$. The proof is complete. $\square$

The next assertion is an obvious corollary of the previous theorem.

Proposition 2.6. Assume GCH. Let $X$ be a compact Hausdorff space. Suppose that $X = \bigcup \{X_\alpha : \alpha \in \lambda\}$, where $\{X_\alpha : \alpha \in \lambda\}$ is an increasing family ($\alpha < \beta$ implies $X_\alpha \subseteq X_\beta$) and $\lambda$ is regular. If $iw(X_\alpha) < \kappa$ for all $\alpha \in \lambda$ and $\kappa < \lambda$, then $iw(X) < \kappa$.

At the moment the author does not know the answer to the following questions:
Question 2.7. For the class of compact Hausdorff spaces, does \( iw \) strongly reflect all infinite cardinals? More generally, does \( iw \) strongly reflect all infinite cardinals?

Question 2.8. Let \( X \) be a compact Hausdorff space. Suppose that \( X = \bigcup \{ X_\alpha : \alpha \in \lambda \} \), where \( \{ X_\alpha : \alpha \in \lambda \} \) is an increasing family \( (\alpha < \beta \text{ implies } X_\alpha \subseteq X_\beta) \) and \( \lambda \) is regular. If \( iw(X_\alpha) \leq \kappa \) for all \( \alpha \in \lambda \) and \( \lambda \leq \kappa \), then is \( iw(X) < \kappa \)?

In connection with Question 2.7, we now show that \( iw \) strongly reflects all infinite cardinals for the class of dyadic spaces.

Theorem 2.9. For the class of dyadic spaces, \( iw \) strongly reflects all infinite cardinals.

Proof: It suffices to prove that \( iw \) reflects every successor cardinal \( \kappa^+ \). Let \( X \) be a dyadic space such that \( iw(X) \geq \kappa^+ \). By Theorem 1.1 (c), \( w(X) \geq \kappa^+ \); hence, by the Efimov-Gerlitz-Hagler theorem [4], \( X \) contains a topological copy of \( D^{\kappa^+} \). Let \( p \in D^{\kappa^+} \); since \( \chi(p, D^{\kappa^+}) = \kappa^+ \), by Efimov’s theorem [3], there exists \( M \subseteq D^{\kappa^+} \) such that \( M \) is discrete, \( |M| = \kappa^+ \), and \( Y = M \cap \{p\} \) is homeomorphic to \( A(\kappa^+) \), where \( A(\kappa^+) \) denotes the one-point compactification of a discrete space of cardinality \( \kappa^+ \). Therefore, \( \kappa^+ \leq iw(Y) \). \( \Box \)

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References

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