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A GROUP THEORETIC CONDITION IN TOPOLOGICAL DYNAMICS

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Abstract. We relate group theoretic conditions to regional proximality in minimal flows. A generalization of a theorem of the author and Robert and David Ellis is obtained.

The regionally proximal relation on a flow is an indication of how the flow differs from an equicontinuous one. In fact, a flow is equicontinuous if and only if the regionally proximal relation is trivial, and for any flow the equicontinuous structure relation is the closed invariant equivalence relation generated by the regionally proximal relation.

Now, it is a remarkable fact that for in many cases (including when the acting group is abelian) the regionally proximal relation on a minimal flow is a closed equivalence relation (and so coincides with the equicontinuous structure relation). There have been a number of proofs of this, under varying hypotheses.

One of these (in [AEE]) involves a condition on the automorphism group G of the universal minimal flow (M, T). Although I am one of the authors of that paper, (and I have no doubt that it is correct), I have been somewhat dissatisfied with the way it was written. In particular, the meaning of certain sets which are introduced in the course of the proofs is difficult to discern.

The present paper entails a careful reworking of [AEE], in an attempt to extract the essence of the latter. In the process, we will define some new subsets of G, and obtain a generalization of the main result. Of necessity, we will reprove some of the results in [AEE], although some of the proofs will be different.

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We first review some basic concepts and establish our notation. As general references see [A1] and [E]. A flow (X,T) is a (right) topological action of the group T on the compact Hausdorff space X. A subset K of X is said to be minimal if $K \neq \emptyset$ and $\overline{xT} = K$ for every $x \in K$. In this case $x \in K$ is called an almost periodic point. If (X,T) is minimal ($\overline{xT} = X$ for all $x \in X$) we say that (X,T) is a minimal flow.

Let (X,T) be a flow. The regionally proximal relation $Q(X) = Q$ is defined by $Q = \cap\{\overline{UT} | U$ a neighborhood of Δ in $X \times X\}$. Clearly $(x,y) \in Q$ if and only if there are nets $\{x_n\}$ and $\{y_n\}$ in X and $\{t_n\}$ in T with $(x_n, y_n) \to (x,y)$ and $(x_n t_n, y_n t_n) \to \Delta$. Q is reflexive, symmetric, closed, and T-invariant, but in general, is not an equivalence relation. As indicated above, this paper is concerned with a condition on a minimal flow under which Q is in fact an equivalence relation.

An action of T on X extends to an action of βT, the Stone-Čech compactification of T. If $p \in \beta T$ and $\{t_i\}$ is a net in T with $t_i \to p$, then, for $x \in X$, $xp = \lim xt_i$.

If X is a compact Hausdorff space, then 2^X denotes the space of closed subsets of X provided with the Hausdorff topology; 2^X is also a compact Hausdorff space.

If (X,T) is a flow, we may regard $(2^X, T)$ as a flow (where $Kt = \{xt | x \in K\}$). Then βT also acts on 2^X and we write $K \circ p$ for this action on 2^X. In general $Kp = \{xp | x \in K\}$ is a proper subset of $K \circ p$. If $N \subset X$, we define $N \circ p = \overline{N} \circ p$ (so $N \circ p$ is always closed). Note that $y \in N \circ p$ if and only if there are nets $\{x_i\}$ in N and $\{t_i\}$ in T with $t_i \to p$ and $x_i t_i \to y$.

Let (M,T) be the universal minimal flow for the group T. Its defining property is that every minimal flow with acting group T is a factor of (M,T). The universal minimal flow exists and is unique up to isomorphism.

The universal minimal flow has a plentiful supply of automorphisms (homeomorphisms commuting with the action of T on M). In fact, if (m, m') is an almost periodic point of the product flow $(M \times M, T)$ then there is a (necessarily unique) automorphism φ with $\varphi(m) = m'$.

This is the point of departure for the Galois theory of minimal flows, initiated by Ellis, which correlates the subgroups of G, the automorphism group of (M, T), with dynamical proerties of minimal flows. G can be endowed with a compact T_1 topology. This is the topology of graphs: a net $\{\varphi_n\}$ converges to $\varphi \in G$ if there is a net $\{m_i\}$ in M with $(m_i, \varphi_i(m_i)) \to (m, \varphi(m))$.

We fix a minimal right ideal I in βT. If $m \in M$, there is a unique u in I such that $mu = m$. This element u is an idempotent ($u^2 = u$).

Let $m \in M$ with $mu = m$ and let W be an open neighborhood of m. Let $\Sigma_{W,m} = \{\theta \in G | \theta(m) \in W \circ u\}$.

Lemma 1. *$\Sigma_{W,m}$ is closed.*

Proof. Let $\{\varphi_j\}$ be a net in $\Sigma_{W,m}$ with $\varphi_j \to \varphi$. Then there is a net $\{m_j\}$ in M with $m_j \to m$ and $\varphi_j(m_j) \to \varphi(m)$. Let $r_j \in I$ such that $m_j = mr_j$. If $r_j \to r$, then $mr = m$ so $r = u$. Now $\varphi_j(m_j) = \varphi_j(mr_j) \in W \circ u \circ r_j = W \circ r_j \to W \circ u$ so $\varphi(m) \in W \circ u$. Therefore $\varphi \in \Sigma_{W,m}$. □

If K is a subset of the topological space X, we write $clintK$ for the closure of the interior of K.

If $m \in M$, let $L(m) = \cap_{m \in W} clint\Sigma_{W,m}$ and $\Delta(m) = \cap_{m \in W} \Sigma_{W,m}$. Clearly $L(m) \subset \Delta(m)$. The key result (proved below) is that $L(m) \neq \emptyset$.

If $m \in M$ with $mu = m$ then $\theta \in \Delta(m)$ if and only if there are nets $\{m_i\}$ in M and $\{s_i\}$ in T with $s_i \to u$, $m_i \to m$, and $m_i s_i \to \theta(m)$.

Lemma 2. *$int\Sigma_{W,m} \neq \emptyset$.*

Proof. We first suppose m is an almost periodic point for (G, M), the (left) action of G on M. Then there is a finite subset F of G such that $\overline{Gm} \subset FW$. Therefore $Gm \subset F(W \circ u)$, and it follows easily that $G = F\Sigma_{W,m}$. Therefore $int\Sigma_{W,m} \neq \emptyset$.

Now let $n \in M$, and suppose $nv = n$. Let W be an open set containing n. Since the elements of T are automorphisms of (G, M), there is an almost periodic point m (for (G, M)) in W. We may suppose $mu = m$. Then $m = \beta(nu)$ for some $\beta \in G$. Then, by the first part of this proof, $G(nu) = G\beta(nu) = Gm \subset F(W \circ u)$ for a finite subset F of G. Then $Gn \subset F(W \circ v)$, and as above $int\Sigma_{W,n} \neq \emptyset$. □

Corollary 3. *If $m \in M$, $L(m) \neq \emptyset$.*

In the following lemma, $\theta \in G$, and u and v are idempotents in I.

Lemma 4. *Let $m \in M$ with $mu = m$. (i): $\theta \in \Delta(m)$ if and only if there are nets $\{m_i\}$ in M and $\{s_i\}$ in T with $s_i \to u$, $m_i \to m$, and $m_i s_i \to \theta(m)$.*

(ii): Suppose $\theta \in \Delta(m)$ and $\{s_i\}$ is a net in T with $s_i \to v$. Then there is a net $\{m_i\}$ with $m_i \to m$ and a subnet of $\{s_i\}$ (still written $\{s_i\}$) such that $m_i s_i \to \theta(mv)$.

(iii): If $\theta \in \Delta(m)$, then $\theta(mv) \in Q(m)$.

Proof. (i): $\theta \in \Delta(m)$ if and only if $\theta(m) \in W \circ u$ for every neighborhood W of m. This immediately implies the sufficiency of the condition. Conversely, if $\theta \in \Delta(m)$, and W is a neighborhood of m, let $\{m_i\}$ be a net in W and $\{s_i\}$ a net in T with $s_i \to u$. Then (a subnet of) $m_i s_i \to m' \in \overline{W}$. Since W is an arbitrary neighborhood of m, we obtain nets as asserted.

(ii): If W is a neighborhood of m, $\theta(m) \in W \circ u$ so $\theta(mv) \in W \circ v$. The proof now proceeds as in (i).

(iii): Let $n = \theta(mv)$. If $\{m_i\}$ and $\{s_i\}$ are as in (ii), we have $(m_i, n) \to (m, n)$ and $(m_i, n)s_i \to (n, nv) = (n, n)$. Therefore $(m, \theta(mv)) = (m, n) \in Q$. □

Let $G' = \cap\{N | N \text{ a closed neighborhood of } id\}$ (where id is the identity of G). Note that $\psi \in G'$ if and only if there is a net $\{\psi_i\}$ in G with $\psi_i \to \psi$ and also $\psi_i \to id$. It is not difficult to show that G' is a closed normal subgroup of G.

Let H be the set of $\varphi \in G$ such that the graph of φ is contained in Q. (Equivalently $(m, \varphi(m)) \in Q$ for some $m \in M$). H is a closed subset of G, but is in general not a subgroup of G.

It is easy to see that $G' \subset H$. Moreover, if the almost periodic points in $M \times M$ are dense (in particular if T is abelian) $G' = H$.

In the next lemma, A denotes a closed subgroup of G.

Lemma 5. *Let $m \in M$.*

(i) $\Delta(m) \subset H$.

(ii) $G'L(m) = L(m)$.

(iii) If AH is a group, then $AH = AHL(m) = AH\Delta(m)$.

(iv) If $AH = AG'$, then $A\Delta(m) = AL(m) = AH$.

Proof. (i) follows from Lemma 4.

(ii): It is straightforward to show (see [AEE]) that if W is an open subset of G, then $G'clW \subset clW$. If we apply this to $W = int\Sigma_{W,m}$, then (ii) follows.

(iii): Since $L(m) \subset \Delta(m) \subset H$ and AH is a group, these equalities follow.

(iv): Since $AH = AG'$, AH is a group, so $AH = AHL(m) = AG'L(m) = AL(m) \subset A\Delta(m) \subset AH$. □

Let (X, T) be a minimal flow, and let $\pi : M \to X$ be a homomorphism. The (Ellis) group of (X, T), denoted $\mathcal{G}(X) = \{\alpha \in G | \pi\alpha = \pi\}$. Note that $\mathcal{G}(X)$ is a closed subgroup of G. Moreover every closed subgroup of G is the Ellis group of some minimal flow.

For the remainder of this paper, (X, T) is a minimal flow, $\pi : M \to X$ is a homomorphism, and $A = \mathcal{G}(X)$.

The group theoretic condition considered in [AEE] is $H \subset AG'$, and it was shown that for a minimal flow satisfying this condition, Q is an equivalence relation. By (iv) of Lemma 5, this implies $H \subset A\Delta(m)$ (equivalently $AH = A\Delta(m)$) for all $m \in M$, and it is this latter condition which we will assume.

Theorem 6. *Suppose $A\Delta(m) = AH$ for all $m \in M$, and let $x \in X$.*

(i): Suppose $y \in Q(x)$, and let $m \in M$ with $\pi(m) = x$. Then there is an $n \in M$ with $(m, n) \in Q$ and $\pi(n) = y$.

(ii): $Q(x) = \pi\{\delta(mv) | \delta \in \Delta(m),\ v$ an idempotent in $I\}$.

Proof. (i): Suppose $mu = m$ (so $xu = x$). Let $(m', n') \in Q$ with $\pi(m', n') = (x, y)$. Suppose $n'v = n'$ (so $yv = y$). Then $(m'u, n'u) \in Q$ and $\pi(m'u, n'u) = (x, yu)$. Then $n'u = h(m'u)$ where $h \in H$ so $h = \alpha\theta$ with $\alpha \in A$ and $\theta \in \Delta(m'u)$. Now $\pi\theta(m'u) = \pi h(m'u) = \pi(n'u) = yu$ and $\pi\theta(m'v) = yv = y$. Also $\theta(m'v) \in \Delta(m'u)(m'v) \in Q(m'u).$, so $(m'u, \theta(m'v)) \in Q$, and $\pi(m'u, \theta(m'v)) = (x, y)$. There is an $a \in A$ such that $m = a(m'u)$. Let $n = a\theta(m'v)$. Then $(m, n) = (a(m'u), a(\theta(m'v)),$ so $(m, n) \in Q$. Finally $\pi(m, n) = (x, y)$ as desired.

(ii): It follows from Lemma 4 that this set is contained in $Q(x)$. Let $y \in Q(x)$ and let $n \in M$ such that $\pi(n) = y$ with $(m, n) \in Q$. Then $n = h(mv)$ for some $h \in H$ and idempotent v in I.

Now $h = \alpha\delta$ for some $\alpha \in A$ and $\delta \in \Delta(m)$. Then $y = \pi(n) = \pi\alpha\delta(mv) = \pi\delta(mv)$. □

Lemma 7. *Suppose $H \subset A\Delta(m)$ for all $m \in M$. Let $(x, y) \in Q$, and suppose $xu = x$. Let $\{s_i\}$ be a net in T with $s_i \to u$. Then there is a subnet of $\{s_i\}$ (still written $\{s_i\}$) and a net $\{y_i\}$ with $y_i \to y$, and $y_i s_i \to x$.*

Proof. Let $m \in M$ with $mu = m$ and $\pi(m) = x$. Let $n \in M$ with $\pi(n) = y$ and $(m, n) \in Q$. Let $\delta \in \Delta(n)$ such that $\pi(\delta(nu)) = x$. Recall (Lemma 4) that $(n, \delta(nu)) \in Q$. Let $m' = \delta(nu)$. Then (Lemma 4 again) there is a net $\{n_i\}$ with $n_i \to n$ and $n_i s_i \to m'$. Thus we have $(m', n_i) \to (m', n)$ and $(m', n_i)s_i \to (m', m')$. Let $y_i = \pi(n_i)$, and we have $y_i \to y$ and $y_i s_i \to x$. □

Let (X, T) be a flow. For $n \geq 2$, we define the n-th regionally proximal relation by $Q^{(n)} = \cap\{\overline{VT} | V$ a neighborhood of the diagonal in $X^n\}$. Thus $Q^{(2)} = Q$. Note that if $(x_1, \ldots, x_n) \in Q^{(n)}$ then if $1 \leq i, j \leq n$, $(x_i, x_j) \in Q$, but this says more; the same net in T brings points points close to (x_i, x_j) to the diagonal in $X \times X$, for all i and j.

Theorem 8. *Suppose $H \subset A\Delta(m)$ for all $m \in M$. Suppose $x_1, \ldots, x_n \in X$ with $(x_1, x_j) \in Q$, for $j = 2, \ldots, n$. Then $(x_1, \ldots, x_n) \in Q^{(n)}$. In particular, Q is an equivalence relation.*

Proof. Suppose $x_1 u = x_1$, and let $\{s_i\}$ be a net in T with $s_i \to u$. By Lemma 7, there is a net $\{x_i\}$ with $x_i \to x_2$ and a subset of $\{s_i\}$ (as above, still written $\{s_i\}$) such that $x_i s_i \to x_1$. Then we have $(x_1, x_i) \to (x_1, x_2)$ and $(x_1, x_i)s_i \to (x_1, x_1)$. Now $(x_1, x_3) \in Q$, so we may proceed as above (using the subnet $\{s_i\}$ obtained in the proof) to obtain a further subnet $\{s_i\}$ and a net $y_i \to x_3$ so that $(x_1, y_i)s_i \to (x_1, x_3)$ and $(x_1, y_i)s_i \to (x_1, x_1)$. Thus we have (using the subnet obtained at this stage) $(x_1, x_i, y_i) \to (x_1, x_2, x_3)$ and $(x_1, x_i, y_i)s_i \to (x_1, x_1, x_1)$. Proceeding in this manner (successively refining the previous subnet $\{s_i\}$), it follows that $(x_1, \ldots, x_n) \in Q^{(n)}$. □

When Q is an equivalence relation, it is not difficult to show that AH is a group (see [A3], Theorem 6). Therefore, Theorem 8 implies that if $H \subset A\Delta(m)$ then $AH = A\Delta(m)$ is a group. It would be desirable to find a direct proof.

Recall that a flow (X, T) is called weakly mixing in the product flow $(X \times X, T)$ is topologically transitive. If the group T is abelian, the minimal flow (X, T) is weakly mixing if and only if $G = AG'$ ([A2]), and Ellis has proposed this group theoretic condition as the definition of weak mixing in general. However, in general the two definitions are not equivalent ([A3]), and we call the group theoretic definition Ellis weak mixing. We show that Ellis weak mixing implies weak mixing. (This was first proved by Glasner ([G2])).

Theorem 9. *If $G = AG'$, then (X, T) is weakly mixing.*

Proof. Since $G = AG'$, $H \subset AG'$ and (Lemma 5) $H \subset A\Delta(m)$ for all $m \in M$. Moreover ([A3]), Theorem 9), $Q = X \times X$. It follows from Theorem 8 that $Q^{(n)} = X^n$ and therefore (see below) (X, T) is weakly mixing. □

For completeness, we show that $Q^{(n)} = X^n$ for all n implies weak mixing, although the proof is essentially contained in Glasner's book ([G1]),Theorem II.2.1).

Let W be a non-empty open invariant subset of $X \times X$ and let U and V be open non-empty subsets of X with $U \times V \subset W$. Let $x, y \in X$. Let $x, y \in X$. Now $X = \cup_{i=1,\ldots,n} V t_i$ for some $t_1, \ldots, t_n$ in T. Let $\Sigma = Ut_1 \cup \cdots \cup Ut_n$, and let $(x_1, \ldots, x_n) \in \Sigma$ (so $(x_1, \ldots, x_n) \in Q^{(n)}$). Then there are nets in X^n and T such that $(x_{1j}, \ldots, x_{nj}) \to (x_1, \ldots, x_n)$ with $(x_{1j}, \ldots, x_{nj})s_j \to (x, \ldots, x)$. We may assume (by choosing a cofinal set) that $ys_j^{-1} \in Vt_{i_0}$ where $1 \leq i_0 \leq n$. Then $(x_{i_0 j}, ys_j^{-1}) \in (U \times V)t_{i_0} \subset Wt_{i0} = W$. Also $(x_{i_0 j}s_j, y) \to (x, y)$ and $(x_{i_0 j}s_j, y) = (x_{i_0 j}, ys_j^{-1})s_j \in Ws_j = W$, so $(x, y) \in \overline{W}$. Since x and y were arbitrary $\overline{W} = X \times X$, and (X, T) is weakly mixing.

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