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THE UNIVERSAL COVER OF THE QUOTIENT OF A LOCALLY DEFINED GROUP

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Abstract. We present a new method to compute the (generalized) universal cover of a quotient V/G of a locally defined group V via a closed subgroup G, and give some applications. For example we show that if G is locally generated (e.g. if G is connected) then V/G is locally defined. We give some answers to the question of when the universal cover of a quotient of a topological vector spaces is again a topological vector space.

In [2] a generalized covering group theory was developed using a generalized notion of cover, namely a quotient epimorphism $\psi : G \to H$ between topological groups G and H having central, prodiscrete kernel. The covering group theory is carried out for a large category of groups called *coverable groups*, which are by definition quotients of *locally defined* groups by a closed normal subgroup. A locally defined group, in turn, is a topological group V such that given a local group homomorphism $h : U \to G$ from a symmetric neighborhood U of the identity in V to a topological group G, there is a unique extension of h to a homomorphism on the entire group V. Coverable groups include all topological groups

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that are connected and locally simply connected in the traditional sense, all metrizable, connected, locally connected groups, and even some totally disconnected groups (see [2] for more details). Locally defined groups are the universal objects in the category of coverable groups and (generalized) covers. Specifically, every coverable group G is covered by a unique (up to isomorphism) locally defined group $\widetilde{G}$. The covering map $\phi : \widetilde{G} \to G$ has the traditional universal property, i.e., for any cover $\eta : K \to G$ of G by a coverable group K, there is a unique cover $\tau : \widetilde{G} \to K$ such that $\phi = \eta \circ \tau$ (Theorem 7, [2]). In addition, for any locally defined group V and homomorphism $f : V \to G$ there is a unique *lift* $\widetilde{f} : V \to \widetilde{G}$ of f; that is, $\widetilde{f}$ is a homomorphism such that $f = \phi \circ \widetilde{f}$. If f is open then $\widetilde{f}(V)$ is dense in $\widetilde{G}$ (Theorem 97, [2]).

In this note we present a new method for constructing the universal cover of a coverable group C using any expression of C as V/G, where V is locally defined and G is a closed normal subgroup of V. We apply this method to quotients of topological vector spaces (which are locally defined by Proposition 131, [2]). Note that every topological group in this paper is assumed to be Hausdorff, with identity denoted by e or 0, and homomorphisms between topological groups are always continuous. The following kind of group plays a significant role in this paper.

Definition 1. A topological group G is called totally asunder if the identity e in G is the intersection of open normal subgroups.

Note that a preprodiscrete group (i.e. having a basis at the identity of open normal subgroups) is totally asunder, but there exist totally asunder non-discrete closed subgroups of l^2 (see Proposition 7), which cannot be preprodiscrete because l^2 has no non-trivial small subgroups. Also, every totally asunder group is totally separated in the sense that every point is the intersection of open and closed sets, but we do not know whether the opposite implication is true. On the other hand every totally asunder group is totally disconnected, but the converse is false. For example, any totally disconnected locally generated group (such as $\mathbb{Q}$ or the complete examples in [8]) has no non-trivial open subgroups and so cannot be totally asunder. Finally, as we show in the proof of Proposition 12 , the proof of Corollary 2.3 in [6] in fact implies that every line-free, weakly closed subgroup Γ of a topological vector V space is totally asunder; if Γ is closed, cocompact and line-free the same conclusion follows from Proposition 12 and Proposition 5.

We begin with a general result:

Theorem 2. *Let V be a locally defined group and G be a closed normal subgroup of V. Then $\widetilde{V/G}$ is isomorphic to $H := \varprojlim V/H_\alpha$, where $\{H_\alpha\}$ is the collection of subgroups H_α that are (relatively) open in G and normal in V, partially ordered by reverse inclusion.*

Proof. By Proposition 80 of [2] we know that $\widetilde{V/G}$ is isomorphic to

$$\varprojlim(\widetilde{V/G})/K_\alpha,$$

where $\{K_\alpha\}$ is the collection of all subgroups that are (relatively) open in the central prodiscrete subgroup $K := \ker\phi$, partially ordered by reverse inclusion, where $\phi : \widetilde{V/G} \to V/G$ is the universal cover. Let K_α be a fixed open subgroup of K. Then ϕ factors as $\phi = \tau \circ \eta$, where $\eta : \widetilde{V/G} \to (\widetilde{V/G})/K_\alpha$ and $\tau : (\widetilde{V/G})/K_\alpha \to ((\widetilde{V/G})/K_\alpha)/(K/K_\alpha) = V/G$ are quotient epimorphisms. Now $\ker\tau = K/K_\alpha$ is discrete; that is, τ is a cover (in the traditional sense). Since V is locally defined there exists a unique homomorphism $\psi_\alpha : V \to (\widetilde{V/G})/K_\alpha$ such that $\tau \circ \psi_\alpha = \pi$, where $\pi : V \to V/G$ is the quotient epimorphism. Since $\ker\tau$ is discrete, ψ_α is open. Now $\widetilde{V/G}$, being locally defined, is locally generated (Corollary 62, [2]). Since η is a continuous surjection, $(\widetilde{V/G})/K_\alpha$ is also locally generated. Therefore ψ_α, being open, is also a surjection. In addition, the fact that $\ker\tau$ is discrete implies that $\ker\psi_\alpha$ is an open subgroup H_α of G (which is normal in V), and $(\widetilde{V/G})/K_\alpha = V/H_\alpha$.

Now begin with a fixed (relatively) open subgroup H_α of G that is normal in V. Then as in the previous case we have a factoring $\pi = \gamma \circ \beta$, by quotient epimorphisms $\beta : V \to V/H_\alpha$ and $\gamma : V/H_\alpha \to V/G$, and $\ker\gamma$ is discrete. Hence γ is a (traditional) cover, and so there exists a unique cover $\theta : (\widetilde{V/G}) \to V/H_\alpha$ such that $\gamma \circ \theta = \phi$. Since $\ker\gamma$ is discrete, $K_\alpha := \ker\theta$ is an open subgroup of K, and $V/H_\alpha = (\widetilde{V/G})/K_\alpha$. In other words, the two families $\{V/H_\alpha\}$ and $\{(\widetilde{V/G})/K_\alpha\}$ are one and the same, and hence have the same inverse limit. □

Corollary 3. *Let V be a locally defined topological group, G be a closed normal subgroup of V, and $\phi : \widetilde{V/G} \to V/G$ be the universal cover of V/G. Let $\widetilde{\pi} : V \to \widetilde{V/G}$ be the lift of the quotient epimorphism $\pi : V \to V/G$. Then*

1. *$\widetilde{\pi}(V)$ is dense in $\widetilde{V/G}$*
2. *$\ker \widetilde{\pi}$ is the intersection I of all (relatively) open subgroups of G that are normal in V*
3. *$\widetilde{\pi}$ is one-to-one if and only if G is totally asunder and central in V.*

Proof. The first part of the corollary is an immediate consequence of Theorem 97 in [2], since π is open. For part 2 note that by Theorem 2, $\widetilde{V/G}$ is isomorphic to $H := \varprojlim V/H_\alpha$, where $\{H_\alpha\}$ is the collection of open subgroups in G that are normal in V. By uniqueness of $\widetilde{\pi}$, $\widetilde{\pi}$ must be the homomorphism induced on the inverse limit $\widetilde{V/G}$ by the quotient maps $\psi_\alpha : V \to V/H_\alpha$, the kernel of which is precisely I.

Now if G is totally asunder and central in V, every subgroup of G is normal in V, and by part 2, $\ker \widetilde{\pi}$ is trivial. Conversely, if $\ker \widetilde{\pi}$ is trivial, then part 2 implies that G is totally asunder. In addition, $\widetilde{\pi}(G) \subset \ker \phi$, which is central in $\widetilde{V/G}$ and therefore $\widetilde{\pi}(G)$ is central in $\widetilde{\pi}(V) \subset \widetilde{V/G}$. Since $\widetilde{\pi}$ is injective, G is central in V. □

Corollary 4. *If G is a closed, locally generated subgroup of a locally defined group V then $\widetilde{V/G} = V/G$, i.e., V/G is locally defined.*

Proof. Since G is locally generated, evidently G has only one open subgroup, G. Hence by Theorem 2, $\widetilde{V/G} = V/G$ and by Theorem 4, [2], this means V/G is locally defined. □

If V is a topological vector space then there are several simplifications in these results. Of course all consideration of normality and centrality can be removed. It is also true that every topological vector space is locally arcwise connected ([5]), and in the locally connected case the proof of Theorem 2 can be simplified somewhat.

It is worthwhile to ask, given a closed, line-free subgroup G of a topological vector space V, when does $\widetilde{V/G}$ have the structure of a topological vector space?

Proposition 5. *Let V be a topological vector space and G be a closed subgroup of V.*

1. *If G is line-free and $\widetilde{V/G}$ is a topological vector space then the lift $\widetilde{\pi}$ of $\pi : V \to V/G$ to $\widetilde{V/G}$ is an injective linear map. In particular $\widetilde{\pi}(V)$ is a dense vector subspace of $\widetilde{V/G}$ and G is totally asunder.*
2. *If G is totally asunder then $\widetilde{V/G}$ is a topological vector space if and only if for every open set $U \ni 0$ in V and (relatively) open subgroup H in G, there exists some (relatively) open subgroup H_1 of G such that for all $-1 \leq t \leq 1$, $tH_1 \subset U + H$.*

Proof. To prove part 1 note that $\widetilde{\pi} : V \to \widetilde{V/G}$ is a (continuous) homomorphism into the topological vector space $\widetilde{V/G}$. For any $v \in V$, the function $\eta_v : \mathbb{R} \to \widetilde{V/G}$ given by $\eta_v(t) = \widetilde{\pi}(tv)$ is a one-parameter subgroup, and hence is linear. Therefore for any t and v we have

$$\widetilde{\pi}(tv) = \eta_v(t) = t\eta_v(1) = t\widetilde{\pi}(v)$$

and so $\widetilde{\pi}$ is linear, since $\widetilde{\pi}$ is already a homomorphism of the additive groups. Therefore $\ker \widetilde{\pi}$ must be a linear subspace of V that is contained in G (because $\pi = \phi \circ \widetilde{\pi}$). Since G is line-free, $\ker \widetilde{\pi} = 0$. The rest of the statement follows from Corollary 3.

To prove the second part, let σ be the topology on V induced by the lift $\widetilde{\pi} : V \to \widetilde{V/G}$ of the quotient homomorphism $\pi : V \to V/G$. Examination of the preimage topology from Theorem 2 shows that a basis at 0 for σ consists of sets of the form $U + H$, where H is a (relatively) open subgroup in G and $U \ni 0$ is an open subset in V. Since $\widetilde{\pi}$ is an injective group homomorphism with dense image, $\widetilde{V/G}$ is a topological vector space if and only if (V, σ) is a topological vector space. However, this is true if and only if for each such $U + H$ there exists some (relatively) open subgroup H_1 of G such that for all $-1 \leq t \leq 1$, $tH_1 \subset U + H$. □

Being totally asunder is not by itself sufficient in Proposition 5 for $\widetilde{V/G}$ to be a topological vector space, as we will now show.

Proposition 6. *Let $\{b_1, ...\}$ be a basis for a topological vector space V and G be the closure of the subgroup generated by $\{b_1, ...\}$ in V. Then G consists of all elements of V with integer coordinates relative to $\{b_1, ..., \}$ and G is weakly closed and totally asunder.*

Proof. The coordinate maps f_i with respect to a basis $\{b_i\}$ in a topological vector space are by definition continuous linear functionals. Certainly the subgroup G_I of all elements of V with integer coordinates relative to $\{b_i\}$ is a dense subgroup of G. On the other hand it is easy to see that G_I is closed relative to the weakest topology in which every f_i is continuous. Hence G_I is weakly closed and therefore closed in V, so $G_I = G$. Now $0 \in G = G_I$ is the intersection of the subgroups $\ker f_i \subset G$. But all these groups are open in G. Hence G is totally asunder. □

Proposition 7. *Let $\{k_n\}$ be a sequence of positive real numbers such that $k_n \searrow 0$ and $\sum_{n=1}^{\infty} k_n^2 = \infty$. Let G be the closure of the subgroup of l^2 generated by the elements $b_n := k_n e_n$, where $\{e_n\}$ is the standard basis of l^2. Then G is a weakly closed, totally asunder subgroup of l^2 and $\widetilde{l^2/G}$ is not a topological vector space.*

Proof. By Proposition 6, G is weakly closed and totally asunder. We need only show that $\widetilde{l^2/G}$ is not a topological vector space. For every k, let G_k denote the closure of the group generated by $\{b_n\}_{n\geq k}$. Let H be an open subgroup of G. Then for some open ball U at 0, $G \cap U \subset H \cap U$. For some k, U contains every b_n with $n \geq k$. The corresponding group G_k is contained in H. Since every open subgroup H of G contains an open subgroup G_k, according to Proposition 5 the group $\widetilde{l^2/G}$ is a topological vector space only if for any open ball U at 0 of radius r and G_k, for some l we have $tG_l \subset G_k + U$ for all $-1 \leq t \leq 1$. But for any $l \geq k$ we can find $m > l$ such that $g := \sum_{n=l}^{m} b_n \in G_l$ has length $\|g\| \geq 2r$. But then the orthogonality of $\{b_n\}$ implies that

$$r \leq d(\frac{1}{2}g, G_l) = d(\frac{1}{2}g, G_k)$$

and therefore $\frac{1}{2}g \notin G_k + U$. □

Remark 8. A similar statement, construction and proof are also valid for all $l^p, p \geq 1$.

Remark 9. Let G be a closed subgroup of a topological vector space V. We now know that if G is totally asunder, $\widetilde{V/G}$ need not be a topological vector space. However, if we strengthen "totally asunder" to "prodiscrete" then $\pi : V \to V/G$ is a cover, and since V is locally defined it follows that $\widetilde{V/G}$ is isomorphic to the topological vector space V. This provides another way to see that the groups constructed by Proposition 7 are not prodiscrete.

We finish with a positive answer to the question of when the universal cover of a quotient of a topological vector space is again a topological vector space. First we recall an essential result from [6].

Corollary 10. ([6]) *If Γ is a subgroup of a topological vector space X then the character group $\widehat{X/\Gamma}$ separates the points of X/Γ if and only if Γ is weakly closed in X.*

Notation 11. If G is an abelian topological group, by $\overline{G}$ we denote the completion of G. Note that if V is a topological vector space then $\overline{V}$ is also a topological vector space.

Proposition 12. *Let G be a closed, cocompact line-free subgroup of a topological vector space V. Then G is weakly closed and pre-prodiscrete relative to the weak topology $\sigma := \sigma(G^*)$ on G defined by*

$$G^* := \{\phi \in V^* : \phi(G) \subset \mathbb{Z}\} \ \ ([6]).$$

Moreover, V/G is isomorphic to $(V,\sigma)/(G,\sigma)$ and $\overline{(V,\sigma)}/\overline{(G,\sigma)}$, and $\overline{G} := \overline{(G,\sigma)}$ is prodiscrete. Hence $\overline{V} \to \overline{V}/\overline{G} = V/G$ is the universal cover, and $\overline{V}$ is a topological vector space isomorphic to $\mathbb{R}^\lambda$ with the product topology, for some indexing set λ.

Proof. Since G is cocompact, by Pontryagin duality the characters of V/G separate the points of V/G, and the proof that G is weakly closed is finished by Corollary 10 above. The family G^* and the topology σ were defined in [6], and it was shown in Corollary 2.2, [6] that G is 0-dimensional with respect to σ and hence totally disconnected in the original topology of V. But a little more than this is true. The sets $\ker\eta$, $\eta \in G^*$, are open (normal) subgroups of (G,σ), and these subgroups form a subbasis at 0 for the topology σ, and therefore (G,σ) is preprodiscrete, and $\overline{G}$ is prodiscrete. Evidently the identity map $\phi : V/G \to (V,\sigma)/(G,\sigma)$ is a continuous bijection (the topology of σ is weaker) and because V/G is compact, $(V,\sigma)/(G,\sigma)$ is also compact and ϕ is an isomorphism. The inclusion $(V,\sigma) \to \overline{(V,\sigma)}$ has dense image and induces a continuous homomorphism $\psi : (V,\sigma)/(G,\sigma) \to \overline{(V,\sigma)}/\overline{(G,\sigma)}$, with compact, dense image. Hence ψ is an epimorphism. Since $(V,\sigma) \cap \overline{(G,\sigma)} = (G,\sigma)$, ψ is one-to-one. Once again, the fact that $(V,\sigma)/(G,\sigma)$ is compact implies that ψ is an isomorphism. Now (V,σ) is a topological vector space, and hence so is $\overline{(V,\sigma)}$. That is, $\overline{V}$ is locally defined, and since $\overline{G}$ is prodiscrete, the quotient epimorphism $\overline{V} \to \overline{V}/\overline{G} = V/G$ is a cover. The universal cover of V/G is the unique (up to isomorphism) cover of V/G by a locally defined group ([2]), and therefore $\overline{V} \to V/G$ must be the universal cover. In [3] the universal cover of a compact, arcwise connected group was shown to be $\mathbb{R}^\lambda$. □

Remark 13. In general case in Proposition 12, $\overline{(V,\sigma)}$ is not isomorphic to V.

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