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# SELECTIVE SCREENABILITY GAME AND COVERING DIMENSION

LILJANA BABINKOSTOVA

ABSTRACT. We introduce an infinite two-person game inspired by the selective version of R. H. Bing's notion of screenability. We show how, for metrizable spaces, this game is related to covering dimension.

## 1. INTRODUCTION

Let $X$ be a topological space. In [3], R. H. Bing introduced the following notion of *screenability*: For each open cover $\mathcal{U}$ of $X$ there is a sequence $(\mathcal{V}_n : n < \infty)$ such that for each $n$, $\mathcal{V}_n$ is a family of pairwise disjoint open sets; for each $n$, $\mathcal{V}_n$ refines $\mathcal{U}$ and $\cup_{n<\infty}\mathcal{V}_n$ is an open cover of $X$. In [1], David F. Addis and John H. Gresham introduced the selective version of screenability: For each sequence $(\mathcal{U}_n : n < \infty)$ of open covers of $X$ there is a sequence $(\mathcal{V}_n : n < \infty)$ such that for each $n$, $\mathcal{V}_n$ is a family of pairwise disjoint open sets; for each $n$, $\mathcal{V}_n$ refines $\mathcal{U}_n$ and $\cup_{n<\infty}\mathcal{V}_n$ is an open cover of $X$. It is evident that selective screenability implies screenability.

Selective screenability is an example of the following selection principle which was introduced in [2]: Let $S$ be a set and let $\mathcal{A}$ and $\mathcal{B}$ be families of collections of subsets of the set $S$.[1] Then $\mathsf{S}_c(\mathcal{A}, \mathcal{B})$

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[1]Thus, if $\mathcal{U}$ is a member of $\mathcal{A}$ or of $\mathcal{B}$, then $\mathcal{U}$ is a collection of subsets of $S$.

denotes the statement that for each sequence $(\mathcal{U}_n : n < \infty)$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n < \infty)$ such that

(1) for each $n$, $\mathcal{V}_n$ is a family of pairwise disjoint sets;
(2) for each $n$, $\mathcal{V}_n$ refines $\mathcal{U}_n$; and
(3) $\cup_{n<\infty}\mathcal{V}_n$ is a member of $\mathcal{B}$.

With $\mathcal{O}$ denoting the collection of all open covers of topological space $X$, $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ is selective screenability.

Addis and Gresham noted that countable dimensional metrizable spaces are selectively screenable and asked if the converse is true. Roman Pol, in [7], showed that the answer is no. We will now show that the countable dimensional metric spaces are exactly characterized by a game-theoretic version of selective screenability.

The following game, denoted $\mathsf{G}_c(\mathcal{A}, \mathcal{B})$, is naturally associated with $\mathsf{S}_c(\mathcal{A}, \mathcal{B})$. Players ONE and TWO play as follows: In the $n$-th inning, ONE first chooses $\mathcal{O}_n$, a member of $\mathcal{A}$, and then TWO responds with $\mathcal{T}_n$ which is pairwise disjoint and refines $\mathcal{O}_n$. A play $(\mathcal{O}_1, \mathcal{T}_1, \cdots, \mathcal{O}_n, \mathcal{T}_n, \cdots)$ is won by TWO if $\cup_{n<\infty}\mathcal{T}_n$ is a member of $\mathcal{B}$; else, ONE wins. We can consider versions of different lengths of this game. For an ordinal number $k$, let $\mathsf{G}_c^k(\mathcal{A}, \mathcal{B})$ be the game played as follows: In the $l$-th inning ($l < k$), ONE first chooses $\mathcal{O}_l$, a member of $\mathcal{A}$, and then TWO responds with a pairwise disjoint $\mathcal{T}_l$ which refines $\mathcal{O}_l$. A play

$$\mathcal{O}_0, \mathcal{T}_0, \cdots, \mathcal{O}_l, \mathcal{T}_l, \cdots\ l < k$$

is won by TWO if $\cup_{l<k}\mathcal{T}_l$ is a member of $\mathcal{B}$; else, ONE wins. Thus, the game $\mathsf{G}_c(\mathcal{A}, \mathcal{B})$ is $\mathsf{G}_c^\omega(\mathcal{A}, \mathcal{B})$.

## 2. Main Results

From now on we assume that the spaces we work with are metrizable. We will see how selective screenability is related to covering dimension by showing that

(1) a metrizable space is countable-dimensional if and only if TWO has a winning strategy in the game $\mathsf{G}_c^\omega(\mathcal{O}, \mathcal{O})$ (Theorem 2.2);
(2) for each nonnegative integer $n$, a metrizable space $X$ is $\leq n$-dimensional if and only if TWO has a winning strategy in $\mathsf{G}_c^{n+1}(\mathcal{O}, \mathcal{O})$ (Theorem 2.4).

We will use the following result:

**Lemma 2.1** ([6, Theorem 2, p. 226]). *Let $X$ be a space and let $Y$ be a subspace of $X$. Let $(V_i : i \in I)$ be a collection of subsets of $Y$ open in $Y$. Then there is a collection $(U_i : i \in I)$ of open subsets of $X$ such that for each $i \in I$, we have $V_i = Y \cap U_i$, and for each finite subset $F$ of $I$, if $\cap_{i\in F} V_i = \emptyset$, then $\cap_{i\in F} U_i = \emptyset$.*

**Theorem 2.2.** *Let $X$ be a metric space.*

(1) *If $X$ is countable dimensional, then TWO has a winning strategy in $\mathsf{G}_c^{\omega}(\mathcal{O}, \mathcal{O})$.*
(2) *If TWO has a winning strategy in $\mathsf{G}_c^{\omega}(\mathcal{O}, \mathcal{O})$, then $X$ is countable dimensional.*

*Proof of (1):* Let $X$ be countable dimensional, i.e., $X = \cup_{n<\infty} X_n$ where each $X_n$ is zero-dimensional. We will define a Markov strategy (for definition, see [4]) $\sigma$ for player TWO: For an open cover $\mathcal{U}$ of $X$ and $n < \infty$, $\mathcal{U}$ is an open cover of $X_n$. Since $X_n$ is zero-dimensional, find a pairwise disjoint family $\mathcal{V}$ of subsets of $X_n$ open in $X_n$ such that $\mathcal{V}$ covers $X_n$ and refines $\mathcal{U}$. By Lemma 2.1, choose a pairwise disjoint family $\sigma(\mathcal{U}, n)$ of open subsets of $X$ refining $\mathcal{U}$ such that each element $V$ of $\mathcal{V}$ is of the form $U \cap X_n$ for some $U \in \sigma(\mathcal{U}, n)$. Now TWO plays as follows: In inning 1, ONE plays $\mathcal{U}_1$, and TWO responds with $\sigma(\mathcal{U}_1, 1)$, thus covering $X_1$. When ONE has played $\mathcal{U}_2$ in the second inning, TWO responds with $\sigma(\mathcal{U}_2, 2)$, thus covering $X_2$, and so on. And in the $n$-th inning, when ONE has chosen the cover $\mathcal{U}_n$ of $X$, TWO responds with $\sigma(\mathcal{U}_n, n)$, covering $X_n$. This strategy evidently is a winning strategy for TWO.

*Proof of (2):* Let $\sigma$ be a winning strategy for TWO. Let $\mathcal{B}$ be a base for the metric space $X$. For each $n$, let $\mathcal{B}_n$ be the family $\{B \in \mathcal{B} : diam(B) < \frac{1}{n}\}$. Consider the plays of the game in which, in each inning, ONE chooses for some $n$ a cover of the form $\mathcal{B}_n$ of $X$.

Define a family $(C_\tau : \tau \in {}^{<\omega}\mathbb{N})$ of subsets of $X$ as

(1) $C_\emptyset = \cap\{\cup\sigma(\mathcal{B}_n) : n < \infty\}$;
(2) for $\tau = (n_1, \cdots, n_k)$, $C_\tau = \cap\{\cup\sigma(\mathcal{B}_{n_1}, \cdots, \mathcal{B}_{n_k}, \mathcal{B}_n) : n < \infty\}$.

We will show that $X = \cup\{C_\tau : \tau \in {}^{<\omega}\mathbb{N}\}$. Suppose, to the contrary, that $x \notin \cup\{C_\tau : \tau \in {}^{<\omega}\mathbb{N}\}$. Let us choose an $n_1$ such that $x \notin \sigma(\mathcal{B}_{n_1})$. With $n_1, \cdots, n_k$ chosen such that $x \notin \sigma(\mathcal{B}_{n_1}, \cdots, \mathcal{B}_{n_k})$, let us choose an $n_{k+1}$ such that $x \notin \sigma(\mathcal{B}_{n_1}, \cdots, \mathcal{B}_{n_{k+1}})$, and so on. Then

$$\mathcal{B}_{n_1}, \sigma(\mathcal{B}_{n_1}), \mathcal{B}_{n_2}, \sigma(\mathcal{B}_{n_1}, \mathcal{B}_{n_2}), \cdots$$

is a $\sigma$-play lost by TWO, contradicting the fact that $\sigma$ is a winning strategy for TWO.

Also, we will show that each $C_\tau$ is zero-dimensional. Let $x \in C_\tau$ and let $\tau = (n_1, \cdots, n_k)$ be given. Thus, $x$ is a member of $\cap\{\cup\sigma(\mathcal{B}_{n_1}, \cdots, \mathcal{B}_{n_k}, \mathcal{B}_n) : n < \infty\}$. For each $n$, choose a neighborhood $V_n(x) \in \sigma(\mathcal{B}_{n_1}, \cdots, \mathcal{B}_{n_k}, \mathcal{B}_n)$. Since for each $n$ we have $diam(V_n(x)) < \frac{1}{n}$, the set $\{V_n(x) \cap C_\tau : n < \infty\}$ is a neighborhood basis for $x$ in $C_\tau$. Also, we have that each $V_n(x)$ is closed in $C_\tau$ because of disjointness of TWO's chosen sets. The set $V = \cup\sigma(\mathcal{B}_{n_1}, \cdots, \mathcal{B}_{n_k}, \mathcal{B}_n) \setminus V_n(x)$ is open in $X$ and so $C_\tau \setminus V_n(x) = C_\tau \cap V$ is open in $C_\tau$. Thus, each element of $C_\tau$ has a basis consisting of clopen sets. □

Observe that in the proof of Theorem 2.2 we show:

**Corollary 2.3.** *Let $X$ be a metric space. The following are equivalent.*

(1) *TWO has a winning strategy in $\mathsf{G}_c^\omega(\mathcal{O}, \mathcal{O})$.*
(2) *TWO has a winning Markov strategy in $\mathsf{G}_c^\omega(\mathcal{O}, \mathcal{O})$.*

The proof of the following theorem uses the ideas in the proof of Theorem 2.2.

**Theorem 2.4.** *Let $X$ be a metric space. The following are equivalent.*

(1) *If $X$ is $\leq n$-dimensional then TWO has a winning strategy in $\mathsf{G}_c^{n+1}(\mathcal{O}, \mathcal{O})$.*
(2) *If TWO has a winning strategy in $\mathsf{G}_c^{n+1}(\mathcal{O}, \mathcal{O})$, then $X$ is $\leq n$-dimensional.*

From this theorem, we obtain that the metric space $X$ is $n$-dimensional if and only if TWO has a winning strategy in $\mathsf{G}_c^{n+1}(\mathcal{O}, \mathcal{O})$ but not in $\mathsf{G}_c^n(\mathcal{O}, \mathcal{O})$.

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Department of Mathematics; Boise State University; Boise, ID 83725

*E-mail address*: liljanab@math.boisestate.edu