NOTES ON \textit{g-METRIZABLE SPACES}

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\textsc{Abstract.} A space is called a g-metrizable space if it is a regular space with a $\sigma$-locally finite weak base (see F. Siwiec, “On Defining a Space by a Weak Base”). In this paper, we discuss spaces with a $\sigma$-HCP (wHCP) weak base and give answers or partial answers to questions posed by A. V. Arhangel’skii during a seminar in 2004 at Ohio University, by S. Lin in a personal communication, and by Y. Tanaka in “$\sigma$-Hereditarily Closure Preserving k-Networks and g-Metrizability.”

1. Introduction

Weak base was introduced by A. V. Arhangel’skii [1] in 1966. Frank Siwiec [14] defined g-metrizable spaces as a spaces with a $\sigma$-locally finite weak base. Yoshio Tanaka [16], L. Foged [4], Shou Lin [8], and Chuan Liu and Mu Min Dai [11] have made much contribution on this field. We discuss topological spaces with a $\sigma$-HCP (wHCP) weak base and give answers or partial answers to Lin’s, Arhangel’skii’s and Tanaka’s questions.

In this paper all spaces are regular and $T_1$; all mappings are continuous and onto. $\mathbb{N}$ denotes the natural numbers. Readers may refer to [6] for unstated definitions.

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2. g-METRIZABLE SPACES

**Definition 2.1.** Let $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ be a cover of a space $X$ such that for each $x \in X$,

1. $\mathcal{P}_x$ is a network of $x$ in $X$;
2. If $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

$\mathcal{P}$ is called a weak base [1] for $X$ if whenever $G \subset X$ satisfying for each $x \in G$ there is $P \in \mathcal{P}_x$ with $P \subset G$, then $G$ is open in $X$; $\mathcal{P}$ is called an sn-network [9] for $X$ if each element of $\mathcal{P}_x$ is a sequential neighborhood of $x$ in $X$ (i.e., every convergent sequence with the limit point $x$ is eventually in the element) for each $x \in X$.

A space $X$ is called a $g$-metrizable space [14] (resp., an sn-metrizable space [5]) if it has a $\sigma$-locally finite weak base (resp., sn-network), and a space is $g$-first countable [1], (resp., sn-first countable) if each $\mathcal{P}_x$ is countable. A space is called $g$-second countable [14] if it has a countable weak base.

A collection $\{C_\alpha : \alpha \in I\}$ of subsets of $X$ is called hereditarily closure preserving, HCP (weakly hereditarily closure preserving, wHCP) [3] if for any $J \subset I$, $\{B_\alpha : B_\alpha \subset C_\alpha, \alpha \in J\}$ is closure preserving ($\{x_\alpha : x_\alpha \in C_\alpha, \alpha \in J\}$ is closed discrete).

Tanaka [16] asked the following.

**Question 2.2.** If $X$ has a $\sigma$-HCP weak base, is $X$ g-metrizable?

By slightly modifying Liang-Xue Peng’s proof in [13], we can obtain the following.

**Lemma 2.3.** Suppose that $X$ has a $\sigma$-closure preserving weak base, then $X$ is hereditary meta-Lindelöf.\(^1\)

Tanaka [16] proved that a Lindelöf space with a $\sigma$-HCP weak base is g-second countable and that the following proposition holds under (CH). By using Lemma 2.3, we may omit (CH).

**Proposition 2.4.** $X$ is g-second countable if and only if $X$ is a separable space with a $\sigma$-HCP weak base.

$X$ has property (**) if for any non-isolated point $x$ of $X$, there is countable subset $D \subset X$ so that $x \in \text{cl}(D \setminus \{x\})$. A space $X$ has property (*) [10] if for any non-isolated point $x$ of $X$, there is

\(^1\)Lin informed the author that he also obtained this result.
a non-trivial sequence converging to \( x \). Obviously, a space having countable tightness\(^2\) or property (*) has property (**).

**Lemma 2.5.** Let \( X \) have a \( \sigma \)-HCP weak base. Then \( X \) has property (*) if it has property (**).

**Proof:** For a non-isolated point \( x \in X \), there is a countable subset \( D \subset X \) with \( x \in cl(D\{x\}) \). By Lemma 2.3, \( cl(D) \) is a Lindelöf space. Tanaka [16] proved that every \( \omega_1 \)-compact space with a \( \sigma \)-HCP weak base is g-first countable. \( cl(D\{x\}) \) is a sequential space; \( X \) has property (*). \( \square \)

The author [10] proved that \( X \) is a g-metrizable space if and only if \( X \) has a \( \sigma \)-HCP weak base and property (*); by Lemma 2.5, we have

**Theorem 2.6.** \( X \) is g-metrizable if and only if \( X \) has a \( \sigma \)-HCP weak base and property (**).

**Corollary 2.7.** A space \( X \) is g-metrizable if and only if \( X \) has a \( \sigma \)-HCP weak base and has countable tightness.

Next, we discuss spaces with a \( \sigma \)-wHCP weak base. In [3], it was proved that a k-space with a \( \sigma \)-wHCP base is metrizable, but not every (paracompact) space with a \( \sigma \)-wHCP base is metrizable. Thus, every space with a \( \sigma \)-wHCP weak base need not to be g-metrizable. We don’t know if a k-space with a \( \sigma \)-wHCP weak base is g-metrizable or not. But we have following.

**Theorem 2.8.** (CH) Suppose \( X \) is a separable space with a \( \sigma \)-wHCP weak base, then \( X \) is g-second countable, hence g-metrizable.

First, let us prove a lemma.

**Lemma 2.9.** Suppose that \( X \) has a \( \sigma \)-wHCP weak base, then \( X \) has a \( \sigma \)-wHCP k-network\(^3\) and a \( \sigma \)-compact-finite k-network.

**Proof:** Let \( B = \bigcup_{n \in \mathbb{N}} B_n \) is a \( \sigma \)-wHCP weak base. We may assume \( B_n \subset B_{n+1} \). For each \( n \in \mathbb{N} \), let \( D_n = \{ x \in X : B_n \text{ is not point-finite at } x \} \). Let \( B'_n = \{ B \setminus D_n : B \in B_n \} \cup \{ \{ x \} : x \in D_n \} \).

\(^2\)A space \( X \) has countable tightness if whenever \( x \in cl(A) \) for \( x \in X \) and a subset \( A \) of \( X \), there is a countable subset \( C \subset A \) such that \( x \in cl(C) \).

\(^3\)A cover \( \mathcal{P} \) of subsets of \( X \) is a k-network if, whenever \( K \subset U \) with \( K \) compact and \( U \) open in \( X \), there is a finite subfamily \( \mathcal{F} \subset \mathcal{P} \) such that \( K \subset \bigcup \mathcal{F} \subset U \).
Then $B'_n$ is compact-finite for each $n$. In fact, let $K$ be a compact subset of $X$; it is easy to see that $K \cap D_n$ is finite. Notice that \{\{B \setminus D_n : B \in B_n\} is wHCP and point-finite, and $K$ meets at most finitely many elements of \{\{B \setminus D_n : B \in B_n\}. Hence, $B' = \bigcup_{n \in \mathbb{N}} B'_n$ is a $\sigma$-compact-finite network. Any compact subset of $X$ has a countable network; hence, it is metrizable. By Proposition A(3) in [17], $B$ is a $\sigma$-wHCP k-network. Now we prove that $B'$ is a k-network. Let $K \subset U$ with $K$ compact and $U$ open; there are $m \in \mathbb{N}$ and a finite subfamily $P$ of $B_m$ such that $K \subset \bigcup P \subset U$. Let $F = \{P \setminus D_m : P \in P\} \cup \{\{x\} : x \in K \cap D_m\}$, then $F \subset B'_m \subset B'$ and $K \subset \bigcup F \subset U$. \hfill \Box

Now we give a proof of Theorem 2.8.

Since $X$ is separable, by (CH), the character of $X$, $\chi(X) \leq \omega_1$. Let $B = \bigcup_{n \in \mathbb{N}} B_n = \bigcup \{P_x : x \in X\}$ is a $\sigma$-wHCP weak base. We may assume $B_n \subset B_{n+1}$. First, we prove that $X$ is g-first countable.

For $x \in X$, if $\{x\}$ is open, then $X$ is g-first countable at $x$. If $\{x\}$ is not open, $B_n \cap P_x$ is locally countable at $x$ for $n \in \mathbb{N}$. Suppose not. Let $\{V_\alpha : \alpha < \omega_1\}$ be the local base at $x$. Notice that for any neighborhood $V$ of $x$, $V \cap (P \setminus \{x\}) \neq \emptyset$ for $P \in P_x$. Then, by induction, there are a subset $S = \{x_\alpha : \alpha < \omega_1\}$ of $X$ and a subcollection $\{B_\alpha : \alpha < \omega_1\}$ of $B_n \cap P_x$ such that $x_\alpha \in V_\alpha \cap B_\alpha$, where $x_\alpha \neq x$, and the $B_\alpha$'s are distinct. $x$ is an accumulation of $S$, so $S$ is not closed. Since $B_n$ is wHCP, $S$ is a closed discrete subset; this is a contradiction. Hence, $X$ is g-first countable. By Lemma 2.9, $X$ has a $\sigma$-compact-finite k-network. Under (CH), a separable, sequential space with a $\sigma$-compact-finite k-network is an $\aleph_0$-space\footnote{A space with a countable k-network.} [12]. $X$ is a g-first countable, $\aleph_0$-space; hence, $X$ is g-second countable [14].

We don’t know if we can omit (CH) or not in the above theorem.

**Question 2.10.** Is a separable space with a $\sigma$-wHCP weak base g-second countable?

We define iterates of the operator seq $cl$ inductively for a space $X$ as follows:

1. $\text{seq } cl^0(S) = S$;
2. $\text{seq } cl(S) = \{x : x$ is a limit point of $S\}$;
(3) if $\alpha$ is an ordinal, let $\text{seq } cl^{\alpha+1}(S) = \text{seq } cl(\text{seq } cl^\alpha(S))$;

(4) if $\alpha$ is a limit ordinal, let $\text{seq } cl^\alpha = \bigcup_{\beta<\alpha} \text{seq } cl^\beta(S)$.

We define iterates of the operator $\text{seq } cl$ inductively for a space $X$ as follows: $\text{seq } cl^0(S) = S$; $\text{seq } cl(S) = \{ x : x \text{ is a limit point of } S \}$; if $\alpha$ is an ordinal, let $\text{seq } cl^{\alpha+1}(S) = \text{seq } cl(\text{seq } cl^\alpha(S))$; if $\alpha$ is a limit ordinal, let $\text{seq } cl^\alpha = \bigcup_{\beta<\alpha} \text{seq } cl^\beta(S)$. If $X$ is sequential space, the sequential order of $X$ is the least ordinal $\alpha$ so that for every subset $S$ of $X$ we have $cl(S) = \text{seq } cl^\alpha(S)$. A subset $D$ of $X$ is $\omega_1$-compact if any subset of $D$ with cardinality $\omega_1$ has a cluster point.

**Lemma 2.11.** Let $X$ have a $\sigma$-wHCP weak base. If $A \subset X$ is $\omega_1$-compact, then $\text{seq } cl(A)$ is $\omega_1$-compact.

**Proof:** Assume to the contrary that there is a discrete subset $\{x_\alpha : \alpha < \omega_1\}$ in $\text{seq } cl(A) \setminus A$. For $\alpha < \omega_1$, let $\{x_n(\alpha)\} \subset A$ be a sequence converging to $x_\alpha$. Let $B = \bigcup_{n \in \mathbb{N}} B_n$ be a $\sigma$-wHCP weak base of $X$. We may assume $B_n \subset B_{n+1}$. For each $\alpha$, there is $B_\alpha \in B$ such that $x_\alpha \in B_\alpha$, $B_\alpha$ contains a tail of $\{x_n(\alpha)\}$, and all $B_\alpha \cap \{x_\beta : \beta \neq \alpha\} = \emptyset$. Without loss of generality, we assume $B_\alpha$ contains $\{x_n(\alpha)\}$ and $B_\alpha \in B_n$ for some $n$.

**Case 1.** $|\{x_n(\alpha) : n \in \mathbb{N}, \alpha < \omega_1\}| = \omega_1$.

By induction, there is an uncountable subset $S = \{x_\beta : \beta < \omega_1\}$ of $\{x_n(\alpha) : n \in \mathbb{N}, \alpha < \omega_1\}$ such that $x_\beta \in B_\beta$ and $B_\beta \neq B_\gamma$ if $\beta \neq \gamma$. Since $B_n$ is wHCP, $S \subset A$ is closed discrete; this is a contradiction.

**Case 2.** $|\{x_n(\alpha) : n \in \mathbb{N}, \alpha < \omega_1\}| \neq \omega_1$.

There exists an $\alpha_0$ such that infinitely many $B_\alpha$’s contain a subsequence of $\{x_n(\alpha_0)\}$. Suppose not. For every $\alpha < \omega_1$, there is $m(\alpha)$ such that $\{B_\alpha : \alpha < \omega_1\}$ is point-finite at $x_{m(\alpha)}(\alpha)$. Since $|\{x_{m(\alpha)}(\alpha) : n \in \mathbb{N}, \alpha < \omega_1\}| \neq \omega_1$ and $x_{m(\alpha)}(\alpha) \in B_\alpha$, $\{B_\alpha : \alpha < \omega_1\}$ is not point-finite at some $x_{m(\alpha)}(\alpha)$; this is a contradiction. Thus, infinitely many $B_\alpha$’s contain a subsequence of $\{x_n(\alpha_0)\}$, then $\{x_n(\alpha_0)\}$ has a subsequence that is discrete, a contradiction. Hence, $\text{seq } cl(A)$ is $\omega_1$-compact.

**Theorem 2.12.** Let $X$ be a separable space with a $\sigma$-wHCP weak base. If the sequential order of $X$ is countable, then $X$ is $g$-second countable.
Proof: Let $D \subset X$ with $|D| = \omega$, $cl(D) = X$. Since the sequential order of $X$ is countable, $X = \cup_{n \in \mathbb{N}} seq cl^n(D)$. $D$ is $\omega_1$-compact, and $seq cl^n(D)$ is $\omega_1$-compact for each $n$ by Lemma 2.11; hence, $X$ is $\omega_1$-compact. Let $\mathcal{B} = \cup_{n \in \mathbb{N}} \mathcal{B}_n$ be a $\sigma$-wHCP weak base of $X$. For $x \in X$, if $x$ is not an isolated point, then $\mathcal{B} \cap P_x$ is locally countable at $x$. Suppose not. There is $n \in \mathbb{N}$ such that $\mathcal{B}_n \cap P_x$ is not locally countable at $x$. By induction, we can select an uncountable subset $\{x_\alpha: \alpha < \omega_1\}$ and an uncountable subfamily $\{\mathcal{B}_\alpha: \alpha < \omega_1\} \subset \mathcal{B}_n$ such that $\{x, x_\alpha\} \subset \mathcal{B}_\alpha$, $x_\alpha \neq x_\beta$ if $\alpha \neq \beta$. $\{x_\alpha: \alpha < \omega\}$ is discrete; this is a contradiction because $X$ is $\omega_1$-compact. Hence, $X$ is $g$-first countable.

$X$ is $\omega_1$-compact and has a $\sigma$-wHCP k-network by Lemma 2.9; hence, $X$ is an $\aleph_0$-space. Thus, $X$ is $g$-second countable. □

It is well known that $g$-metrizable spaces are not preserved by perfect mappings. Arhangel’skii [2], in a topology seminar at Ohio University, asked the following question:

**Question 2.13.** Let $X$ be a topological space; if every perfect image of $X$ is $g$-metrizable, is $X$ metrizable?

We shall give an affirmative answer to this question; in fact, we may prove a slightly stronger version.

The sequential fan $S_\omega$ is a perfect image of the Arens’ space $S_2$. It is well known that $S_\omega$ is not $g$-first countable.

**Theorem 2.14.** Let $X$ be a $g$-metrizable space. If every perfect image of $X$ has a $\sigma$-wHCP weak base, then $X$ is metrizable.

**Proof:** First, we prove that a space $Y$ with a $\sigma$-wHCP weak base $\mathcal{B} = \cup_{n \in \mathbb{N}} \mathcal{B}_n$ does not contain a copy of $S_\omega$. Suppose not. There is a non-trivial sequence $\{y_n\}$ converging to a point $y \in Y$ that is not an isolated point and $Y$ is not $g$-first countable at $y$. There is $n \in \mathbb{N}$ and infinitely many weak neighborhoods of $y$ in $\mathcal{B}_n$, each containing a tail of $\{x_n\}$. Since $\mathcal{B}_n$ is wHCP, there is a subsequence $L$ of $\{x_n\}$ such that $L$ is discrete. This is a contradiction.

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5 $S_\omega$ is a space obtained from the topological sum of $\omega$ many convergent sequences by identifying all limit points to a single point.

6 $S_2 = (\mathbb{N} \times \mathbb{N}) \cup \mathbb{N} \cup \{\infty\}$ is the space with each point of $\mathbb{N} \times \mathbb{N}$ isolated. A basic neighborhood of $n \in \mathbb{N}$ consists of all sets of the form $\{n\} \cup \{(m, n): m \geq k\}$. And $U$ is a neighborhood of $\infty$ if and only if $\infty \in U$ and $U$ is a neighborhood of all but finitely many $n \in \mathbb{N}$.
Since no perfect image of $X$ contains a copy $S_\omega$, then $X$ contains no copy of $S_2$. $X$ is a sequential space and every point is a $G_\delta$-set; hence, $X$ is a Fréchet-Urysohn space [15]. Since every Fréchet-Urysohn, g-metrizable space is metrizable [14], then $X$ is metrizable. □

Lin [9] introduced sn-networks to generalize weak bases (it is easy to see that a weak base is an sn-network). An sn-network is a weak base if the topological space is a sequential space. Many results on g-metrizable spaces can be generalized in terms of sn-networks [5]. Dai and the author [11] proved the following.

**Theorem 2.15.** Let $X$ be a $k$-space with a $\sigma$-HCP $k$-network, then $X$ is g-metrizable if $X$ contains no copy of $S_\omega$.

In a personal communication with the author, Lin asked if we can generalize the above theorem as follows:

**Question 2.16.** Let $X$ have a $\sigma$-HCP $k$-network. Does $X$ have a $\sigma$-locally finite sn-network if it contains no copy of $S_\omega$?

We give a negative answer to the question.

**Example 2.17.** There is an $\aleph_0$-space that contains no copy of $S_\omega$, but it is not sn-first countable.

**Proof:** Let $X = \{\infty\} \cup \{x_i(n) : i \in \mathbb{N}, n \in \mathbb{N}\}$, and let $\{f_\alpha : \alpha < 2^{\omega}\}$ be all maps from $\mathbb{N}$ to $\mathbb{N}$. Endow $X$ with topology as follows: each $\{x_i(n)\}$ is open for $i \in \mathbb{N}, n \in \mathbb{N}$; the neighborhood of $\infty$ is $X \setminus \cup \{x_{f(n)}(n) : f \in F \in \{f_\alpha : \alpha < 2^{\omega}\}^{<\omega}\}$. It is easy to see that $x_i(n) \to \infty$ for each $n \in \mathbb{N}$ and any convergent sequence in $X$ is contained in the finite union of $\{\{x_i(n)\} : n \in \mathbb{N}\}$.

(1) $X$ has a countable k-network. $\{\{x_i(n)\} : i \in \mathbb{N}, n \in \mathbb{N}\} \cup \{\{x_i(n) : i > m\} \cup \{\infty\} : m \in \mathbb{N}, n \in \mathbb{N}\}$ is a countable cs*-network

for $X$. Since $X$ is countable, then each point of $X$ is a $G_\delta$-set; hence, $X$ has a countable k-network by Proposition B(1) in [17].

(2) $X$ contains no copy of $S_\omega$. Let $g : \mathbb{N} \to \mathbb{N}$ be a surjection. It is obvious that $\{x_i(n) : i \leq g(n), n \in \mathbb{N}\}$ is not closed in $X$; therefore, $X$ contains no copy of $S_\omega$.

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7A cover $P$ is a cs*-network of $X$ if whenever $\sigma$ is a sequence converging to a point $x$ and $x \in U$ with $U$ open, then for some $P \in P$, $x \in P \subset U$, and $P$ contains a subsequence of $\sigma$. 
(3) $X$ is not sn-first countable. Suppose not. Let $\{P_n : n \in \mathbb{N}\}$ be a decreasing countable sn-network at $\infty$. For each $n \in \mathbb{N}$, $x_i(n) \to \infty$, pick $x_i(n) \in P_n$, then $x_i(n) \to \infty$. This is a contradiction because $X \setminus \{x_i(n)\}$ is an open neighborhood of $\infty$. □

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References


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