Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY PROCEEDINGS
Volume 29, No. 1, 2005
Pages 277-291

A FUNCTIONAL EQUATION FOR THE LEFSCHETZ ZETA FUNCTIONS OF INFINITE CYCLIC COVERINGS WITH AN APPLICATION TO KNOT THEORY

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Abstract. The Weil conjecture is a delightful theorem for algebraic varieties on finite fields and an important model for dynamical zeta functions. In this paper, we prove a functional equation of Lefschetz zeta functions for infinite cyclic coverings which is analogous to the Weil conjecture. Applying this functional equation to knot theory, we obtain a new view point on the reciprocity of the Alexander polynomial of a knot.

Introduction

The Lefschetz zeta function is one of the dynamical zeta functions. Dynamical zeta functions are developed in order to study the number of fixed points or periodic points. These zeta functions are motivated by the Weil conjecture [1]. Therefore, studying dynamical zeta functions is usually modeled after Weil conjecture.

In addition, dynamical zeta functions are useful in geometric topology. For example, if you fix a map to a geometrical one, it can be related to topological invariants (e.g., [5], [7], [8], [15]). That is, a property of a topological invariant can be regarded as a property of the dynamical zeta function.

2000 *Mathematics Subject Classification.* Primary 57M27; Secondary 37C30.

The author was supported in part by JSPS fellowship for young scientists.

Alexander Grothendieck[1] (after the works of André Weil and Jean-Pierre Serre) observed that the Weil conjecture should be the consequence of a certain good cohomology theory, which is called the Weil cohomology, and in particular the functional equation should be derived from the Poincaré duality of the cohomology (see section 1.2). In this paper, we prove an analogous functional equation of Lefschetz zeta functions by following his idea and applying it to knot theory. To do that, we have to resolve the following two problems:

(1) The infinite cyclic covering of a knot complement is a non-compact odd-dimensional manifold but the Weil cohomology requires the even-dimensional Poincaré duality.
(2) This manifold has the boundary, so we need to deal with the relative version of the Poincaré duality (Lefschetz-Poincaré duality).

John Milnor's duality theorem for infinite cyclic coverings [18] resolves the first problem. The device for the second is the Lefschetz zeta function for the boundary, found in Definition 2.2. With those tools, the following theorem is proved in section 2.

Theorem. *Let M be a compact connected manifold of dimension n which may have a boundary ∂M, and $\tilde{M}$ an orientable infinite cyclic covering of M with $\dim H_*(\tilde{M};\mathbb{Q}) < \infty$. Let $f : (\tilde{M}, \partial\tilde{M}) \to (\tilde{M}, \partial\tilde{M})$ be a proper continuous map with degree $\lambda \neq 0$ with respect to the compact support cohomology $H^n_{\mathrm{cpt}}(\tilde{M}, \partial\tilde{M};\mathbb{Q})$. If n is odd, then the Lefschetz zeta function ζ_f and $\zeta_{f|\partial\tilde{M}}$ satisfies the functional equation:*

$$\frac{\zeta_f(1/\lambda z)^2}{\zeta_{f|\partial\tilde{M}}(1/\lambda z)} = \lambda^{\chi} z^{2\chi} \frac{\zeta_f(z)^2}{\zeta_{f|\partial\tilde{M}}(z)},$$

where χ is the Euler characteristic of $\tilde{M}$.

In Theorem 2.1, the Lefschetz zeta function for the boundary is essential. However, it is usually easy to compute it. In section

[1]Grothendieck and various others produced *Séminaire de Géométrie Algébrique du Bois-Marie* [Lectures Notes in Mathematics (151, 152, 153, 224, 225, 269, 270, 288, 305, 340, 589), Berlin: Springer-Verlag, 1970–1977; and *Cohomologie Locale des Faisceaux Cohrents et Thormes de Lefschetz Locaux et Globaux*, Amsterdam: North-Holland, 1968]. The volumes, SGA 1–7, have been scanned and are made available at http://modular.fas.harvard.edu/sga/.

3, we apply the functional equation to knot theory. Computing directly the Lefschetz zeta function for the boundary, we obtain the following:

Corollary [21]**.** *The Alexander polynomial $\Delta_K(z)$ of a knot K is reciprocal.*

This corollary implies that the reciprocity of the Alexander polynomial of a knot is a special case of the functional equation of Lefschetz zeta functions for infinite cyclic coverings. Formally, the reciprocity of the Alexander polynomial was interpreted as symmetry of the covering transformations. However, not only covering transformations but also more general maps have symmetry, and those symmetries are realized in the functional equation. That is, this functional equation would be useful in studying the symmetry of a knot.

The covering transformation of the infinite cyclic coverings plays an analogous role to the Frobenius automorphism in this paper. The similarity between the Lefschetz zeta function and the Reidemeister torsion (also the Alexander polynomial) was first pointed out by Milnor [18] and studied by some other investigators from many viewpoints. For example, John M. Franks [6] studied the Lefschetz zeta function of a flow on S^3 which is associated with a knot, and as a result he re-proved the reciprocity of the Alexander polynomial. We also refer to Masanori Morishita's article [19] for some other analogies with Iwasawa theory.

Grothendieck also studied p-adic cohomology theory to approach the Weil conjecture. That study immediately asks whether p-adic (co)homology theory is also helpful for knot theory. The answer is given in [20]. The p-adic coefficient Alexander module is helpful to investigate what the zeros of $\Delta_K(z)$ mean.

The first section is a preliminary section. In 1.1, we recall the Lefschetz zeta function. In 1.2, we review the Weil conjecture. In 1.3, we recall a duality theorem for infinite cyclic coverings of manifolds. This theorem was obtained by Milnor [18]. The proof of the main theorem, Theorem 2.1, is in section 2. We prove this theorem, using the duality of infinite cyclic coverings in place of the Weil cohomology's duality. In section 3, we apply the functional equation of Theorem 2.1 to knot theory.

1. Preliminaries

1.1. Dynamical zeta functions

Dynamical zeta functions are types of zeta function which have been developed in order to study the number of fixed points and periodic points of a map. The most fundamental dynamical zeta function is defined by Michael Artin and Barry Mazur [1]. This zeta function, called the Artin-Mazur zeta function, contains all the information about the numbers of fixed points for any iterations. While the Artin-Mazur zeta function counts the periodic points geometrically, the Lefschetz zeta function counts them homologically.

Definition 1.1. Let M be a manifold with $\dim H_*(M,\mathbb{Q}) < \infty$, and $f : M \to M$ a continuous map on M. The *Lefschetz zeta function* is defined by

$$\zeta_f(z) = \exp \sum_{n=1}^{\infty} \frac{\Lambda(f^n)}{n} z^n,$$

where

$$\Lambda(f) = \sum_{i=0}^{\dim M} (-1)^i \operatorname{Tr}(f_{*i} : H_i(M;\mathbb{Q}) \to H_i(M;\mathbb{Q}))$$

is the Lefschetz number of f.

The Lefschetz zeta function has some better properties than the Artin-Mazur zeta function. Not all Artin-Mazur zeta functions have a strictly positive radius of convergence, but Lefschetz zeta functions always have. This means that the Artin-Mazur zeta function is a "formal function," but the Lefschetz zeta function is always a function. Moreover, the Lefschetz zeta function is always a rational function. (Artin-Mazur zeta functions for Axiom A diffeomorphisms are rational functions [14].) This rationality is a remarkable property so that Lefschetz zeta functions can be related to characteristic polynomials. (The proof is written in [22], for example.)

Proposition 1.2. *The Lefschetz zeta function admits a rational expression*

$$\zeta_f(z) = \prod_{i=0}^{\dim X} \det(\operatorname{Id} - z f_{*i})^{(-1)^{i+1}} ,$$

where Id *is defined as the identity map. In particular, it is always a rational function.*

Remark 1.3. Since we use homology with rational coefficients, homology groups can be replaced by cohomologies.

1.2. Weil conjecture

Our theorem is an analogy to the Weil conjecture. Here, we recall briefly the Weil conjecture and related topics.

The congruence zeta function arises from the study of the number of solutions of a congruence

$$f(x_1, \ldots, x_n) \equiv 0 \qquad \text{mod } p,$$

where p is a prime and $f(x_1, \ldots, x_n)$ is a polynomial of integral coefficients. It is natural to look for solutions not only in the prime field $\mathbb{F}_p$ but also in all of its finite extensions $\mathbb{F}_{p^m}$. Abstracting this idea to algebraic varieties, the congruence zeta function of algebraic variety is defined as follows.

Definition 1.4. Let $\mathbb{F}_p$ be a finite field with q elements and V an algebraic variety of dimension n defined over $\mathbb{F}_p$. Let $\mathbb{F}_{p^m}$ be the extension field of $\mathbb{F}_p$ of degree m and N_m the number of $\mathbb{F}_{p^m}$-rational points of V. Then the function $Z(u, V)$ of u, defined by

$$Z(u, V) = \exp \sum_{m=1}^{\infty} \frac{N_m}{m} u^m ,$$

is called the *congruence zeta function* of the algebraic variety V.

The congruence zeta function $Z(u, V)$ has the following properties. These properties were conjectured by Weil [24] and finally proved by Deligne [4]. But this theorem is still called the Weil conjecture.

Proposition 1.5 (Weil conjecture)**.** *If V is a nonsingular projective variety over k, then $Z(u, V)$ has the following properties:*
(1) Rationality: $Z(u, V)$ is a rational function in u.
(2) Functional equation: $Z(u, V)$ satisfies the functional equation

$$Z(\frac{1}{q^n u}, V) = \pm q^{n\chi/2} u^{\chi} Z(u, V),$$

where the integer χ is the Euler characteristic of V.
(3) Riemann hypothesis: $Z(u, V)$ can be factored as

$$Z(u, V) = \prod_{i=0}^{2n} P_i(u)^{(-1)^{i+1}} ,$$

where $P_i(u) = \prod_{j=1}^{B_i}(1 - \alpha_j^{(i)} u)$ and $\alpha_j^{(i)}$ satisfies $|\alpha_j^{(i)}| = q^{i/2}$.

Now let us review how the Weil conjecture was solved. Weil initially suggested the possibility of using the Lefschetz fixed point formula to approach the Weil conjecture [25]. It means that the congruence zeta functions can be interpreted as the Lefschetz zeta function of the Frobenius action. (In view of this, he should get credit for the definition of the Lefschetz zeta function.)

Inspired by works of Serre, Grothendieck formulated the cohomology theory which is required to realize the Weil conjecture, which is called Weil cohomology (see below), and studied the étale cohomology toward realizing the Weil cohomology [**?**]. Actually, his étale cohomology proved the functional equation.

Definition 1.6 (Weil cohomology)**.** Let K be a field of characteristic 0. A contravariant functor $V \to H^*(V)$ is called a *Weil cohomology* with coefficients in K if it has the following three properties:

(1) *Poincaré duality*: If $n = \dim V$, then an orientation isomorphism $H^{2n}(V) \cong K$ exists and the cup product $H^j(V) \times H^{2n-j}(V) \to H^{2n}(V) \cong K$ induces a non-degenerate pairing.

(2) *Künneth formula*: For any V_1 and V_2 the mapping $H^*(V_1) \otimes H^*(V_2) \to H^*(V_1 \times V_2)$ defined by $a \otimes b \mapsto Proj_1^*(a) \cdot Proj_2^*(b)$ is an isomorphism.

(3) *Cycle map*: Let $C^j(V)$ be the group of algebraic cycles of codimension j on V. There exists a fundamental class homomorphism $FUND : C^j(V) \to H^{2j}(V)$ for all j which is functorial, compatible with products via the Künneth map, and sends a 0-cycle to its degree as an element of $H^{2n}(V)$.

The Weil cohomology was modeled after the classical cohomology theory of smooth compact varieties over the complex numbers $\mathbb{C}$. Actually, in this case, the cohomology theory satisfies the condition of the Weil cohomology.

The expository articles [12] and [16] are helpful to see how the Weil cohomology derives the Weil conjecture. The point is that the even-dimensional Poincaré duality derives the functional equation. (In view of this, Klaudiusz Wójcik's functional equation in [26] might have been already known in Grothendieck's project. However, his computation is helpful in this paper.)

1.3. Duality theorem for infinite cyclic coverings

To prove functional equations, we need an even-dimensional Poincaré duality. But the infinite cyclic covering of a knot complement is three dimensional and apparently does not admit the functional equation. However, Milnor [18] proved a duality theorem, which looked like even-dimensional Poincaré duality for odd-dimensional manifolds. In this section, we give a brief review of this duality.

Let M be a manifold. The *infinite cyclic covering* of M is defined as covering space $\tilde{X}$ which is determined by some homomorphism of the fundamental group $\pi_1(X)$ onto an infinite cyclic group. These spaces have two ends.

Let N_α and N'_α be neighborhoods of two ends ϵ and ϵ', and $\{N_\alpha \cup N'_\alpha\}$ be directed. The direct limit of the Mayer-Vietoris sequence:

$$\cdots \longrightarrow H^{i-1}(\tilde{M}, N_\alpha \cap N'_\alpha) \xrightarrow{\delta^*} H^i(\tilde{M}, N_\alpha \cup N'_\alpha) \longrightarrow H^i(\tilde{M}, N_\alpha) \oplus H^i(\tilde{M}, N'_\alpha) \longrightarrow \cdots,$$

prove that the connecting homomorphism

$$\delta^*(= \varinjlim \delta^*) : H^{i-1}(\tilde{M}) \to H^i_{\mathrm{cpt}}(\tilde{M})$$

is an isomorphism. This isomorphism is essential.

From the Poincaré duality theorem for an oriented n-dimensional manifold, the cup product

$$\cup : H^i_{\mathrm{cpt}}(\tilde{M}) \times H^{n-i}(\tilde{M}, \partial\tilde{M}) \to H^n_{\mathrm{cpt}}(\tilde{M}, \partial\tilde{M}) \cong \mathbb{Q}$$

provides a non-degenerate pairing.

By the above argument, we have the following.

Proposition 1.7 (Duality theorem for infinite cyclic coverings [18])**.** *Let M be a compact connected n-dimensional manifold with boundary, and $\tilde{M}$ an orientable infinite cyclic covering of M. If $H_*(\tilde{M};\mathbb{Q})$ is finitely generated over $\mathbb{Q}$, then the cup product:*

$$\cup : H^{i-1}(\tilde{M};\mathbb{Q}) \times H^{n-i}(\tilde{M}, \partial\tilde{M};\mathbb{Q}) \to H^{n-1}(\tilde{M}, \partial\tilde{M};\mathbb{Q}) \cong \mathbb{Q}$$

provides a non-degenerate pairing.

Remark 1.8. In the original paper, M is assumed to be triangulated. However, from the study of Robion C. Kirby and Laurence

C. Siebenmann [11], it is not necessary. It was pointed out in [9] and [10].

2. Functional equation for infinite cyclic coverings

In this section, we prove Theorem 2.1, replacing the Poincaré duality of the Weil cohomology with Milnor's duality theorem (Proposition 1.7). However, one problem remains. We need to deal with the relative version of Milnor duality because the infinite cyclic covering has the boundary. For that reason, we introduce other Lefschetz zeta functions in this section (Definition 2.2). After that, we prove the following theorem.

Theorem 2.1. *Let M be a compact connected manifold of dimension n which may have a boundary ∂M, and $\tilde{M}$ an orientable infinite cyclic covering of M with $\dim H_*(\tilde{M};\mathbb{Q}) < \infty$. Let $f : (\tilde{M}, \partial\tilde{M}) \to (\tilde{M}, \partial\tilde{M})$ be a proper continuous map with degree $\lambda \neq 0$ with respect to the compact support cohomology $H^n_{\mathrm{cpt}}(\tilde{M}, \partial\tilde{M};\mathbb{Q})$. If n is odd, then the Lefschetz zeta function ζ_f and $\zeta_{f|\partial\tilde{M}}$ satisfies the following functional equation:*

$$\frac{\zeta_f(1/\lambda z)^2}{\zeta_{f|\partial\tilde{M}}(1/\lambda z)} = \lambda^{\chi} z^{2\chi} \frac{\zeta_f(z)^2}{\zeta_{f|\partial\tilde{M}}(z)},$$

where $\zeta_{f|\partial\tilde{M}}(z)$ is the restricted Lefschetz zeta function to the boundary (cf. Definition 2.2), and χ is the Euler characteristic of $\tilde{M}$.

In particular, in the case where $\partial\tilde{M} = \emptyset$, we have

$$\zeta_f(\frac{1}{\lambda z}) = \pm\lambda^{\chi/2} z^{\chi} \zeta_f(z)$$

(cf. Proposition 1.5).

We define two other Lefschetz zeta functions (cf. Definition 1.1). These zeta functions let us use the property of the Lefschetz numbers in the proof of Theorem 2.1.

Definition 2.2. Suppose that (M, A) is a pair of manifolds with $\dim H_*(M, A;\mathbb{Q}) < \infty$ and $\dim H_*(A;\mathbb{Q}) < \infty$. Let $f : (M, A) \to (M, A)$ be a continuous map. The *relative Lefschetz zeta function* is defined by

$$\zeta_f^{\mathrm{rel}}(z) = \exp \sum_{n=1}^{\infty} \frac{\Lambda^{\mathrm{rel}}(f^n)}{n} z^n,$$

where $\Lambda^{\mathrm{rel}}(f)=\sum_{i=0}^{\dim M}(-1)^i \operatorname{Tr}(f_{*i}: H_i(M,A;\mathbb{Q})\to H_i(M,A;\mathbb{Q}))$. The *restricted Lefschetz zeta function to* A is defined by

$$\zeta_{f|A}(z)=\exp\sum_{n=1}^{\infty}\frac{\Lambda((f|A)^n)}{n}z^n,$$

where $\Lambda(f|A)=\sum_{i=0}^{\dim A}(-1)^i \operatorname{Tr}((f|A)_{*i}: H_i(A;\mathbb{Q})\to H_i(A;\mathbb{Q}))$.

Lemma 2.3. *If two of the three homology groups—$H_*(M;\mathbb{Q})$, $H_*(A;\mathbb{Q})$, and $H_*(M,A;\mathbb{Q})$—are finite dimensional, then all three Lefschetz zeta functions are defined and satisfy the following relation:*

$$\zeta_f(z)=\zeta_f^{\mathrm{rel}}(z)\times\zeta_{f|A}(z).$$

Proof: We can see that $\Lambda(f)=\Lambda^{\mathrm{rel}}(f)+\Lambda(f|A)$, which follows from the exact sequence:

$$\cdots\xrightarrow{\partial_*}H_i(A)\xrightarrow{\alpha_*}H_i(X)\xrightarrow{\beta_*}H_i(X,A)\xrightarrow{\partial_*}\cdots.$$

From the definitions, we can see that $\zeta_f(z)=\zeta_f^{\mathrm{rel}}(z)\times\zeta_{f|A}(z)$. □

We need some formulas to prove Theorem 2.1.

Lemma 2.4 ([26])**.** *Let $\langle\ ,\ \rangle: V\times V'\to\mathbb{Q}$ be a non-degenerate pairing of the $\mathbb{Q}$-vector spaces with dimension n. Let $\lambda\in\mathbb{Q}\setminus\{0\}$, and $f:V\to V, g:V'\to V'$ be endomorphisms such that*

$$\langle f(x),g(y)\rangle=\lambda\langle x,y\rangle$$

for all $x\in V, y\in V'$. Then

(1) $(\det f)(\det g)=\lambda^n$,

(2) $\det(\mathrm{Id}-gt)\det f=(-1)^n\lambda^n t^n\det(\mathrm{Id}-f/\lambda t)$.

Here t is an indeterminacy.

Proof: Since $\lambda\in\mathbb{Q}\setminus\{0\}$, f and g are isomorphisms, and their inverse maps f^{-1} and g^{-1} exist. The equation $\langle x,\lambda g^{-1}(y)\rangle=\langle f(x),y\rangle$ means that the dual map $f^*=\lambda g^{-1}$.

Hence, we obtain

$$\det f=\det f^*=\det\lambda g^{-1}=\lambda^n/\det g.$$

Similarly, we have

$$\begin{aligned}\det(\mathrm{Id}-gt)\det f&=\det(\mathrm{Id}-gt)\det(\lambda g^{-1})\\&=(-1)^n\lambda^n t^n\det(\mathrm{Id}-g^{-1}/t)\\&=(-1)^n\lambda^n t^n\det(\mathrm{Id}-f/\lambda t).\end{aligned}$$

□

Proof of Theorem 2.1: Let $f^* : H^*(\tilde{M};\mathbb{Q}) \to H^*(\tilde{M};\mathbb{Q})$ and $f^*_{\mathrm{rel}} : H^*(\tilde{M},\partial\tilde{M};\mathbb{Q}) \to H^*(\tilde{M},\partial\tilde{M};\mathbb{Q})$ be the induced homomorphisms of f. By the naturality of the cup product and Proposition 1.7,

$$\begin{aligned} f^{*i-1}(x) \cup f^{*n-i}_{\mathrm{rel}}(y) &= f^{*n-1}_{\mathrm{rel}}(x \cup y) \\ &= \lambda(x \cup y) \end{aligned}$$

is a non-degenerate pairing and satisfies the condition of Lemma 2.4. Recall that the degree λ is with respect to the compact support cohomology $H^n_{\mathrm{cpt}}(\tilde{M},\partial\tilde{M})$ and that the connecting homomorphism $\delta^* : H^{n-1}(\tilde{M},\partial\tilde{M}) \to H^n_{\mathrm{cpt}}(\tilde{M},\partial\tilde{M})$ is an isomorphism.

By using Proposition 1.2 (or Remark 1.3) and Lemma 2.4(2),

$$\begin{aligned} \zeta_f(\frac{1}{\lambda z}) &= \prod_{i=0}^{n-1} \left[\det(\mathrm{Id} - f^{*i}/\lambda z)\right]^{(-1)^{i+1}} \\ &= \prod_{i=0}^{n-1} \left[(-\lambda z)^{-b_i} \det(\mathrm{Id} - f^{*n-i-1}_{\mathrm{rel}} z) \det f^{*i}\right]^{(-1)^{i+1}} \\ &\qquad\qquad (b_i : i\text{-th Betti number}) \\ &= (-\lambda z)^{\chi} \prod_{i=0}^{n-1} \left[\det(\mathrm{Id} - f^{*n-i-1}_{\mathrm{rel}} z)\right]^{(-1)^{i+1}} \prod_{i=0}^{n-1} \left[\det f^{*i}\right]^{(-1)^{i+1}} \\ &= (-\lambda z)^{\chi} \left[\prod_{k=0}^{n-1} \left[\det(\mathrm{Id} - f^{*k}_{\mathrm{rel}} z)\right]^{(-1)^{k+1}}\right]^{(-1)^{n-1}} \prod_{i=0}^{n-1} \left[\det f^{*i}\right]^{(-1)^{i+1}} \\ &= (-\lambda z)^{\chi} \left[\zeta_f^{\mathrm{rel}}(z)\right]^{(-1)^{n-1}} \prod_{i=0}^{n-1} \left[\det f^{*i}\right]^{(-1)^{i+1}} . \end{aligned}$$

In the same way, we obtain

$$\zeta_f^{\mathrm{rel}}(\frac{1}{\lambda z}) = (-\lambda z)^{\chi} \left[\zeta_f(z)\right]^{(-1)^{n-1}} \times \prod_{i=0}^{n-1} \left[\det f^{*i}_{\mathrm{rel}}\right]^{(-1)^{i+1}} .$$

By the assumption, n is odd, and using Lemma 2.4(1), we get

$$\begin{aligned}
&\zeta_f^{\mathrm{rel}}(\frac{1}{\lambda z})\zeta_f(\frac{1}{\lambda z})\\
=&(-\lambda z)^{2\chi}\left[\zeta_f^{\mathrm{rel}}(z)\zeta_f(z)\right]\prod_{i=0}^{n-1}\left[\det f_{\mathrm{rel}}^{*i}\right]^{(-1)^{i+1}}\prod_{i=0}^{n-1}\left[\det f^{*i}\right]^{(-1)^{i+1}}\\
=&(\lambda z)^{2\chi}\left[\zeta_f^{\mathrm{rel}}(z)\zeta_f(z)\right]\prod_{i=0}^{n-1}\left[\det f_{\mathrm{rel}}^{*i}\right]^{(-1)^{i+1}}\prod_{k=0}^{n-1}\left[\det f^{*n-k-1}\right]^{(-1)^{k+1}}\\
=&(\lambda z)^{2\chi}\left[\zeta_f^{\mathrm{rel}}(z)\zeta_f(z)\right]\prod_{i=0}^{n-1}\left[\det f_{\mathrm{rel}}^{*i}\det f^{*n-i-1}\right]^{(-1)^{i+1}}\\
=&(\lambda z)^{2\chi}\left[\zeta_f^{\mathrm{rel}}(z)\zeta_f(z)\right]\prod_{i=0}^{n-1}\left[\lambda^{b_i}\right]^{(-1)^{i+1}}\\
=&(\lambda z)^{2\chi}\left[\zeta_f^{\mathrm{rel}}(z)\zeta_f(z)\right]\lambda^{-\chi}\\
=&\lambda^{\chi}z^{2\chi}\left[\zeta_f^{\mathrm{rel}}(z)\zeta_f(z)\right].
\end{aligned}$$

From Lemma 2.3, we complete the proof. □

3. Application to knot theory

The Alexander polynomial is one of the most important invariants of knots. In this section, we apply Theorem 2.1 to the infinite cyclic covering of a knot complement and reduce the functional equation to the reciprocity of the Alexander polynomial.

Let $K \subset S^3$ be an oriented (tame) knot and N a tubular neighborhood of K, and put $X = S^3 \setminus \mathrm{Int}\, N$. The infinite cyclic covering of X, i.e., the covering associated with the kernel of the Abelianization homomorphism $\pi_1(X) \to H_1(X) \cong \mathbb{Z}$, will be denoted by X_∞. By the theory of covering, the infinite cyclic group $\Pi = \langle t \rangle$ acts on $H_1(X_\infty)$, where generator t is chosen to be the positive linking number with oriented knot K.

Identifying the integral group ring of Π with the ring of Laurent polynomials $\mathbb{Z}[t^{\pm 1}] = \mathbb{Z}[t, t^{-1}]$, $H_1(X_\infty)$ is a finitely generated $\mathbb{Z}[t^{\pm 1}]$-module. Similarly, $H_1(X_\infty; \mathbb{Q})$ is a finitely generated $\mathbb{Q}[t^{\pm 1}]$-module.

Let M be a module over a commutative ring R. A finite *presentation* for M is an exact sequence

$$F \xrightarrow{\alpha} E \xrightarrow{\phi} M \longrightarrow 0,$$

where E and F are free R-modules with finite bases $\{e_1, \dots, e_m\}$ and $\{f_1, \dots, f_n\}$. If α is represented by an $m \times n$ matrix, then the matrix A is a *presentation matrix* for M.

By adjoining rows of zeros if necessary, we may suppose that A is $m \times n$ with $m \geq n$. Then the *r-th elementary ideal* of M is the ideal in R generated by all the $(n-i+1) \times (n-i+1)$ minors of A.

Definition 3.1. The *r-th Alexander ideal* of an oriented knot K is r-th elementary ideal of $\mathbb{Z}[t^{\pm 1}]$-module $H_1(X_\infty)$. The r-th Alexander polynomial of K is a generator of the smallest principal ideal of $\mathbb{Z}[t^{\pm 1}]$ which contains the r-th Alexander ideal. The first Alexander polynomial is called the *Alexander polynomial* and is denoted by $\Delta_K(t)$.

Proposition 3.2 ([13]). *Let K be a knot in S^3 and $t : X_\infty \to X_\infty$ be the covering transformation of X_∞. Then*
(1) $H_1(X_\infty; \mathbb{Q})$ is a finite-dimensional vector space over the field $\mathbb{Q}$, and
(2) the characteristic polynomial of the linear map $t_* : H_1(X_\infty; \mathbb{Q}) \to H_1(X_\infty; \mathbb{Q})$ is, up to a multiplication of a unit of $\mathbb{Q}[t^{\pm 1}]$, equal to the Alexander polynomial of K.

Lemma 3.3. *Let K be a knot in S^3 and $t : X_\infty \to X_\infty$ be the covering transformation of X_∞. Then the Alexander polynomial $\Delta_K(z)$ and the Lefschetz zeta function satisfy the relation:*

$$\zeta_t(z) = \frac{1}{a_n} \frac{z^{b_1} \Delta_K(z^{-1})}{1-z},$$

where the Alexander polynomial is normalized as $\Delta_K(z) = a_0 + a_1 z + \cdots + a_n z^n$ with $a_0 a_n \neq 0$, and $b_1 (= n)$ is the first Betti number of X_∞.

Proof: $H_i(X_\infty) = 0$ for $i \geq 2$ (see [3]). From propositions 1.2 and 3.2, we complete the proof. $\square$

From this lemma, Theorem 2.1 implies the following:

Corollary 3.4 ([21] (see also [2], [6], [17], [23])**.** *The Alexander polynomial* $\Delta_K(z)$ *of a knot* K *is reciprocal, i.e.,* $\Delta_K(z) = z^{b_1}\Delta_K(z^{-1})$.

Proof: Proposition 3.2(1) means that X_∞ satisfies the requirement for Theorem 2.1. In our condition, $\partial X_\infty \cong S^1 \times \mathbb{R}$. Therefore, we can see that $\zeta_{t|\partial X_\infty}(z) = 1$. Applying Theorem 2.1 to Lemma 3.3, the proof is complete. □

Remark 3.5. From this corollary, we can rewrite the relation of Lemma 3.3 as the following:

$$\zeta_t(z) = \frac{1}{\Delta_K(0)}\frac{\Delta_K(z)}{1-z}.$$

This implies that the Alexander polynomial of a knot contains all the information about the Lefschetz numbers $\Lambda(t^n)$ of the covering transformations.

Acknowledgment. I would like to express my gratitude to Professor Kazuo Masuda, Professor Hiroyuki Ochiai, Professor Hitoshi Murakami, Professor Sadayoshi Kojima, Professor Masanori Morishita, and Professor Akio Kawauchi for their valuable advice.

References

[1] M. Artin and B. Mazur, "On Periodic Points," *Ann. of Math. (2)* **81** (1965), no. 1, 82–99.

[2] Richard C. Blanchfield, "Intersection Theory of Manifolds with Operators with Applications to Knot Theory," *Ann. of Math. (2)* **65** (1957), no. 2, 340–356.

[3] Gerhard Burde and Heiner Zieschang, *Knots.* de Gruyter Studies in Mathematics, 5. Berlin: Walter de Gruyter & Co., 1985.

[4] Pierre Deligne, "La conjecture de Weil: I," *Inst. Hautes Études Sci. Publ. Math.* **43** (1974), 273–307.

[5] Alexander Fel′shtyn, *Dynamical Zeta Functions, Nielsen Theory and Reidemeister Torsion.* Memoirs of the American Mathematical Society, 147, no. 699. Providence, RI: AMS, 2000.

[6] John M. Franks, "Knots, Links, and Symbolic Dynamics," *Ann. of Math. (2)* **113** (1981), no. 3, 529–552.

[7] Michael Hutchings and Yi-Jen Lee, "Circle-valued Morse theory and Reidemeister torsion," *Geom. Topol.* **3** (1999), 369–396 (electronic).

[8] ———, "Circle-valued Morse theory, Reidemeister torsion, and Seiberg-Witten invariants of 3-manifolds," *Topology* **38** (1999), no. 4, 861–888.

[9] Akio Kawauchi, "On quadratic forms of 3-manifolds," *Invent. Math.* **43** (1977), no. 2, 177–198.

[10] ———, "The signature invariants of infinite cyclic coverings of closed odd-dimensional manifolds," *Algebraic and Topological Theories (Kinosaki, 1984).* Tokyo: Kinokuniya, 1986. 52–85.

[11] Robion C. Kirby and Laurence C. Siebenmann, *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations.* Annals of Mathematics Studies, No. 88. Princeton, N.J.: Princeton University Press, 1977.

[12] S. L. Kleiman, "Algebraic cycles and the Weil conjectures," *Dix Esposés sur la Cohomologie des Schémas.* Amsterdam: North-Holland, 1968. 359–386.

[13] W. B. Raymond Lickorish, *An Introduction to Knot Theory.* Graduate Texts in Mathematics, 175. New York: Springer-Verlag, 1997.

[14] Anthony Manning, "Axiom A diffeomorphisms have rational zeta functions," *Bull. London Math. Soc.* **3** (1971), 215–220.

[15] Thomas Mark, "Torsion, TQFT, and Seiberg-Witten invariants of 3-manifolds," *Geom. Topol.* **6** (2002), 27–58 (electronic).

[16] B. Mazur, "Eigenvalues of Frobenius acting on algebraic varieties over finite fields," *Algebraic Geometry–Arcata 1974.* Ed. Robin Hartshorne. Proceedings of Symposia in Pure Mathematics, 29. Providence, R.I.: Amer. Math. Soc., 1975. 231–261.

[17] John Milnor, "A Duality Theorem for Reidemeister Torsion," *Ann. of Math. (2)* **76** (1962), no. 1, 137–147.

[18] ———, "Infinite cyclic coverings," *Conference on the Topology of Manifolds,* Ed. John G. Hocking. The Prindle, Weber & Schmidt Complementary Series in Mathematics, Vol. 13. Boston: Prindle, Weber & Schmidt, 1968. 115–133.

[19] Masanori Morishita, "Analogies between knots and primes, 3-manifolds and number fields." Preprint.

[20] Akio Noguchi, "Zeros of the Alexander polynomial of knot." Preprint: arXiv:math.GT/0503252.

[21] H. Seifert, "Über das Geschlecht von Knoten," *Math. Ann.* **110** (1934), 571–594.

[22] S. Smale, "Differentiable dynamical systems," *Bull. Amer. Math. Soc.* **73** (1967), 747–817.

[23] G. Torres and R. H. Fox, "Dual Presentations of the Group of a Knot," *Ann. of Math. (2)* **59** (1954), no. 2, 211–218.

[24] André Weil, "Numbers of solutions of equations in finite fields," *Bull. Amer. Math. Soc.* **55** (1949), 497–508.

[25] ———, "Abstract versus classical algebraic geometry," *Proceedings of the International Congress of Mathematicians, 1954, Amsterdam.* Vol. III. Amsterdam: North-Holland, 1956. 550–558.

[26] Klaudiusz Wójcik, "Lefschetz zeta function and the periodic points," *Bull. Polish Acad. Sci. Math.* **43** (1995), no. 1, 59–64.

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