Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 29, No. 1, 2005
Pages 333-341

ORIENTABLE ONE-CIRCUIT DOUBLE COVERS

CARL R. SEAQUIST, KASIA BINAM,
ROB STREET, AND GALEN E. TURNER, III

Abstract. We define an orientable one-circuit double cover of a graph as a walk that starts and ends at the same vertex and traverses each edge of the graph exactly twice, once in each direction, but never in immediate succession. We characterize the graphs that have an orientable one-cycle double cover.

1. Introduction

An orientable k-cycle double cover of a graph G is defined to be a set $\mathcal{F} = \{C_1, ..., C_k\}$ of oriented cycles of G so that each edge of G occurs in exactly two of the cycles, $C_1, ..., C_k$, once in each orientation [7] and [10]. Using this definition as motivation, we define an orientable one-circuit double cover of a graph as a walk that starts and ends at the same vertex and traverses each edge of the graph exactly twice, once in each direction, but never in immediate succession. We then give necessary and sufficient conditions for a graph to have an orientable one-circuit double cover. Thus, we answer many of the open questions about orientable one-circuit double covers asked by Doug Engel [2], e.g., which of the platonic solids have orientable one-cycle double covers. Note that Engel calls an orientable

2000 *Mathematics Subject Classification.* Primary 0510.
Key words and phrases. celtic magnecurve, orientable double cover.
The second author was supported in part by the generosity of the Allensworth Endowment for undergraduate research and the Honors College of Texas Tech University.

one-circuit double cover a *celtic magnecurve*. The problem of determining if a graph has an orientable one-circuit double cover (or equivalently, the celtic magnecurve problem) can be thought of as a generalization of the Königsberg Bridge problem originally given to Euler [2]. Our problem instead of simply finding a walk starting and ending at the same point that crosses each bridge once (called an Eulerian circuit), seeks to find a walk starting and ending at the same point that crosses each bridge twice, once in each direction, so that once a bridge is crossed it is not immediately re-crossed until another different bridge is crossed. An orientable one-circuit double cover is a more general concept than the orientable bi-Eulerian circuit defined in [8], because it allows more generality in how the walk can traverse a vertex. The order in which edges of a particular vertex are traversed by an orientable bi-Eulerian circuit induces a cyclic permutation (or rotation) of the edges, whereas an orientable one-circuit double cover can in general induce a permutation on the edges that is not cyclic. In [2], Engel states that the celtic magnecurve problem is reminiscent of the famous cycle double cover conjecture [6] [5]; however, note that a graph with a cut-edge can have a celtic magnecurve but cannot have a cycle double cover.

2. Preliminaries

Unless we state otherwise, the terminology used will follow [7]. In particular, a graph may have loops and multiple edges. We will only be concerned with connected graphs. The *Betti number* of a graph G, denoted by $\beta(G)$, is $|E(G)| - |V(G)| + 1$. We say that the graph C is a *cotree* of a graph G if there is a spanning tree T of G so that $V(C) = V(G)$ and $E(C) = E(G) - E(T)$. A component C' of a graph G is *odd* if $|E(C')|$ is odd. Following Nguyen Huy Xuong [9], we define the *Betti deficiency* of a graph G, denoted by $\xi(G)$, to be

$$\xi(G) = \min_{C \subset G}\{\xi(C) \mid \xi(C) \text{ is the number of odd components of cotree } C \text{ of } G\}.$$

This paper makes extensive use of the ideas of Xuong [8] and the notion of a 2-cell embedding of a graph in an orientable surface. By *orientable surface* we mean a closed orientable two manifold, i.e., a sphere or a sphere with one or more handles. The *genus* of an orientable surface is the number of handles the surface has. Each such surface can be represented by a convex polygon with an even number of sides. Pairs of sides are identified in such a way so that two paired sides are always oriented oppositely going clockwise around the polygon. See [4, Chapter 1] for details. We say a subset G of a surface S is a 2-cell embedding of a graph, if G is a topological representation [3] of the graph and if the components of the complement of G in S, called *faces* of G, are all homeomorphic to an open disk. We will use G to refer both to a graph and to its topological representation. The *maximum genus* of a graph, $\gamma_M(G)$, is the largest genus of an orientable surface for which there exists a 2-cell embedding. A graph G is *upper embeddable* if $\gamma_M(G) = [\beta(G)/2]$ where $[x]$ is the greatest integer less than or equal to x. Equivalently, a graph G is upper embeddable if there is a 2-cell embedding of G in an orientable surface with only one or two faces depending on whether $\beta(G)$ is even or odd, respectively. In [8, Theorem 3], it is shown that $\gamma_M(G) = \frac{1}{2}(\beta(G) - \xi(G))$.

Fundamental to the characterization of graphs with an orientable one-circuit double cover is the observation that when a graph is upper embeddable with one face, then it has an orientable one-circuit double cover, which can be found simply by tracing the edges as they occur around the single face. This observation follows directly from [8], where Xuong approaches the problem of how to determine the maximum genus of a graph by defining an orientable bi-Eulerian circuit. Given a graph G, an *orientable bi-Eulerian circuit* is a triple $C = (2E, \pi, \rho)$ where $2E$ is a set that contains both orientations of each edge of G, π is a pairing of the opposite orientations of each edge represented formally as a product of pairwise disjoint transpositions, and ρ is a cyclic permutation of the orientations of the edges. Following Xuong, we define a *rotation of a vertex* v to be a cyclic permutation assigned to members of $2E$ that are incident on v. A rotation of a vertex v, $(...AB'...)$, can be viewed as directing a walker entering a vertex v along the oriented edge A, which has its head at v, on how to exit the vertex. In this case, the walker is directed to exit v along the oriented edge B', which has its tail at v. A *rotation* σ *of a graph* results from assigning each vertex of a graph a rotation. Since rotations of distinct vertices must be pairwise disjoint cycles, this can be viewed formally as the product (in any order) of the rotations of all the vertices of the graph after each vertex is assigned a rotation.

Given a graph G, along with an orientable bi-Eulerian circuit $C = (2E, \pi, \rho)$, a new graph G_C is defined with vertices that are the orbits of $\sigma = \rho\pi$ and with edges that are the orbits of π. Note that σ can be interpreted as inducing a rotation on G_C. The orientable bi-Eulerian circuit C of a graph G is said to be associated with G if G_C is isomorphic to G. Several comments about a graph for which there is an associated bi-Eulerian circuit are useful here. First, the cyclic permutation of the orientations of the edges is an orientable one-circuit double cover of the graph. Second, the cyclic permutation of the edges also defines an identification of edges of an even sided polygon and thus defines an orientable surface S_C in which the graph can be embedded with one face. Third, $\sigma = \rho\pi$ defines a rotation of G. In other words, σ tells a walker along the orientable one-circuit double cover of G on which edge he must exit if he enters the vertex on a particular edge. The following theorem is proved in [8, Proposition 1].

Theorem 2.1 (Xuong)**.** *A graph G is upper embeddable if and only if there exists an orientable bi-Eulerian circuit C such that G_C is isomorphic to G (or $G - \{A\}$ where $A \in E(G)$) depending on whether $\beta(G)$ is even or odd, respectively.*

3. Main Theorems

If a graph G has vertices of only degree 2 or 3, then each possible permutation of the edges of a vertex is precisely a cyclic permutation. In this case, orientable one-circuit double covers correspond exactly with two cell embeddings of G with only one face and we get the following theorem.

Theorem 3.1. *For a graph G with the degree of all vertices either 2 or 3 then G has an orientable one-circuit double cover if and only if there is a two cell embedding of G with only one face in some surface.*

Proof: Assume that G has an orientable one-circuit double cover. Then, since each vertex is of order 2 or 3, the orientable one-circuit double cover induces an oriented cyclic order on each vertex of G

and therefore a rotation σ on G. Associate with each edge of G two oppositely oriented directed edges. The resulting directed graph we will call $\overrightarrow{G}$. In the Euclidean plane consider a regular polygon with $2|E(G)|$ number of sides. Label these sides in a clockwise order around the polygon with the directed edges of $\overrightarrow{G}$ in the order that they occur in the orientable one-circuit double cover. Then identify the sides of the polygon that are labeled by oppositely directed orientations of the same edge. Since each edge of $E(G)$ appears exactly twice in the orientable one-circuit double cover (once for each directed edge), this establishes an involution π, which matches opposite orientations of the same edge, of the sides of the polygon and thus defines an orientable surface. This labeling of the polygon based on the orientable one-circuit double cover also establishes a cyclic permutation ρ of the directed edges. Thus, $C = (2E, \pi, \rho)$ is an orientable bi-Eulerian circuit and the associated graph is (G_C, σ) where $\sigma = \rho\pi$ and where $V(G_C)$ is the orbits of σ and $E(G_C)$ is the orbits of π. Now G_C is isomorphic to G. Thus, by Theorem 2.1, G is upper embeddable and since $\beta(G)$ is even, it is embeddable in an orientable surface with exactly one face: namely, in the surface described by the edge identification on the polygon established by the orientable one-circuit double cover.

Let G be a graph with a two cell embedding with only one face in a surface. Then by the Theorem 2.1, there is an orientable bi-Eulerian circuit for G that can be expressed by giving ρ a cyclic permutation of the directed edges of $\overrightarrow{G}$. But ρ is an orientable one-circuit double cover. □

Combining this result with results from [8], we have the following theorem.

Theorem 3.2. *For a graph G with the degree of all vertices either 2 or 3, the following are equivalent:*

(1) *G has an orientable one-circuit double cover;*
(2) *there is a two cell embedding of G with only one face in some surface;*
(3) *G is upper embeddable and $\beta(G)$ is even;*
(4) *$\xi(G) \leq 1$ and $\beta(G)$ is even.*

We apply the theorem to get the following results.

Remark 3.3. The edge graphs of the tetrahedron, the cube, and the dodecahedron have no orientable one-circuit double covers. This is because they are all planar cubic graphs with an even number of faces and so have odd Betti numbers; therefore, by Theorem 3.2, they have no orientable one-circuit double covers.

When G has vertices of degree greater than three then an orientable one-circuit double cover will induce a permutation on the edges incident on a vertex, but this permutation may not be a cyclic permutation. When the permutation induced on each vertex is cyclic then the orientable one-circuit double cover defines an orientable bi-Eulerian circuit that is associated with the original graph; otherwise, it does not. In the case when it does not, the graph G_C will have additional vertices. Each additional vertex will correspond to a cyclic permutation where the product of the cyclic permutations is the original permutation on the vertex in G. Because the additional vertices of G_C can be viewed as derived from the original vertex by some sort of "splitting," we define formally how to obtain a graph G', which will be isomorphic to G_C, from a graph G by splitting edges away from a vertex v. We say a graph G' has been *obtained from a graph G by splitting edges away from a vertex $v \in V(G)$* if $V(G') = (V(G) - \{v\}) \cup \{v', v''\}$ where neither v' nor v'' is in $V(G)$ and an edge is in $E(G')$ if and only if the corresponding edge is in $E(G)$ and every edge in $E(G)$ that is incident on v has a unique corresponding edge incident on either v' or v'' and there is no edge between v' and v''. Notice that the degree of v in G is the sum of the degrees of v' and v'' in G'. The definition we are using is that of "splitting edges away from a vertex" given in [10, p. 296] where there is no edge between v' and v'', rather than that of "vertex splitting" given in [7, p. 174] where there is an edge between v' and v''.

We will obtain a new graph G' from a graph G by splitting edges away from a vertex v only when the order of v is greater than 3 and the orders of v' and v'' are both at least 2. We say that a graph G'' is obtained from G by successive splittings of edges away from vertices if G is G'' or if there is a finite sequence of graphs $G_0, ..., G_n$ so that $G = G_0$, $G'' = G_n$, and each graph G_{i+1} is obtained from G_i by splitting edges away from a vertex in G_i. Note that if G' is obtained from G by successive splitting of edges away from vertices, then G has an orientable one-circuit double cover when G' has an orientable one-circuit double cover.

If $\beta(G)$ is even and $\xi(G) \leq 1$, then G is upper embeddable in a surface with one face and therefore has an associated orientable bi-Eulerian circuit, which defines an orientable one-circuit double cover on G. If $\beta(G)$ is odd and G has a vertex v with degree greater than 3 such that a graph G' can be obtained from G by splitting two edges away from v so that $\xi(G') \leq 1$, then $\beta(G')$ is even and $\xi(G') \leq 1$, and so G' has an orientable one-circuit double cover and so G has an orientable one-circuit double cover. Thus, we have the following theorem about the existence of orientable one-circuit double covers. Note that if G has a vertex of degree 1, then it is trivial that it has no orientable one-circuit double cover.

Theorem 3.4. *Let G be a graph with the degree of all vertices greater than 1. Then G has an orientable one-circuit double cover if there is a graph G' that can be obtained from G by successive vertex splittings so that $\beta(G')$ is even and $\xi(G') \leq 1$.*

Remark 3.5. The edge graphs of both the octahedron and the icosahedron have orientable one-circuit double covers because in both cases a graph G' can be obtained by splitting one vertex where $\xi(G') = 0$ and $\beta(G')$ is even; therefore, by Theorem 3.4, both have orientable one-circuit double covers. On the octahedron such a walk is given by the cyclic permutation of vertices: (1 2 3 1 4 3 2 1 3 6 2 5 4 1 5 2 6 4 5 6 3 4 6 5); see Figure 1. Such a walk on an icosahedron is given by (1 2 3 1 4 3 2 1 3 4 1 5 4 9 3 8 2 6 5 1 6 2 7 6 11 5 6 7 2 8 3 9 4 10 5 11 7 8 9 10 4 5 10 9 12 7 11 10 12 8 7 12 10 11 12 9 8 12 11 6); see Figure 1.

We now state the necessary conditions for the existence of an orientable one-circuit double cover.

Theorem 3.6. *Let G be a graph. If G has an orientable one-circuit double cover, then there is a graph G' that can be obtained from G by successive (or no) vertex splittings so that G' is upper embeddable in an orientable surface with 1 face or equivalently so that $\beta(G')$ is even and $\xi(G') \leq 1$.*

Proof: Assume we have an orientable one-circuit double cover on G represented by a permutation of $2|E(G)|$ directed edges $\rho = (...A...A'...)$ where A and A' represent the same edge traversed in

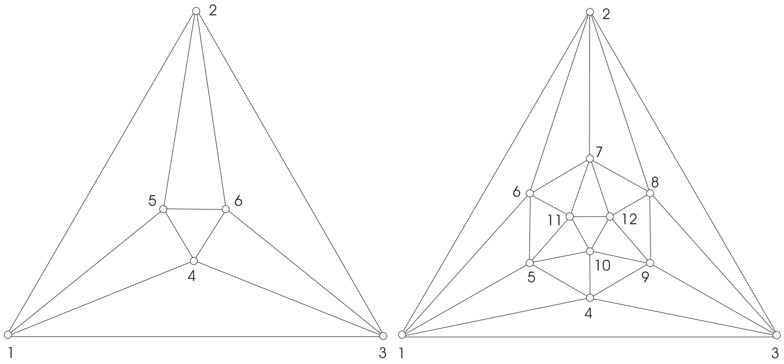

FIGURE 1. Edge graphs of the octahedron and icosahedron.

opposite directions. Let $\pi = (AA')(BB')...$ be the pairing of directed edges. Consider $\sigma = \rho\pi$. Then the cyclic permutations of σ represent permutations of edges of the vertices of G. Now consider the graph G_C where each vertex of G_C is a cyclic permutation of σ and an edge exists between two vertices if there is an edge that appears in the two corresponding cycles. Now G_C is isomorphic to a graph G' that can be obtained from G by successively splitting vertices. Also note that G' is embeddable with a single face in the surface formed from labeling a regular polygon in the order dictated by the orientable one-circuit double cover and identifying the paired sides. Thus, G' is upper embeddable with only one face and so $\beta(G')$ is even and $\xi(G') \leq 1$ by results in [8]. $\square$

REFERENCES

[1] Kasia Binam, C. R. Seaquist, Rob Street, and Galen E. Turner, III, "Celtic Magnecurves." 22 Aug. 2004 <http://www.math.ttu.edu/˜ seaqucr/magne/fgraph1.pdf>.

[2] Doug Engel, "A Celtic Magnecurve Problem," *Amer. Math. Monthly* **107** (June-July 2000), 563–565.

[3] Jonathan L. Gross and Thomas W. Tucker, *Topological Graph Theory.* Wiley-Interscience Series in Discrete Mathematics and Optimization. New York: John Wiley & Sons, 1987.

[4] William S. Massey, *Algebraic Topology: An Introduction.* 1967. Graduate Texts in Mathematics, Vol. 56. New York-Heidelberg: Springer-Verlag, 1977.

[5] P. D. Seymour, "Sums of Circuits," *Graph Theory and Related Topics*. Ed. John Adrian Bondy and U. S. R. Murty. New York-London: Academic Press, 1979. 341–355.

[6] G. Szekeres, "Polyhedral decompositions of cubic graphs," *Bull. Austral. Math. Soc.* **8** (1973), 367–387.

[7] Douglas B. West, *Introduction to Graph Theory*. 2nd ed. Upper Saddle River, NJ: Prentice Hall, 2001.

[8] Nguyen Huy Xuong, "How to determine the maximum genus of a graph," *J. Combin. Theory Ser. B* **26** (1979), no. 2, 217–225.

[9] ———, "Upper-embeddable graphs and related topics," *J. Combin. Theory Ser. B* **26** (1979), no. 2, 226–232.

[10] Cun-Quan Zhang, *Integer Flows and Cycle Covers of Graphs*. Monographs and Textbooks in Pure and Applied Mathematics, 205. New York: Marcel Dekker, 1997.

(Seaquist) Department of Mathematics and Statistics; Texas Tech University; Lubbock, TX 79409-1042
E-mail address: carl.seaquist@ttu.edu

(Binam) Department of Mathematics and Statistics; Texas Tech University; Lubbock, TX 79409-1042
E-mail address: kbinam@ttacs.ttu.edu

(Street) Lubbock-Cooper High School; Lubbock-Cooper ISD; Rt. 6, Box 400; Lubbock, TX 79423
E-mail address: rdstreet@hotmail.com

(Turner) Department of Mathematics and Statistics; Louisiana Tech University; Ruston, LA 71272
E-mail address: gturner@latech.edu