Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY PROCEEDINGS
Volume 30, No. 1, 2006
Pages 1-23

HOMOGENEOUS CIRCLE-LIKE CONTINUA ARE C-DETERMINED

GERARDO ACOSTA

Abstract. For a metric continuum X we denote by $C(X)$ the hyperspace of subcontinua of X with the Hausdorff metric. A class $\mathcal{G}$ of continua is said to be C-*determined* provided that if $X, Y \in \mathcal{G}$ and the hyperspaces $C(X)$ and $C(Y)$ are homeomorphic, then continua X and Y are homeomorphic. In 1978, Sam B. Nadler, Jr. in [Hyperspaces of Sets. Monographs and Textbooks in Pure and Applied Mathematics, Vol. 49] asked if the class of circle-like continua is C-determined. In this paper, we provide a partial positive answer to this question by showing that both the class of arcwise connected circle-like continua and the class of homogeneous circle-like continua are C-determined. By considering Knaster continua, we present two other classes of circle-like continua which are C-determined.

1. Introduction

A *continuum* is a nonempty compact connected metric space. The *hyperspace of subcontinua* of a given continuum X is denoted by $C(X)$. We consider that $C(X)$ is metrized by the Hausdorff metric H. If two continua X and Y are homeomorphic, we write $X \approx Y$. Note that if $X \approx Y$, then $C(X) \approx C(Y)$. A class $\mathcal{G}$ of continua is said to be C-*determined* provided that if $X, Y \in \mathcal{G}$ and

2000 *Mathematics Subject Classification.* Primary 54B20; Secondary 54B15, 54F15, 54F50.

Key words and phrases. arc-like continuum, C-determined, circle-like continuum, homogeneous space, Knaster continua, manifold interior, semi-boundary, unique hyperspace.

$C(X) \approx C(Y)$, then $X \approx Y$ ([25, Definition 0.61]). Classes of continua which are C-determined have been studied in [1], [2], [3], [4], [5], [10] and [21]. In [18], it is shown that the class of fans is not C-determined.

A continuum X is said to be *arc-like* (respectively, *circle-like*) if for any $\varepsilon > 0$ there is a map f from X onto an arc A (respectively, onto a circle A) such that the diameter of $f^{-1}(y)$ is less than ε, for any $y \in A$. In [17], it is shown that the class of arc-like continua is not C-determined. In [25, Questions 0.62], Sam B. Nadler, Jr. asks if the class of circle-like continua is C-determined. Some partial results in the positive have been obtained in [2] and [5]. The main purpose of this paper is to present new classes of circle-like continua which are C-determined. Among other results, we show that the class of homogeneous circle-like continua is C-determined.

The paper is divided into seven sections. After the Introduction, we present in section 2 some terminology and results that will be used later. We refer the reader to [25] and [26] for notions not defined and used in the paper. In section 3, for a continuum X and a proper subcontinuum K of X, we recall a technique used in [1] for detecting either a 2-cell $\mathcal{D}$ in $C(X)$, such that K belongs to the manifold interior of $\mathcal{D}$, or triods arbitrarily close to K, with respect to the Hausdorff metric. We use this technique in section 4 to prove Theorem 4.2, which is the first important result of the paper. In section 5, we show that the class of arcwise connected circle-like continua is C-determined. In section 6, we show that the class of homogeneous circle-like continua is C-determined. Finally, in section 7, we consider the class of Knaster continua and from it we construct two classes of circle-like continua and show that they are C-determined.

2. Terminology and Facts

All spaces are assumed to be metric. Given a space X, we denote by $B_X(x, \varepsilon)$ the open ball in X centered at the point $x \in X$ and having the radius $\varepsilon > 0$. For a subset A of X, we define $N_X(A, \varepsilon) = \bigcup_{a \in A} B_X(a, \varepsilon)$, and we use the symbols $\mathrm{cl}_X(A)$ and $\mathrm{bd}_X(A)$ to denote the closure and the boundary of A in X, respectively. The symbol $|A|$ denotes the cardinality of A. The letter I

represents the closed unit interval $[0,1]$ and the letter S the unit circle.

For $n \in \mathbb{N}$ an *n-od in X* is an element $B \in C(X)$ for which there exists $A \in C(B)$, called a *core* of B, such that $B - A$ has at least n components. A 3-od is also called a *triod*. If X contains no triods, then we say that X is *atriodic*. An *n-cell* is a space homeomorphic to I^n. By [28, p. 177] and [15, p. 64], it follows that a continuum X contains an n-od if and only if $C(X)$ contains an n-cell. Using this result it follows that if X and Y are continua such that $C(X) \approx C(Y)$, then X is atriodic if and only if Y is atriodic.

If D is a 2-cell in a space X, then the symbols ∂D and D° are used to denote the *manifold boundary* and the *manifold interior* of D, respectively. Note that if $f\colon I^2 \to D$ is a homeomorphism, then $\partial D = f\left(\mathrm{bd}_{\mathbb{R}^2}(I^2)\right)$ and $D^\circ = D - \partial D$.

Theorem 2.1. [3, Theorem 3] *Let X be a continuum such that $C(X)$ contains a 2-cell $\mathcal{D}$, and let a point $p \in X$ satisfy $\{p\} \in \mathcal{D}^\circ$. Then, for each $\varepsilon > 0$, there is a triod T in X such that $H(T, \{p\}) < \varepsilon$.*

Throughout this paper we are going to consider the hyperspace $F_1(X)$ of singletons of the continuum X. We assume that $F_1(X)$ is a subspace of $C(X)$. Clearly, $X \approx F_1(X)$. A proper and nondegenerate subcontinuum A of a continuum X is said to be *terminal in X* provided that, for each $B \in C(X)$ such that $B \cap A \neq \emptyset$ either $A \subset B$ or $B \subset A$.

Theorem 2.2. [3, Lemma 6] *Let X and Y be continua such that $C(X) \approx C(Y)$ and let $h\colon C(Y) \to C(X)$ be a homeomorphism. If E is terminal in X, then E is not an element of $h(F_1(Y))$.*

For a continuum X and $A \in C(X)$ we put

$$C(A, X) = \{D \in C(X)\colon A \subset D\}.$$

If $A = \{p\} \in F_1(X)$ we simply write $C(p, X)$ instead of $C(\{p\}, X)$. A *map* is a continuous function. If $A, B \in C(X)$ and $A \subsetneq B$, then an *order arc from A to B in $C(X)$* is a map $\lambda\colon I \to C(X)$ such that $\lambda(0) = A$, $\lambda(1) = B$ and $\lambda(s) \subsetneq \lambda(t)$ if $s < t$ (see [25, definitions 1.2 and 1.7]).

3. Locating 2-cells or triods

In this section, the letter X denotes a continuum and the letter K a proper subcontinuum of X. We recall a technique used in section 3 of [1] for detecting either a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathcal{D}^{\circ}$ or triods arbitrarily close to K, with respect to the Hausdorff metric. In order to do this, we require the following notion due to Alejandro Illanes.

Definition 3.1. [16, p. 63] The *semi-boundary of* K is the set $SB(K) = \{B \in C(K)\colon$ there is a map $\alpha : I \to C(X)$ such that $\alpha(0) = B$ and $\alpha(t) - K \neq \emptyset$, for each $t > 0\}$.

In [16, Theorem 1.2] it is shown that $K \in SB(K)$ and that if $B \in SB(K)$, then there exists a minimal element (with respect to the inclusion) $C \in SB(K)$ such that $C \subset B$. It is also shown that $SB(K) = \{B \in C(K)\colon$ there is an order arc $\alpha\colon I \to C(X)$ such that $\alpha(0) = B$ and $\alpha(t) - K \neq \emptyset$, for each $t > 0\}$. In Chapter X of [19] more properties of the semi-boundary are presented.

The following result is easy to prove.

Lemma 3.2. *Suppose that $A \in C(X)$ is such that $A \cap K \neq \emptyset$ and $A - K \neq \emptyset$. If C is a component of $A \cap K$, then $C \in SB(K)$.*

We denote by $m(K)$ the set of minimal elements in $SB(K)$. By considering the cardinality of $m(K)$, it is possible to detect either a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathcal{D}^{\circ}$ or triods arbitrarily close to K. Theorem 3.3 generalizes [1, Theorem 3.6].

Theorem 3.3. *Let $n \in \mathbb{N}$. If $m(K)$ has at least n mutually disjoint elements then, for each $\varepsilon > 0$, there is an n-od T in X such that $H(T, K) < \varepsilon$.*

Proof: Let us assume that $E_1, E_2, \ldots, E_n$ are mutually disjoint elements in $SB(K)$. In order to construct an n-od T with the desired properties, fix $i \in \{1, 2, \ldots, n\}$ and let $\alpha_i\colon I \to C(X)$ be an order arc such that $\alpha_i(0) = E_i$ and $\alpha_i(t) - K \neq \emptyset$ for any $t > 0$. Since the set

$$\mathcal{U}_1 = \left\{A \in C(X)\colon A \subset (X - (E_2 \cup E_3 \cup \cdots \cup E_n)) \cap N_X\left(K, \frac{\varepsilon}{n}\right)\right\}$$

is open in $C(X)$, $E_1 \in \mathcal{U}_1$, and α_1 is continuous, there is $t > 0$ such that $F_1 = \alpha_1(t) \in \mathcal{U}$. Then $E_1 \subsetneq F_1$, $F_1 - K \neq \emptyset$, $F_1 \cap$

$(E_2 \cup E_3 \cup \cdots \cup E_n) = \emptyset$, and $F_1 \subset N_X\left(K, \frac{\varepsilon}{n}\right)$. This implies that $H(K, K \cup F_1) < \frac{\varepsilon}{n}$. Now, considering the open set

$$\mathcal{U}_2 = \left\{A \in C(X)\colon A \subset (X - (F_1 \cup E_3 \cup E_4 \cup \cdots \cup E_n)) \cap N_X\left(K, \frac{\varepsilon}{n}\right)\right\}$$

in $C(X)$ that contains E_2, we can find, in a similar way, an element $F_2 \in C(X)$ such that $E_2 \subsetneq F_2$, $F_2 - K \neq \emptyset$, $F_2 \cap (F_1 \cup E_3 \cup E_4 \cup \cdots \cup E_n) = \emptyset$ and $H(K \cup F_1, K \cup F_1 \cup F_2) < \frac{\varepsilon}{n}$. Proceeding in this way, we find subcontinua $F_1, F_2, \ldots, F_n$ of X such that if $i \in \{1, 2, \ldots, n\}$, then $E_i \subsetneq F_i$ and $F_i - K \neq \emptyset$. Moreover, $F_1 \cap (E_2 \cup E_3 \cup \cdots \cup E_n) = \emptyset$, $H(K, K \cup F_1) < \frac{\varepsilon}{n}$ and, for any $i \geq 2$,

$$F_i \cap (F_1 \cup F_2 \cup \cdots \cup F_{i-1} \cup E_{i+1} \cup E_{i+2} \cup \cdots \cup E_n) = \emptyset$$

and

$$H(K \cup F_1 \cup F_2 \cup \cdots \cup F_{i-1}, K \cup F_1 \cup F_2 \cup \cdots \cup F_i) < \frac{\varepsilon}{n}.$$

Then $T = K \cup F_1 \cup F_2 \cup \cdots \cup F_n$ is an n-od, with core K, such that $H(T, K) < \varepsilon$. □

Theorem 3.4. [1, Theorem 3.15] *If K is decomposable and $m(K)$ has at least two elements, then either there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathcal{D}^\circ$ or, for each $\varepsilon > 0$, there is a triod T in X such that $H(T, K) < \varepsilon$.*

Corollary 3.5. *If X is atriodic, K is decomposable, and $m(K)$ has at least two elements, then there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathcal{D}^\circ$.*

When K is indecomposable, we have the following result that generalizes [1, Theorem 3.17].

Theorem 3.6. *If K is indecomposable and $C(X) - \{K\}$ is arcwise connected, then for each $\varepsilon > 0$ and each $n \in \mathbb{N}$, there is an n-od T in X such that $H(T, K) < \varepsilon$.*

Proof: Fix $a \in K$. Since $C(X) - \{K\}$ is arcwise connected and $\{a\}, X \in C(X) - \{K\}$ there is a map $\alpha\colon I \to C(X) - \{K\}$ such that $\alpha(0) = \{a\}$ and $\alpha(1) = X$. Consider the map $\beta\colon I \to C(X)$ defined by $\beta(t) = \bigcup \alpha([0, t])$. Note that $\beta(0) = \{a\}$, $\beta(1) = X$, and $\beta(s) \subset \beta(t)$ if $s \leq t$. Define $t_0 = \max\{t \in I\colon \beta(t) \subset K\}$ and $E_a = \beta(t_0)$. Since K is a proper subcontinuum of X, we have $t_0 < 1$. Note that if $t \in [0, t_0]$, then $\alpha(t) \subset \beta(t) \subset \beta(t_0) = E_a \subset K$. In particular, for

$t = 0$ it follows that $a \in E_a$. Moreover $\alpha([0, t_0]) \subset C(K)$ is arcwise connected. Thus, if $\bigcup \alpha([0, t_0]) = K$, then by [25, Theorem 1.50], we have $K \in \alpha([0, t_0])$. Since this is a contradiction, we infer that $\bigcup \alpha([0, t_0]) \neq K$, so $E_a \subsetneq K$. Since $t_0 < 1$ the map $\gamma = \beta_{|[t_0,1]}$ satisfies that $\gamma(t_0) = E_a$ and $\gamma(t) - K \neq \emptyset$ for each $t > t_0$. Thus, $E_a \in SB(K)$.

We have shown that for any $a \in K$, there is $E_a \in SB(K) - \{K\}$ such that $a \in E_a$. Clearly, E_a is contained in the composant of a in K. Now let $\varepsilon > 0$ and $n \in \mathbb{N}$. Since X contains uncountably many composants, we can take n points $a_1, a_2, \ldots, a_n$ in different composants of K. For each $i \in \{1, 2, \ldots, n\}$ let $E_i \in SB(K) - \{K\}$ be such that $a_i \in E_i$. Since composants of X are mutually disjoint, the sets $E_1, E_2, \ldots, E_n$ are mutually disjoint as well. The rest follows from Theorem 3.3. □

Now assume that $|m(K)| = 1$. The next result deals with the case in which $m(K) = \{K\}$.

Theorem 3.7. *$m(K) = \{K\}$ if and only K is terminal in X.*

Proof: Assume first that $m(K) = \{K\}$. If K is not terminal in X, then there is $A \in C(X)$ such that $A \cap K \neq \emptyset$, $A - K \neq \emptyset$, and $K - A \neq \emptyset$. Fix a component C of $A \cap K$ and note that $C \subsetneq K$. By Lemma 3.2 $C \in SB(K)$, and since $m(K) = \{K\}$, it follows that $K \subset C$. Then $K \subset C \subsetneq K$, a contradiction, so K is terminal in X.

Assume now that K is terminal in X. If $m(K) \neq \{K\}$, then there is $E \in m(K)$ such that $E \subsetneq K$. Fix $k \in K - E$ and let $\alpha\colon I \to C(X)$ be an order arc such that $\alpha(0) = E$ and $\alpha(t) - K \neq \emptyset$ for any $t > 0$. Since the set $\mathcal{U} = \{B \in C(X) \colon B \subset X - \{k\}\}$ is open in $C(X)$ and contains E, there is $\varepsilon > 0$ such that $B_{C(X)}(E, \varepsilon) \subset \mathcal{U}$. Since α is continuous, there is $t_1 > 0$ such that $A = \alpha(t_1) \in B_{C(X)}(E, \varepsilon)$. Thus, $k \notin A$. It follows that $A \cap K \neq \emptyset$, $A - K \neq \emptyset$, and $K - A \neq \emptyset$, contradicting the terminality of X. □

4. The Classes $\mathfrak{F}(X)$ and $\mathcal{U}(X)$

For a given continuum X consider the class $\mathfrak{F}(X)$ of continua with the following properties:

(1) no two distinct members of $\mathfrak{F}(X)$ are homeomorphic;
(2) $C(Y) \approx C(X)$ for each $Y \in \mathfrak{F}(X)$;

(3) $\mathfrak{F}(X)$ is the maximal class, satisfying conditions (1) and (2); i.e., if Z is a continuum such that $C(Z) \approx C(X)$, then $Z \approx Y$ for some $Y \in \mathfrak{F}(X)$.

We say that X has *unique hyperspace* ([3, Definition 1]) provided that $\mathfrak{F}(X) = \{X\}$. If the class $\mathfrak{F}(X)$ is finite and consists of more than one element, then we say that X has *almost unique hyperspace* ([2, Definition 1.1]).

Theorem 4.1. [3, Theorem 2] *Hereditarily indecomposable continua have unique hyperspace.*

For a continuum X consider the class $\mathcal{U}(X)$ of all proper and nondegenerate subcontinua K of X with the following properties: K is decomposable, $|m(K)| = 1$ and the only element in $m(K)$ is a proper subcontinuum of K. The next result is of particular interest in this paper. It will be used to obtain families of atriodic continua with either unique or almost unique hyperspace.

Theorem 4.2. *Let X and Y be (nondegenerate) continua such that $C(X) \approx C(Y)$ and let $h\colon C(Y) \to C(X)$ be a homeomorphism. Then the following assertions are true*

(4.2.1) *X is atriodic if and only if Y is atriodic;*

(4.2.2) *if X is atriodic, then $h(F_1(Y)) \subset F_1(X) \cup \{X\} \cup \mathcal{U}(X)$;*

(4.2.3) *if X is atriodic and indecomposable, then $h(F_1(Y)) \subset F_1(X) \cup \mathcal{U}(X)$;*

(4.2.4) *if X is atriodic and $\mathcal{U}(X) = \mathcal{U}(Y) = \emptyset$, then $X \approx Y$.*

Proof: Assertion (4.2.1) was stated before. To show (4.2.2), let us assume that X is atriodic. Fix $y \in Y$, put $K = h(\{y\})$ and consider that K is a proper and nondegenerate subcontinuum of X. By [25, Lemma 11.2] $C(Y) - \{\{y\}\}$ is arcwise connected, so $C(X) - \{K\}$ is arcwise connected as well. Thus, if K is indecomposable, by Theorem 3.6, X contains a triod. Since this contradicts (4.2.1), K is decomposable. If $m(K)$ has at least two elements then, by Corollary 3.5, there is a 2-cell $\mathcal{D}$ in $C(X)$ such that $K \in \mathcal{D}^\circ$. Then $\mathcal{E} = h^{-1}(\mathcal{D})$ is a 2-cell in $C(Y)$ such that $\{y\} \in \mathcal{E}^\circ$. Then, by Theorem 2.1, Y contains a triod. Since this contradicts (4.2.1), $m(K)$ has exactly one element. Let us assume that $m(K) = \{E\}$. If $E = K$ then, by Theorem 3.7, K is terminal in X. This contradicts Theorem 2.2 since $K = h(\{y\})$. Thus, E is a proper subcontinuum of K. This shows that $K \in \mathcal{U}(X)$, so (4.2.2) holds.

To show (4.2.3), let X be atriodic and indecomposable. Assume that there is $y \in Y$ such that $h(\{y\}) = X$. Since $C(Y) - \{\{y\}\}$ is arcwise connected ([25, Lemma 11.2]), it follows that $C(X) - \{h(\{y\})\} = C(X) - \{X\}$ is arcwise connected. However, since X is indecomposable, by [25, Theorem 1.51], $C(X) - \{X\}$ is not arcwise connected. This contradiction, together with (4.2.2), shows that $h(F_1(Y)) \subset F_1(X) \cup \mathcal{U}(X)$, so (4.2.3) holds.

To show (4.2.4), assume that X atriodic and $\mathcal{U}(X) = \mathcal{U}(Y) = \emptyset$. Then, by (4.2.2), $h(F_1(Y)) \subset F_1(X) \cup \{X\}$ and since Y is connected and nondegenerate, we have $h(F_1(Y)) \subset F_1(X)$. Since Y is atriodic and $h^{-1}\colon C(X) \to C(Y)$ is a homeomorphism, by (4.2.2), $h^{-1}(F_1(X)) \subset F_1(Y) \cup \{Y\}$. This implies that $h^{-1}(F_1(X)) \subset F_1(Y)$, so $F_1(X) = h(h^{-1}(F_1(X)) \subset h(F_1(Y)) \subset F_1(X)$ and then $h(F_1(Y)) = F_1(X)$. Thus, $X \approx Y$. □

For a homeomorphism $h\colon C(Y) \to C(X)$ condition $h(F_1(Y)) \subset F_1(X)$ implies that Y is homeomorphic to a nondegenerate subcontinuum of X. This follows from the fact that $Y \approx F_1(Y) \approx h(F_1(Y))$ and $h(F_1(Y)$ is a nondegenerate subcontinuum of $F_1(X)$, which is homeomorphic to X.

Theorem 4.3. *Let X be an atriodic continuum such that $F_1(X) \cup \{X\} \cup \mathcal{U}(X) \approx X$. Assume that for any nondegenerate subcontinuum Z of X such that $Z \not\approx X$ and for any $W \in \mathfrak{F}(Z)$, we have $X \not\approx W$. Then X has unique hyperspace.*

Proof: Let Y be a continuum such that $C(X) \approx C(Y)$ and let $h\colon C(Y) \to C(X)$ be a homeomorphism. By (4.2.2), we have $h(F_1(Y)) \subset F_1(X) \cup \{X\} \cup \mathcal{U}(X)$. Thus, $Y \approx Z$, for some nondegenerate subcontinuum Z of X. Assume $Z \not\approx X$. Since $C(X) \approx C(Z)$, it follows that $X \approx W$, for some $W \in \mathfrak{F}(Z)$. This is a contradiction, so $Z \approx X$ and then $Y \approx X$. □

Corollary 4.4. *Let X be an atriodic continuum such that $F_1(X) \cup \{X\} \cup \mathcal{U}(X) \approx X$. If any nondegenerate subcontinuum Z of X such that $Z \not\approx X$ has unique hyperspace, then X has unique hyperspace as well.*

Corollary 4.5. *Let X be an atriodic continuum such that $\mathcal{U}(X) = \emptyset$. Assume that for any nondegenerate subcontinuum Z of X such that $Z \not\approx X$ and for any $W \in \mathfrak{F}(Z)$ we have $X \not\approx W$. Then X has unique hyperspace.*

Corollary 4.6. *Let X be an atriodic continuum such that $\mathcal{U}(X) = \emptyset$. If any nondegenerate subcontinuum Z of X such that $Z \not\approx X$ has unique hyperspace, then X has unique hyperspace as well.*

In [22, Theorem 1], it was shown that if X is an arc-like continuum with exactly two arc-components, then one of them is an arc and the other is a ray, i.e., a homeomorphic image of the real half-line $[0, \infty)$. Then X is a compactification of a ray with an arc as its remainder, i.e., an *Elsa continuum*, according to Nadler's terminology in [24]. In [3, Theorem 4], it is shown that compactifications of the space $[0, \infty)$ with nondegenerate remainder have unique hyperspace. Thus, Elsa continua, in particular, have unique hyperspace. As an immediate consequence, it follows that the class of Elsa continua is C-determined. In [6, Theorem 4.7], it is shown that there are uncountably many Elsa continua, no two of which are homeomorphic.

5. Arcwise connected circle-like continua

In this section, we present a class of circle-like continua which is C-determined. Let us note that there are circle-like continua that do not have unique hyperspace. To see this, simply take X as the unit circle and Y as the unit interval. Then $C(X) \approx C(Y)$ and $X \not\approx Y$. Now consider the subset $V = \left\{\left(x, \sin\left(\frac{1}{x}\right)\right) : 0 < x \leq 1\right\}$ of the plane $\mathbb{R}^2$ as well as subcontinua

$$X_0 = V \cup (\{0\} \times [-1, 1]), \quad X_1 = V \cup (\{0\} \times [-2, 1]), \quad Y_1 = X_1 \cup T_1$$

where T_1 is an arc with end-points $a_0 = (1, \sin 1)$ and $b_0 = (0, -2)$ such that $T_1 \cap X_1 = \{a_0, b_0\}$. The continuum Y_1 is called the *Warsaw circle*, X_0 is the $\sin(\frac{1}{x})$-continuum, and X_1 is the $\sin(\frac{1}{x})$-continuum with the limit arc extended with an arc through the end-point $c_0 = (0, -1)$. Note that $Y_1 = X_0/\{a_0, c_0\} = X_1/\{a_0, b_0\}$. Since X_0 is an Elsa continuum, it has unique hyperspace. One is tempted to think that both X_1 and Y_1 have unique hyperspace. However, this is not the case and indeed the following result is true.

Theorem 5.1. [2, Theorem 3.1] *Let X be a continuum with the following properties:*

(5.1.1) *X is irreducible between points p and q;*
(5.1.3) *$C(p, X)$ and $C(q, X)$ are arcs in $C(X)$;*

(5.1.3) *whenever a continuum $X' = P \cup X \cup Q$ is obtained by joining to X two disjoint arcs P and Q such that $P \cap X = \{p\}$ and $Q \cap X = \{q\}$, where p and q are end-points of P and Q, respectively, it follows that $X \approx X'$.*

Let M be an arc with end-points p and q such that $X \cap M = \{p, q\}$. If $Y = X \cup M$, then $C(X) \approx C(Y)$.

Hence, by the previous theorem, $C(X_1) \approx C(Y_1)$. Thus, the Warsaw circle, which is a non-locally connected and arcwise connected circle-like continuum, does not have unique hyperspace. However, such continuum has almost unique hyperspace, since it is possible to show that $\mathfrak{F}(X_1) = \mathfrak{F}(Y_1) = \{X_1, Y_1\}$. Hence, it is natural to consider the class $\mathcal{AC}_c$ of circle-like continua which are arcwise connected and ask if its elements have almost unique hyperspace and behave like X_1 and Y_1.

A *generalized Warsaw circle* is an arcwise connected, circle-like continuum which is different from a circle. In [23, Theorem 6], Nadler proved that a continuum X is a generalized Warsaw circle if and only if there is a one-to-one and onto map $f : [0, \infty) \to X$ such that

$$f([0,1]) = \mathrm{cl}_X(f([t,\infty))) - f([t,\infty)) \text{ for each } t > 1.$$

Let $\mathcal{W}_c$ be the class of generalized Warsaw circles. By the previous characterization, each element X of $\mathcal{W}_c$ can be written in the form $X = V \cup R \cup M$, where $V \cup R$ is a compactification of the ray $V = [a, \infty)$ with the remainder an arc $R = bc$ and M is an arc with end-points a and b such that $M \cap (V \cup R) = \{a, b\}$. Note that $X = (V \cup R)/\{a, b\}$. The following result is stated in [2, Theorem 4.7]. Since its proof is not explicitly given in [2] and is not difficult using Theorem 4.2, we present the proof here.

Theorem 5.2. *Let $X_1 = V \cup R$ be a compactification of the space $V = [a, \infty)$ with an arc $R = bc$ as the remainder. Let us assume that M is an arc with end-points a and b such that $M \cap X_1 = \{a, b\}$ and define $X = X_1 \cup M$. If Y is a continuum such that $C(X) \approx C(Y)$ then either $Y \approx X$ or $Y \approx X_1 \cup T$, where T is an arc with b as an end-point of T such that $X_1 \cap T = \{b\}$.*

Proof: Let X and Y be as assumed. It is easy to see that $\mathcal{U}(X) = C(c, R) - \{\{c\}\}$, so if $h\colon C(Y) \to C(X)$ is a homeomorphism, by (4.2.2), $h(F_1(Y)) \subset F_1(X) \cup C(c, R) \cup \{X\}$. Since $\{X\}$ is isolated with respect to $F_1(X) \cup C(c, R)$, we have $h(F_1(Y)) \subset F_1(X) \cup C(c, R)$. Note that $C(c, R)$ is an order arc in $C(X)$ from $\{c\}$ to R such that $C(c, R) \cap F_1(X) = \{c\}$. Thus, $F_1(X) \cup C(c, R)$ is locally connected at any point of $C(c, R) - \{\{c\}\}$. Moreover, if Γ is a nondegenerate subcontinuum of $F_1(X) \cup C(c, R)$ that intersects $C(c, R) - \{\{c\}\}$, then Γ is locally connected at any point of $\Gamma \cap (C(c, R) - \{\{c\}\})$. We claim that

1) $h(F_1(Y)) \cap C(c, R) = \{\{c\}\}$.

To prove 1), assume, on the contrary, that there is a point $y \in Y$ such that $A = h(\{y\}) \in C(c, R) - \{\{c\}\}$. Then $h(F_1(Y))$ is a nondegenerate subcontinuum of $F_1(X) \cup C(c, R)$ that intersects $C(c, R) - \{\{c\}\}$, so $h(F_1(Y))$ is locally connected at A. This implies that Y is locally connected at y, so Y is connected im kleinen at y. However, since R is the remainder of the compactification X_1 of V, by [13, Theorem 2], $C(X)$ is not connected im kleinen at A. Then $C(Y)$ is not connected im kleinen at $\{y\}$, so by [12, Corollary 4], Y is not connected im kleinen at y. This contradiction shows 1).

By 1), we have $h(F_1(Y)) \subset F_1(X)$. Now, since X is not locally connected, by [25, Theorem 1.92], Y is not locally connected as well. Thus, Y is homeomorphic to a nondegenerate subcontinuum of X which is not locally connected. Hence, either $Y \approx X$, $Y \approx X_1$ or $Y \approx X_1 \cup T$, where T is an arc with b as an end-point of T such that $X_1 \cap T = \{b\}$. Note that X_1 is an Elsa continuum, so by [3, Theorem 4], X_1 has unique hyperspace. Hence, if $Y \approx X_1$, it follows that Y has unique hyperspace. Thus, $X \approx X_1$, which is a contradiction. This shows that either $Y \approx X$ or $Y \approx X_1 \cup T$, where T is as indicated. □

Let $X = X_1 \cup M$ be as assumed in Theorem 5.2. Combining theorems 5.1 and 5.2, it follows that $\mathfrak{F}(X) = \{X, X_1 \cup T\}$, where T is as described in Theorem 5.2. Hence, the elements of $\mathcal{W}_c$ have almost unique hyperspace. Using this and the fact that each element of $\mathcal{AC}_c$ is either a circle or a generalized Warsaw circle, we obtain the following result, which is a partial answer to the question whether the class of circle-like continua is C-determined.

Theorem 5.3. [2, Theorems 4.12 and 4.13] *The class $\mathcal{AC}_c$ of arcwise connected circle-like continua is C-determined. Moreover, if X and Y are circle-like continua such that $C(X) \approx C(Y)$, then X is arcwise connected if and only if Y is arcwise connected.*

Since there are uncountably many Elsa continua, no two of which are homeomorphic, the class $\mathcal{W}_c$ of generalized Warsaw circles contains uncountable many elements, no two of which are homeomorphic.

6. Homogeneous circle-like continua

A space X is *homogeneous* if for any two points $x, y \in X$ there exists a homeomorphism $f\colon X \to X$ such that $f(x) = y$. Let us consider the classes $\mathcal{HI}_c$ and $\mathcal{HOM}_c$ of hereditarily indecomposable circle-like continua and of homogeneous circle-like continua, respectively. The circle is the only element of $\mathcal{HOM}_c$ which is arcwise connected. R. H. Bing constructed a planar hereditarily indecomposable circle-like continuum which is not arc-like. He called it the *pseudo-circle*. Lawrence Fearnley [11] and James T. Rogers, Jr. [27] proved that the pseudo-circle is not homogeneous. They also showed that if X is a homogeneous, hereditarily indecomposable circle-like continuum then X is arc-like, so X is a pseudo-arc. Thus, among hereditarily indecomposable circle-like continua, the pseudo-arc is the only one which is homogeneous.

A *pseudo-solenoid* is any hereditarily indecomposable circle-like continuum which is not arc-like. Note that a pseudo-circle is a planar pseudo-solenoid. Since the pseudo-arc is the only hereditarily indecomposable arc-like continuum ([7, Theorem 1]), it follows that each element of $\mathcal{HI}_c$ is either a pseudo-solenoid or a pseudo-arc. Moreover, since any proper and nondegenerate subcontinuum of a circle-like continuum is arc-like, each proper and nondegenerate subcontinuum of a pseudo-solenoid is a pseudo-arc. Although pseudo-solenoids are not homogeneous, all of them have unique hyperspace by Theorem 4.1.

Theorem 6.1. *The class $\mathcal{HI}_c$ of hereditarily indecomposable circle-like continua is C-determined.*

An *arc-continuum* is a continuum such that all its proper and nondegenerate subcontinua are arcs. Solenoids different from the circle are homogeneous arc-continua. Using Corollary 4.5, we have the following result.

Theorem 6.2. *Solenoids different from the circle have unique hyperspace.*

Proof: Let X be a solenoid different from the circle. Since any proper and nondegenerate subcontinuum of X is an arc with two minimal elements in its semi-boundary, we have $\mathcal{U}(X) = \emptyset$. Moreover, if Z is a nondegenerate subcontinuum of X which is not homeomorphic to X, then Z is an arc and $X \not\approx W$ for any $W \in \mathfrak{F}(Z) = \{I, S\}$. Then, by Corollary 4.5, X has unique hyperspace. □

Besides solenoids, there are other indecomposable arc-continua, like Knaster continua. We will analyze this class in section 7. In [2, Theorem 2.3], it is shown, without using Theorem 4.2, that indecomposable arc-continua have unique hyperspace. The proof for solenoids different from the circle and for Knaster continua is shorter, as we saw in the proof of Theorem 6.2 and as we will see in the proof of Theorem 7.1.

An *arc of pseudo-arcs* is any arc-like continuum which admits a continuous decomposition into pseudo-arcs with the decomposition space an arc. Bing and F. B. Jones have shown that an arc of pseudo-arcs exists and that any two of them are homeomorphic ([9, Theorem 10]). Thus, the arc of pseudo-arcs is unique. Note that each proper and nondegenerate subcontinuum of the arc of pseudo-arcs is either a pseudo-arc or an arc of pseudo-arcs. Using this fact and Theorem 4.2, we have the following result.

Theorem 6.3. *The arc of pseudo-arcs has unique hyperspace.*

Proof: Let X be the arc of pseudo-arcs and $\mathcal{P}$ be a continuous decomposition of X into pseudo-arcs with the decomposition space an arc. Without loss of generality, we can assume that the decomposition space is I. Let $g\colon X \to I$ be the quotient map. We claim that

1) if A is a proper and nondegenerate subcontinuum of X such that $g(A)$ is nondegenerate and $0, 1 \notin g(A)$, then $|m(A)| \geq 2$.

To show 1), let A be as assumed. Then $g(A)$ is an arc in I with end-points s and t and $0 < s < t < 1$. Hence, $m(g(A)) = \{\{s\}, \{t\}\}$ and since $s \neq t$, there exist two disjoint subcontinua B_1 and B_2 of I such that $s \in B_1, t \in B_2$, $B_1 - g(A) \neq \emptyset$, and $B_2 - g(A) \neq \emptyset$. Then $g^{-1}(B_1)$ and $g^{-1}(B_2)$ are two disjoint subcontinua of X such that $g^{-1}(B_1) \cap A \neq \emptyset$, $g^{-1}(B_2) \cap A \neq \emptyset$, $g^{-1}(B_1) - A \neq \emptyset$, and $g^{-1}(B_2) - A \neq \emptyset$. Let C_1 be a component of $g^{-1}(B_1) \cap A$ and C_2 be a component of $g^{-1}(B_2) \cap A$. By Lemma 3.2, it follows that $C_1, C_2 \in SB(A)$. Since $g^{-1}(B_1) \cap g^{-1}(B_2) = \emptyset$, we have $C_1 \cap C_2 = \emptyset$. Thus, $|m(A)| \geq 2$, so 1) holds.

Put $T_0 = g^{-1}(0)$, $T_1 = g^{-1}(1)$ and let $A \in \mathcal{U}(X)$. Note that $g(A)$ is either a one-point set or a proper and nondegenerate subcontinuum of I. In the former case, $A \subset g^{-1}(t)$ for some $t \in I$, so A is hereditarily indecomposable. Hence, $A \notin \mathcal{U}(X)$. In the second case, by 1), $g(A) \in [C(0, I) \cup C(1, I)] - \{\{0\}, \{1\}\}$. This implies that

2) $\mathcal{U}(X) \cup \{X\} = [C(T_0, X) \cup C(T_1, X)] - \{T_0, T_1\}$.

Note that $C(T_0, X)$ and $C(T_1, X)$ are order arcs in $C(X)$ from T_0 to X and from T_1 to X, respectively. Moreover $C(T_0, X) \cap C(T_1, X) = \{X\}$, so $C(T_0, X) \cup C(T_1, X)$ is an arc in $C(X)$ with end-points T_0 and T_1. Moreover $[C(T_0, X) \cup C(T_1, X)] \cap F_1(X) = \emptyset$.

Let Y be a continuum such that $C(X) \approx C(Y)$ and let $h\colon C(Y) \to C(X)$ be a homeomorphism. Since X is atriodic, by (4.2.2) and 2),

$$h(F_1(X)) \subset F_1(X) \cup \mathcal{U}(X) \cup \{X\} \subset F_1(X) \cup C(T_0, X) \cup C(T_1, X).$$

Now since X is not locally connected, by [25, Theorem 1.92], Y is not locally connected. Thus, $h(F_1(Y)) \cap [C(T_0, X) \cup C(T_1, X)] = \emptyset$ since otherwise $h(F_1(Y)) \subset C(T_0, X) \cup C(T_1, X)$ and then $h(F_1(Y))$ is locally connected, so Y is locally connected too. This shows that $h(F_1(Y)) \subset F_1(X)$ and then Y is homeomorphic to a nondegenerate subcontinuum of X, so Y is either a pseudo-arc or an arc of pseudo-arcs. In the former case, it follows, by Theorem 4.1, that X is a pseudo-arc as well. Since this is a contradiction, $Y \approx X$. □

A continuum X is said to be a *solenoid of pseudo-arcs* if X is circle-like and admits a continuous decomposition into pseudo-arcs with the decomposition space a solenoid. If X is a solenoid of pseudo-arcs and the decomposition space is a circle, then X is said to be a *circle of pseudo-arcs*. It is known that for any solenoid Σ, there is a unique solenoid of pseudo-arcs X whose decomposition space is Σ ([20, Theorem 3]). Such a solenoid of pseudo-arcs is homogeneous ([29, Theorem 9]). Note that each proper and nondegenerate subcontinuum of a solenoid of pseudo-arcs is either a pseudo-arc or an arc of pseudo-arcs. Using this fact and Corollary 4.6, we have the following result.

Theorem 6.4. *Solenoids of pseudo-arcs have unique hyperspace.*

Proof: Let X be a solenoid of pseudo-arcs and $\mathcal{P}$ be a continuous decomposition of X into pseudo-arcs with the decomposition space, Σ, a solenoid. Let $g\colon X \to \Sigma$ be the quotient map and A be a proper and nondegenerate subcontinuum of X. Then $g(A)$ is either a one-point set or an arc in Σ. In the former case, $A \subset g^{-1}(t)$ for some $t \in \Sigma$, so A is hereditarily indecomposable. This implies that $A \notin \mathcal{U}(X)$. Assume then that $g(A)$ is an arc in Σ. Then proceeding as in the proof of claim 1) of Theorem 6.3, it follows that $|m(A)| \geq 2$, so $A \notin \mathcal{U}(X)$. This shows that $\mathcal{U}(X) = \emptyset$. By theorems 4.1 and 6.3, any nondegenerate subcontinuum of X which is not homeomorphic to X has unique hyperspace. Thus, by Corollary 4.6, X has unique hyperspace. □

Charles L. Hagopian and Rogers [14] showed that every homogeneous circle-like continuum is either a pseudo arc, a solenoid, or a solenoid of pseudo-arcs (with the circle and the circle of pseudo-arcs considered as special cases of solenoids and solenoids of pseudo-arcs, respectively). Since each of such continua, different from the circle, has unique hyperspace, we obtain the following result, which is another partial answer to the question whether the class of circle-like continua is C-determined.

Theorem 6.5. *The class of homogeneous circle-like continua is C-determined.*

Moreover, since the elements of the classes $\mathcal{HI}_c$ and $\mathcal{HOM}_c$, different from the circle, have unique hyperspace, by Theorem 5.3, we have the following result.

Theorem 6.6. *The class $\mathcal{AC}_c \cup \mathcal{HI}_c \cup \mathcal{HOM}_c$ is C-determined.*

7. Knaster continua

In terms of inverse limits, a continuum X is arc-like (respectively, circle-like) if X can be written as the inverse limit of arcs (respectively, circles) with surjective bonding maps. By a *Knaster continuum*, we understand the inverse limit of copies of I with surjective open bonding maps. For a continuum X and a point $x \in X$, we say that x is an *end-point* of X if x is an end-point of any arc in X that contains x. If K is a Knaster continuum, then K is an indecomposable arc-continuum with either exactly one end-point or exactly two end-points. Moreover, if K has exactly two end-points, p and q, then such points belong to different composants of K. Using Theorem 4.2, we have the following result.

Theorem 7.1. *Knaster continua have unique hyperspace.*

Proof: Let K be a Knaster continuum and Y be a continuum such that $C(K) \approx C(Y)$. Since K is not locally connected, by [25, Theorem 1.92], Y is not locally connected as well. Let $h\colon C(Y) \to C(K)$ be a homeomorphism. Since K is atriodic, it follows, by (4.2.2), that $h(F_1(Y)) \subset F_1(K) \cup \{K\} \cup \mathcal{U}(K)$. Let us assume that K contains exactly one end-point, p. Then $\mathcal{U}(K) \cup \{K\} = C(p, K) - \{\{p\}\}$, since any proper and nondegenerate subcontinuum of K that does not contain p is an arc in K with exactly two minimal elements in its semi-boundary. Note that $C(p, K)$ is an order arc in $C(K)$ from $\{p\}$ to K such that $C(p, K) \cap F_1(K) = \{p\}$. Thus, $F_1(K) \cup \{K\} \cup \mathcal{U}(K) = F_1(K) \cup C(p, K)$ is locally connected at any point of $C(p, K) - \{\{p\}\}$. Moreover, if Γ is a nondegenerate subcontinuum of $F_1(K) \cup C(p, K)$ that intersects $C(p, K) - \{\{p\}\}$, then Γ is locally connected at any point of $\Gamma \cap (C(p, X) - \{\{p\}\})$. We claim that

1) $h(F_1(Y)) \cap C(p, K) = \{\{p\}\}$.

To show 1), assume, on the contrary, that there is a point $y \in Y$ such that $A = h(\{y\}) \in C(p, K) - \{\{p\}\}$. Then $h(F_1(Y))$ is a nondegenerate subcontinuum of $F_1(K) \cup C(p, K)$ that intersects $C(p, K) - \{\{p\}\}$, so $h(F_1(Y))$ is locally connected at A. This implies that Y is locally connected at y, so Y is connected im kleinen at y. However, since K is a Knaster continuum and $p \in A$, by [13, Theorem 2], $C(K)$ is not connected im kleinen at A. Then $C(Y)$ is not connected im kleinen at $\{y\}$ so, by [12, Corollary 4], Y is not connected im kleinen at y. This contradiction shows 1).

By 1), it follows that $h(F_1(Y)) \subset F_1(K)$. Thus, Y is homeomorphic to a nondegenerate subcontinuum of K, so either Y is an arc or $Y \approx K$. In the former case, it follows that either K is an arc or a circle. In any situation, we obtain a contradiction, so $Y \approx K$.

Now assume that K has exactly two end-points, p and q. Then p and q belong to different composants of K. Moreover, $\mathcal{U}(K) \cup \{K\} = [C(p, K) \cup C(q, K)] - \{\{p\}, \{q\}\}$. Note that $C(p, K)$ and $C(q, K)$ are order arcs in $C(K)$ from $\{p\}$ to K and from $\{q\}$ to K, respectively. Moreover, $C(p, K) \cap C(q, K) = \{K\}$. Then $C(p, K) \cup C(q, K)$ is an arc in $C(K)$ such that $[C(p, K) \cup C(q, K)] \cap F_1(K) = \{\{p\}, \{q\}\}$, so $F_1(K) \cup \{K\} \cup \mathcal{U}(K) = F_1(K) \cup C(p, X) \cup C(q, X)$ is locally connected at any point of $[C(p, K) \cup C(q, K)] - \{\{p\}, \{q\}\}$. Proceeding as in the proof of 1), it can be shown that $h(F_1(Y)) \cap [C(p, K) \cup C(q, K)] = \{\{p\}, \{q\}\}$, so $h(F_1(Y)) \subset F_1(K)$ and from this inclusion, it follows that $Y \approx K$. □

As a consequence of the previous result, the class $\mathcal{K}$ of Knaster continua is C-determined.

Theorem 7.2. *Let X be an indecomposable arc-continuum with no end-points. Then X has unique hyperspace.*

Proof: Let X be as assumed. Since any proper and nondegenerate subcontinuum of X is an arc with two minimal elements in its semi-boundary, we have $\mathcal{U}(X) = \emptyset$. The rest follows from Corollary 4.5. □

The proof of 1) of Theorem 7.1 will be used later. As a particular case of Theorem 7.1, both the buckethandle continuum and the double buckethandle continuum have unique hyperspace. The buckethandle continuum has exactly one end-point, while the double buckethandle continuum has exactly two end-points.

Let K be a Knaster continuum with exactly one end-point, p. Let K_p denote the union of K with an arc attached through p, i.e., $K_p = K \cup A$, where A is an arc with p as one of its end-points and such that $A \cap K = \{p\}$.

Theorem 7.3. *Let K be a Knaster continuum with exactly one end-point, p. Then K_p has unique hyperspace.*

Proof: Put $K_p = K \cup A$, where A is as indicated above. Let a be the other end-point of A. Note that

$$\mathcal{U}(K_p) \cup \{K_p\} = [C(a, K_p) \cup C(K, K_p)] - \{\{a\}, K\}$$

since any other proper and nondegenerate subcontinuum of K_p has exactly two minimal elements in its semi-boundary. Note that $C(a, K_p)$ is an order arc in $C(K_p)$ from $\{a\}$ to K_p, while $C(K, K_p)$ is an order arc in $C(K_p)$ from K to K_p. Note also that $C(a, K_p) \cap F_1(K_p) = \{\{a\}\}$ and $C(K, K_p) \cap [F_1(K_p) \cup C(a, K_p)] = \{K_p\}$. Then $F_1(K_p) \cup \{K_p\} \cup \mathcal{U}(K_p) = F_1(K_p) \cup C(a, K_p) \cup C(K, K_p) \approx K_p$. Let Z be a nondegenerate subcontinuum of K_p such that $Z \not\approx K_p$. Then Z is either an arc or $Z = K$. Then, using Theorem 7.1, it follows that $\mathfrak{F}(Z) = \{I, S\}$ or $\mathfrak{F}(Z) = \{Z\}$. Thus, $K_p \not\approx W$, for any $W \in \mathfrak{F}(Z)$. Then, by Theorem 4.3, K_p has unique hyperspace. $\square$

Let K be a Knaster continuum K with exactly two end-points, p and q. We consider continua K_p, K_q, K_{pq}, $W(K)$, and $V(K)$ defined as follows:

(1) K_p is defined as before, i.e., by attaching an arc through p;
(2) K_q is obtained from K by attaching an arc through q;
(3) K_{pq} is obtained from K_p by attaching an arc through q;
(4) $W(K)$ is obtained from K_{pq} by identifying its two points of irreducibility;
(5) $V(K) = K/\{p, q\}$, i.e., $V(K)$ is obtained from K by identifying p and q.

Note that $K_{pq} = A \cup K \cup B$, where A and B are disjoint arcs, p is an end-point of A, q is an end-point of B, $A \cap K = \{p\}$, and $B \cap K = \{q\}$. Moreover, if a is the other end-point of A and b is the other end-point of B, then $W(K) = K_{pq}/\{a, b\}$. Note also that $W(K) = K_{pq} \cup M$, where M is an arc with end-points a and b such that $K_{pq} \cap M = \{a, b\}$. We also have that $W(K) = K \cup N$, where N is an arc with end-points p and q such that $K \cap N = \{p, q\}$. By Theorem 5.1, we have the following result.

Theorem 7.4. *Let K be a Knaster continuum with exactly two end-points, p and q. Then $C(K_{pq}) \approx C(W(K))$.*

Note that each continuum of the form K_{pq} is arc-like, while each of the form $W(K)$ is circle-like. By the previous result, neither K_{pq} nor $W(K)$ have unique hyperspace. To analyze the structure of the class $\mathfrak{F}(K_{pq})$, we have to analyze first the structure of the class $\mathfrak{F}(K_p)$.

Theorem 7.5. *Let K be a Knaster continuum with exactly two end-points, p and q. Then K_p and K_q have unique hyperspace.*

Proof: Put $K_p = K \cup A$, where A is an arc with p as one of its two end-points and $A \cap K = \{p\}$. Let a be the other end-point of A. It is easy to see that

$$\mathcal{U}(K_p) \cup \{K_p\} = [C(a, K_p) \cup C(q, K_p)] - \{\{a\}, \{q\}\}.$$

Note that $C(a, K_p)$ is an order arc in $C(K_p)$ from $\{a\}$ to K_p, while $C(q, K_p)$ is an order arc in $C(K_p)$ from $\{q\}$ to K_p. Moreover, $C(a, K_p) \cap C(q, K_p) = \{K_p\}$ and $[C(a, K_p) \cup C(q, K_q)] \cap F_1(K_p) = \{\{a\}, \{q\}\}$.

Let Y be a continuum such that $C(K_p) \approx C(Y)$ and let $h\colon C(Y) \to C(K_p)$ be a homeomorphism. By [25, Theorem 1.92] and (4.2.2), Y is not locally connected and $h(F_1(Y)) \subset F_1(K_p) \cup C(a, K_p) \cup C(q, K_p)$. Note that $C(q, K_p) = C(q, K) \cup C(K, K_p)$ and that $C(K, K_p)$ is an order arc in $C(K_p)$ from K to K_p. Proceeding as in the proof of 1) of Theorem 7.1, it follows that $h(F_1(Y)) \cap C(q, K) = \{q\}$. Thus, $h(F_1(Y)) \subset F_1(K_p) \cup C(a, K_q) \cup C(K, K_p)$. This implies, since Y is not locally connected and $F_1(K_p) \cup C(a, K_q) \cup C(K, K_p) \approx K_p$, that either $Y \approx K$ or $Y \approx K_p$. In the former case, we have, by Theorem 7.1, that $K_p \approx K$. This is a contradiction, so $Y \approx K_p$. This shows that K_p has unique hyperspace. The proof for K_q is similar. □

Let $\mathcal{K}_p$ be the class of decomposable continua of the form K_p, where K is a Knaster continuum. As a consequence of theorems 7.3 and 7.5, we have the following result.

Theorem 7.6. *The class $\mathcal{K}_p$ is C-determined.*

The next result shows that continua K_{pq} and $W(K)$ have almost unique hyperspace.

Theorem 7.7. *Let K be a Knaster continuum with exactly two end-points, p and q. Then $\mathfrak{F}(W(K)) = \{K_{pq}, W(K)\}$.*

Proof: The inclusion $\{K_{pq}, W(K)\} \subset \mathfrak{F}(W(K))$ follows from Theorem 7.4. To show the other inclusion, let Y be a continuum such that $C(W(K)) \approx C(Y)$. Take a homeomorphism $h\colon C(Y) \to$

$C(W(K))$. It is easy to see that $\mathcal{U}(W(K)) = \emptyset$ so, by (4.2.2), $h(F_1(Y)) \subset F_1(W(K))$. Using this and the fact that Y is not locally connected, it follows that Y is homeomorphic to a nondegenerate subcontinuum of $W(K)$ which is not locally connected. Then either $Y \approx K$, $Y \approx K_p$, $Y \approx K_q$, $Y \approx K_{pq}$, or $Y \approx W(K)$. If $Y \approx K$ then, since $C(Y) \approx C(W(K))$ and K has unique hyperspace (as does Y), we have $W(K) \approx K$, which is a contradiction. If $Y \approx K_p$, we proceed in a similar way, using the fact that K_p has unique hyperspace, to obtain a contradiction. We proceed in the same fashion if $Y \approx K_q$. Thus, either $Y \approx K_{pq}$ or $Y \approx W(K)$, i.e., $\mathfrak{F}(W(K)) \subset \{K_{pq}, W(K)\}$. □

Let $\mathcal{W}_c$ be the class of continua of the form $W(K)$, where K is any Knaster continuum with exactly two end-points. As an immediate consequence of Theorem 7.7, we obtain the following result, which is another partial answer to the question whether the class of circle-like continua is C-determined.

Theorem 7.8. *The class $\mathcal{W}_c$ is C-determined.*

Now consider the class $\mathcal{K}_{pq}$ of continua of the form K_{pq}, where K is any Knaster continuum with exactly two end-points. By Theorem 7.7, it follows that $\mathfrak{F}(K_{pq}) = \mathfrak{F}(W(K)) = \{K_{pq}, W(K)\}$. Thus, we have the following result.

Theorem 7.9. *The class $\mathcal{K}_{pq}$ is C-determined.*

Recall that $\mathcal{K}$ is the class of Knaster continua. Combining theorems 7.1, 7.6, and 7.10, it follows that the class $\mathcal{K} \cup \mathcal{K}_p \cup \mathcal{K}_{pq}$ is C-determined. We can also combine theorems 7.1, 7.6, and 7.8 to conclude that the class $\mathcal{K} \cup \mathcal{K}_p \cup \mathcal{W}_c$ is C-determined as well.

Now consider the class $\mathcal{V}_c$ of continua of the form $V(K)$, where K is any Knaster continuum with exactly two end-points.

Theorem 7.10. *Let K be a Knaster continuum with exactly two end-points, p and q. Then $V(K)$ has unique hyperspace.*

Proof: Recall that $V(K) = K/\{p, q\}$. Note that any proper and nondegenerate subcontinuum of $V(K)$ is an arc with exactly two minimal elements in its semi-boundary. Thus, $\mathcal{U}(V(K)) = \emptyset$. The rest follows from Corollary 4.5. □

Note that Theorem 7.10 also follows from [2, Theorem 2.3] since $V(K)$ is an indecomposable arc-continuum. As in immediate consequence of Theorem 7.10, we have the following result.

Theorem 7.11. *The class $\mathcal{V}_c$ is C-determined.*

Note that the members of the class $\mathcal{V}_c$ are circle-like. Combining theorems 6.6, 7.8, and 7.11, it follows that the class $\mathcal{AC}_c \cup \mathcal{HI}_c \cup \mathcal{HOM}_c \cup \mathcal{W}_c \cup \mathcal{V}_c$ is C-determined.

Let B be the double buckethandle continuum. Note that $W(K)$ and $V(K)$ are two circle-like continua which are not homeomorphic. One is tempted to think that hyperspaces $C(W(K))$ and $C(V(K))$ are homeomorphic. However, this is not the case, by theorems 7.7 and 7.10.

Note added. This paper is the first of a series of papers devoted to answering the question whether the class of circle-like continua is C-determined. The author concluded this paper during his stay at West Virginia University in Morgantown, West Virginia, and thanks Professor Sam B. Nadler, Jr. for his help with the paper.

References

[1] Gerardo Acosta, *On compactifications of the real line and unique hyperspace*, Topology Proc. **25** (2000), Spring, 1–25.

[2] ———, *Continua with almost unique hyperspace*, Topology Appl. **117** (2002), 175–189.

[3] ———, *Continua with unique hyperspace*, in Continuum Theory. Ed. Alejandro Illanes, Sergio Macías, and Ira Lewis. Lecture Notes in Pure and Applied Mathematics, 230. New York: Marcel Dekker, Inc., 2002. 33–49

[4] ———, *On smooth fans and unique hyperspaces*, Houston J. Math. **30** (2004), no. 1, 99–115.

[5] Gerardo Acosta, Janusz J. Charatonik, and Alejandro Illanes, *Irreducible continua of type λ with almost unique hyperspace*, Rocky Mountain J. Math. **31** (2001), no. 3, 745–772.

[6] Marwan M. Awartani, *An uncountable collection of mutually incomparable chainable continua*, Proc. Amer. Math. Soc. **118** (1993), no. 1, 239–245.

[7] R. H. Bing, *Concerning hereditarily indecomposable continua*, Pacific J. Math. **1** (1951), 43–51.

[8] ———, *Each homogeneous nondegenerate chainable continuum is a pseudo-arc*, Proc. Amer. Math. Soc. **10** (1959), 345–346.

[9] R. H. Bing and F. B. Jones, *Another homogeneous plane continuum*, Trans. Amer. Math. Soc. **90** (1959), no. 1, 171–192.

[10] Carl Eberhart and Sam B. Nadler, Jr., *Hyperspaces of cones and fans*, Proc. Amer. Math. Soc. **77** (1979), no. 2, 279–288.

[11] Lawrence Fearnley, *The pseudo-circle is not homogeneous*, Bull. Amer. Math. Soc. **75** (1969), 554–558.

[12] Jack T. Goodykoontz, Jr., *Connectedness im kleinen and local connectedness in* 2^X *and* $C(X)$, Pacific J. Math. **53** (1974), 387–397.

[13] ———, *More on connectedness im kleinen and local connectedness in* $C(X)$ Proc. Amer. Math. Soc. **65** (1977), no. 2, 357–364.

[14] Charles L. Hagopian and James T. Rogers, Jr., *A classification of homogeneous, circle-like continua*, Houston J. Math. **3** (1977), no. 4, 471–474.

[15] Alejandro Illanes, *Cells and cubes in hyperspaces*, Fund. Math. **130** (1988), no. 1, 57–65.

[16] ———, *Semi-boundaries in hyperspaces*, Topology Proc. **16** (1991), 63–87.

[17] ———, *Chainable continua are not* C*-determined*, Topology Appl. **98** (1999), no. 1-3, 211–216.

[18] ———, *Fans are not* C*-determined*, Colloq. Math. **81** (1999), no. 2, 299–308.

[19] Alejandro Illanes and Sam B. Nadler, Jr., *Hyperspaces: Fundamentals and Recent Advances.* Monographs and Textbooks in Pure and Applied Mathematics, 216. New York: Marcel Dekker, Inc., 1999.

[20] Wayne Lewis, *Homogeneous circle-like continua*, Proc. Amer. Math. Soc. **89** (1983), no. 1, 163–168.

[21] Sergio Macías, *On* $\mathcal{C}$*-determined continua*, Glas. Mat. Ser III **32/(52)** (1997), no. 2, 259–262.

[22] Sam B. Nadler, Jr. *Arc components of certain chainable continua*, Canad. Math. Bull. **14** (1971), 183–189.

[23] ———, *Multicoherence techniques applied to inverse limits*, Trans. Amer. Math. Soc. **157** (1971), 227–234.

[24] ———, *Continua whose cone and hyperspace are homeomorphic*, Trans. Amer. Math. Soc. **230** (1977), 321–345.

[25] ———, *Hyperspaces of Sets.* Monographs and Textbooks in Pure and Applied Mathematics, Vol. 49. New York-Basel: Marcel Dekker, Inc., 1978.

[26] ———, *Continuum Theory: An Introduction.* Monographs and Textbooks in Pure and Applied Mathematics, 158. New York: Marcel Dekker, Inc., 1992.

[27] James T. Rogers, Jr., *The pseudo-circle is not homogeneous*, Trans. Amer. Math. Soc. **148** (1970), 417–428.

[28] ———, *Dimension of hyperspaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **20** (1972), 177–179.

[29] ______, *Solenoids of pseudo-arcs*, Houston J. Math. **3** (1977), no. 4, 531–537.

Instituto de Matemáticas; Circuito Exterior; Ciudad Universitaria; Área de la Investigación Científica; México D.F., 04510; México.
E-mail address: gacosta@matem.unam.mx