Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 30, No. 1, 2006
Pages 25-38

COVERING PROPERTIES AND NEIGHBORHOOD ASSIGNMENTS

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ABSTRACT. We continue the study originated in "Classes defined by stars and neighbourhood assignment" (to appear in Topology and its Applications), where an idea of E. van Douwen used to define D-spaces was developed. Given a topological property (or a class) $\mathcal{P}$, the class $\mathcal{P}^*$ dual to $\mathcal{P}$ (with respect to neighborhood assignments) consists of spaces X such that for any neighborhood assignment $\{O_x : x \in X\}$ there is $Y \subset X$ with $Y \in \mathcal{P}$ and $\bigcup\{O_x : x \in Y\} = X$. The spaces from $\mathcal{P}^*$ are called *dually* $\mathcal{P}$. We show, among other things, that a dually σ-compact Tychonoff space need not be Lindelöf; this solves problems 4.1 and 4.3 from "Classes defined by stars and neighbourhood assignment."

We also consider the *weak dual class* $\mathcal{P}'$ of a given class $\mathcal{P}$; the elements of $\mathcal{P}'$ are the spaces X such that for any neighborhood assignment $\{O_x : x \in X\}$ there is a subspace $Y \subset X$ such that $Y \in \mathcal{P}$ and $\bigcup\{O_x : x \in Y\}$ is dense in X. We establish that pseudocompactness is weakly self-dual in the class of Tychonoff spaces but not in the class of Hausdorff spaces; we also show that the class weakly dual to the class of compact spaces contains Tychonoff spaces which are not countably compact.

2000 *Mathematics Subject Classification.* Primary 54H11, 54C10, 22A05, 54D06; Secondary 54D25, 54C25.

Key words and phrases. assignment, duality, Lindelöf space, neighborhood, weak duality, weakly Lindelöf space.

Research supported by Consejo Nacional de Ciencia y Tecnología (CONACYT) de México, grant 400200-5-38164-E.

1. Introduction

A *neighborhood assignment* in a space X is a family $\mathcal{O} = \{\mathcal{O}_x : x \in X\}$ of open subsets of X such that $x \in O_x$ for any $x \in X$. Neighborhood assignments are useful for defining covering properties. For example, a space X is Lindelöf if and only if, for any neighborhood assignment $\{O_x : x \in X\}$ there is a countable $Y \subset X$ such that $\bigcup\{O_x : x \in Y\} = X$. If we substitute "closed discrete" for "countable," then we obtain the definition of the class of D-spaces introduced by Eric van Douwen [8] and studied in [5], [4], [6], [7], [10], and other papers.

This idea of van Douwen was generalized in [13] by defining, for any class (or property) $\mathcal{P}$, a dual class $\mathcal{P}^*$ which consists of spaces X such that, for any neighborhood assignment $\mathcal{O} = \{O_x : x \in X\}$ there exists a subspace $Y \subset X$ such that $\mathcal{O}(Y) = \bigcup\{O_x : x \in Y\} = X$ and $Y \in \mathcal{P}$. It was proved in [13] that many classical covering properties $\mathcal{P}$ are self-dual in this sense, i.e., $\mathcal{P}^* = \mathcal{P}$. For example, compactness, pseudocompactness, countable compactness, and the linear Lindelöf property are self-dual with respect to neighborhood assignments.

In this paper, we give an example of a Tychonoff pseudocompact non-compact space X which is dually σ-compact; this completely solves problems 4.1 and 4.3 from [13] and shows that neither Lindelöf (Σ-)property nor σ-compactness is self-dual with respect to neighborhood assignments. We also show that, in the class of perfect spaces, the Souslin property is self-dual with respect to neighborhood assignments. The Souslin property being equivalent to the weak Lindelöf property in the class of perfectly normal spaces, the weak Lindelöf property is self-dual in the class of perfectly normal spaces.

Now, if $\mathcal{P}$ is a topological property (or a class) then the weak dual class $\mathcal{P}'$ of the class $\mathcal{P}$ consists of spaces X such that for any neighborhood assignment $\mathcal{O} = \{O_x : x \in X\}$ there exists a subspace $Y \subset X$ with $Y \in \mathcal{P}$ for which the set $\mathcal{O}(Y)$ is dense in X. We consider the weak duals of some topological properties and classes showing that pseudocompactness is weakly self-dual in the class of Tychonoff spaces while not all elements of the weak dual of the class of compact spaces are countably compact.

2. Notation and terminology

All spaces under consideration are assumed to be Hausdorff. Given a space X, the family $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\backslash\{\emptyset\}$. If $x \in X$, then $\tau(x, X) = \{U \in \tau(X) : x \in U\}$. If $\mathcal{O} = \{O_x : x \in X\}$ is a neighborhood assignment in a space X, then $\mathcal{O}(Y) = \bigcup\{O_x : x \in Y\}$ for any $Y \subset X$.

A space X is *weakly Lindelöf* if for any $\mathcal{U} \subset \tau(X)$ with $X = \bigcup \mathcal{U}$, there is a countable $\mathcal{U}' \subset \mathcal{U}$ such that $\bigcup \mathcal{U}'$ is dense in X. A space is *feebly compact* if every locally finite family $\mathcal{U} \subset \tau^*(X)$ is finite. If a space is Tychonoff and feebly compact, then we call it *pseudocompact*.

A space X is *linearly Lindelöf* if every open cover of X, linearly ordered by the subset relation, has a countable subcover. The cardinal $l(X)$ called *the Lindelöf number* of X is the minimal cardinal κ such that every open cover of X has a subcover of cardinality at most κ. The *spread* $s(X)$ of a space X is the supremum of cardinalities of discrete subspaces of X.

The space $\mathbb{D}$ is the doubleton $\{0, 1\}$ with the discrete topology, $\mathbb{R}$ is the real line with the usual order topology, and $\mathbb{Q}$ is the set of rational numbers. The rest of the notation is standard and can be found in [9].

3. Self-duality in some nice classes

It was proved in [13] that a space is Lindelöf whenever it is dually hereditarily Lindelöf. We will show first that being dually Lindelöf is not sufficient for being Lindelöf.

Theorem 3.1. *There exists a Tychonoff dually σ-compact space X which is pseudocompact but not compact (and hence, not Lindelöf).*

Proof: For any set A, the point $u_A \in \mathbb{D}^A$ is defined by $u_A(a) = 0$ for any $a \in A$. Let $\xi_0 = \omega$; if $n \in \omega$ and a cardinal ξ_n is defined, let $\xi_{n+1} = 2^{\xi_n}$. This gives us a sequence $\{\xi_n : n \in \omega\}$ of cardinals; let $\xi = \sup\{\xi_n : n \in \omega\}$.

In the space $\mathbb{D}^\xi$, consider the subspace $X = \{x \in \mathbb{D}^\xi : |x^{-1}(1)| < \xi\}$. Then $X = \bigcup_{n\in\omega} X_n$ where $X_n = \{x \in X : |x^{-1}(1)| \leq \xi_n\}$ for every $n \in \omega$. Fix a neighborhood assignment $\mathcal{O} = \{O_x : x \in X\}$; we claim that there is a σ-compact $S \subset X$ such that $\mathcal{O}(S) = X$.

Of course, it suffices to find a σ-compact $S_n \subset X$ such that $X_n \subset \mathcal{O}(S_n)$ for every $n \in \omega$.

We can assume, without loss of generality, that every O_x is a standard open subset of X; i.e., there is a finite $A_x \subset \xi$ such that $O_x = \{y \in X : y|A_x = x|A_x\}$ for any $x \in X$.

Fix an $m \in \omega$; we are going to show that there is a σ-compact $S_m \subset X$ such that $\mathcal{O}(S_m) \supset X_m$. We have $X = \bigcup_{n\in\omega} Y_n$ where $Y_n = \{x \in X : A_x \subset \xi_n\}$ for any $n \in \omega$. Therefore, it suffices to show that there is a compact $P_{mn} \subset X$ for which $\mathcal{O}(P_{mn}) \supset Z_{mn} = Y_n \cap X_m$ for each $n \in \omega$. So, fix $n \in \omega$; for every $x \in X_m$, let $\pi(x)(\alpha) = x(\alpha)$ for any $\alpha < \xi_n$; this gives a map $\pi : X_m \to \mathbb{D}^{\xi_n}$. We have $|\pi(Z_{mn})| \leq 2^{\xi_n} = \xi_{n+1}$, so choose a set $H_{mn} \subset Z_{mn}$ such that $|H_{mn}| \leq \xi_{n+1}$ and $\pi(H_{mn}) = \pi(Z_{mn})$.

Let $k = \max\{n + 1, m\}$; since $|x^{-1}(1)| \leq \xi_m$ for any point $x \in H_{mn}$, the set $B = \bigcup\{x^{-1}(1) : x \in H_{mn}\}$ has cardinality at most ξ_k. If $A = \xi\backslash B$, then $H_{mn} \subset \mathbb{D}^B \times \{u_A\} \subset X$ so the closure P_{mn} of the set H_{mn} in X is compact.

Given any point $y \in Z_{mn}$, there is $x \in H_{mn}$ such that $\pi(x) = \pi(y)$; it follows from $A_x \subset \xi_n$ that $y|A_x = \pi(y)|A_x = \pi(x)|A_x = x|A_x$ and therefore, $y \in O_x$. This proves that $Z_{mn} \subset \mathcal{O}(H_{mn}) \subset \mathcal{O}(P_{mn})$ for any $m, n \in \omega$. Therefore, the set $P = \bigcup\{P_{mn} : m, n \in \omega\} \subset X$ is σ-compact and $X = \mathcal{O}(P)$ so X is, indeed, dually σ-compact. To finally see that X is pseudocompact, observe that X_0 is a dense countably compact subspace of X; since $X \neq \mathbb{D}^\xi$, the space X is not compact. □

Recall that X is called *a Lindelöf* Σ*-space* if there exists a space Y which maps perfectly onto a second countable space and continuously onto X. If X is the countable intersection of σ-compact subspaces of some larger space, then X is called *a* $K_{\sigma\delta}$*-space.* Continuous images of $K_{\sigma\delta}$-spaces are called K*-analytic spaces.* For more information about these classes, the reader is referred to section 7 of the paper [3] and section 9 of Chapter IV of the book [2].

Corollary 3.2. *A Tychonoff pseudocompact dually Lindelöf* Σ*-space (or even dually* K*-analytic space) need not be Lindelöf.*

Lemma 3.3. *If every open subspace of a space* X *is linearly Lindelöf, then* X *is Lindelöf.*

Proof: Suppose to the contrary, that our lemma is false; then the class $\mathcal{P}$ of non-Lindelöf spaces Y in which every open subspace is linearly Lindelöf is non-empty. For every $Y \in \mathcal{P}$, let κ_Y be the minimal cardinality of an open cover of Y which has no countable subcover. Choose a space $X \in \mathcal{P}$ such that $\kappa = \kappa_X$ is minimal and fix an open cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ of the space X with no countable subcover.

The space X being linearly Lindelöf, the cofinality of the cardinal κ is countable. (This fact is well-known; see e.g., [4, Proposition 2.7] or [12, Proposition 1].) Let $\{\kappa_n : n \in \omega\}$ be an increasing sequence of regular cardinals whose supremum is κ. For each $n \in \omega$, the space $H_n = \bigcup\{U_\alpha : \alpha \in \kappa_n\}$ is open in X and hence linearly Lindelöf. Since $\{U_\alpha : \alpha \in \kappa_n\}$ is an open cover of H_n of cardinality less than κ, by the minimality of κ, there is a countable $\mathcal{C}_n \subset \{U_\alpha : \alpha \in \kappa_n\}$ such that $H_n = \bigcup \mathcal{C}_n$. Then $\mathcal{C} = \bigcup\{\mathcal{C}_n : n \in \omega\}$ is a countable subcover of $\mathcal{U}$ which is a contradiction. □

Corollary 3.4. *If every open subset of a space X is linearly Lindelöf then X is hereditarily Lindelöf.*

Proof: It follows from Lemma 3.3 that any open subset of X is Lindelöf so X is hereditarily Lindelöf (see [9, 3.8A]). □

Corollary 3.5. *If every open subspace of a space X is dually Lindelöf then X is hereditarily Lindelöf.*

Proof: It was established in [13, Proposition 2.7] that any dually Lindelöf space is linearly Lindelöf so every open subspace of X is linearly Lindelöf. Corollary 3.4 does the rest. □

Recall that a space X is *perfect* if any closed $F \subset X$ is a G_δ-subset of X.

Theorem 3.6. *In the class of perfect spaces, countable cellularity is self-dual with respect to neighborhood assignments.*

Proof: Assume that there exists a perfect space X such that $c(X) > \omega$, and for each neighborhood assignment $\mathcal{O} = \{O_x : x \in X\}$, there is a subspace $Y \subset X$ with $c(Y) = \omega$ and $\mathcal{O}(Y) = X$. Suppose that $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$ is a disjoint family of non-empty open subsets of X and let $C = X \setminus \bigcup \mathcal{U}$. Since X is perfect, there is a family $\{V_n : n \in \omega\} \subset \tau(X)$ such that $C = \bigcap\{V_n : n \in \omega\}$. Fix $n \in \omega$; given a point $x \in X$, let $O_x^n = V_n$ if $x \in V_n$; if $x \in X \setminus V_n$, then let $O^n_x = U_\alpha$, where U_α is the unique element of $\mathcal{U}$ containing x. This gives a neighborhood assignment $\mathcal{O}_n = \{O^n_x : x \in X\}$ of the space X.

By our hypothesis, there is a subspace $Y_n \subset X$ such that $c(Y_n) = \omega$ and $\mathcal{O}_n(Y_n) = X$; the family $\mathcal{U}$ being disjoint, the set $\{\alpha : Y_n \cap U_\alpha \neq \emptyset\}$ is countable and hence, the set $I_n = \{\alpha : U_\alpha \not\subset V_n\}$ is also countable for any $n \in \omega$. Then $V_n \cup \bigcup\{U_\alpha : \alpha \in I_n\} = X$ for any $n \in \omega$ and hence, $I = \bigcup_{n\in\omega} I_n$ is a countable set such that $C \cup \bigcup\{U_\alpha : \alpha \in I\} = X$; an immediate consequence is that $I = \omega_1$, which is a contradiction. □

Corollary 3.7. *In the class of perfectly normal spaces, the property of being weakly Lindelöf is self-dual with respect to neighborhood assignments.*

Proof: It is an easy exercise that a perfectly normal space X is weakly Lindelöf if and only if $c(X) = \omega$; apply Theorem 3.6 to conclude the proof. □

Recall that a space X is called *monotonically normal* if there exists an operator $H : \{(x, U) : x \in U \in \tau(X)\} \to \tau(X)$ such that $x \in H(x, U) \subset U$ and $H(x, X\backslash\{y\}) \cap H(y, X\backslash\{x\}) = \emptyset$ for any distinct $x, y \in X$ and, additionally, $x \in V \subset U$ implies $H(x, V) \subset H(x, U)$.

Proposition 3.8. *In the class of countably metacompact spaces the property of being Lindelöf is self-dual with respect to neighborhood assignments.*

Proof: If X is dually Lindelöf, then X is linearly Lindelöf by [13, Proposition 2.7], so if X is also countably metacompact, then we can apply Theorem 2.10 of [1] to conclude that X is Lindelöf. (It is worth mentioning that, for regular spaces, the fact that every countably metacompact linearly Lindelöf space is Lindelöf was established in [12, Corollary to Theorem 1]). □

Corollary 3.9. *In the class of monotonically normal spaces (and, in particular, in the class of generalized ordered spaces), the property of being Lindelöf is self-dual with respect to neighborhood assignments.*

Proof: Every monotonically normal space is even countably paracompact [15, Theorem 2.3], so Proposition 3.8 is applicable. □

Theorem 3.10. *Any monotonically normal weakly Lindelöf space is Lindelöf.*

Proof: Assume that X is a monotonically normal weakly Lindelöf space. Any monotonically normal space is countably paracompact by [15, Theorem 2.3]; we already saw that a countably metacompact linearly Lindelöf space is Lindelöf, so it suffices to prove that X is linearly Lindelöf.

Assume, towards a contradiction, that X is not linearly Lindelöf. Then there exists a regular uncountable cardinal κ such that, for some decreasing family $\mathcal{F} = \{F_\alpha : \alpha < \kappa\}$ of non-empty closed subsets of X, we have $\bigcap \mathcal{F} = \emptyset$.

Fix an operator $H : \{(x, U) : x \in U \in \tau(X)\} \to \tau(X)$ which witnesses monotone normality of the space X and let $\alpha(x) = \min\{\beta < \kappa : x \notin F_\beta\}$ for any $x \in X$. Then $\mathcal{U} = \{H(x, X\backslash F_{\alpha(x)}) : x \in X\}$ is an open cover of the space X. Since X is weakly Lindelöf, we can find a countable set $A \subset X$ such that $G = \bigcup\{H(x, X\backslash F_{\alpha(x)}) : x \in A\}$ is dense in X. Take $\beta < \kappa$ such that $\{\alpha(x) : x \in A\} \subset \beta$; then $G \cap F_\beta = \emptyset$ and hence, $A \cap F_\beta = \emptyset$. Take a point $x \in F_\beta$ and $U \in \tau(x, X)$.

If the set U does not meet A and $V = H(x, U)$, then it follows from $x \notin X\backslash F_\beta$ and the definition of the operator H that $V \cap H(a, X\backslash F_\beta) = \emptyset$ for any $a \in A$. This implies $V \cap G = \emptyset$ which is a contradiction with the set G being dense in X.

Thus, $U \cap A \neq \emptyset$ for every $x \in F_\beta$ and $U \in \tau(x, X)$. This shows that $F_\beta \subset \overline{A}$; any monotonically normal separable space is hereditarily Lindelöf (see Theorem 1 and Theorem A of [11]), so $\{\overline{A}\backslash F_\alpha : \beta \leq \alpha < \kappa\}$ is an open cover of the Lindelöf space $\overline{A}$ which has no countable subcover. This contradiction proves that X is linearly Lindelöf. □

Corollary 3.11. *If X is a monotonically normal space (in particular, if X is a generalized ordered space), then the following conditions are equivalent:*
(1) X is Lindelöf;
(2) X is weakly Lindelöf;
(3) X is dually Lindelöf;
(4) X is dually weakly Lindelöf.

Remark 3.12. Any compact space is dually finite and hence dually hereditarily separable. Thus, any compact space with uncountable

Souslin number shows that a dually hereditarily separable space need not have the Souslin property. In particular, the Souslin property is not self-dual even in the class of compact spaces. An example of A. J. Ostaszewski [14] shows that, in some models of ZFC, there are separable (and hence weakly Lindelöf) perfectly normal spaces which are not Lindelöf. Therefore, Corollary 3.7 cannot be strengthened to deduce the Lindelöf property of a perfectly normal dually weakly Lindelöf space.

4. Weak duals with respect to neighborhood assignments

Given a topological property (or a class) $\mathcal{P}$, consider the class $\mathcal{P}'$ which consists of spaces X such that for any neighborhood assignment $\mathcal{O} = \{O_x : x \in X\}$ there exists a subspace $Y \subset X$ with $Y \in \mathcal{P}$ for which $\mathcal{O}(Y) = \bigcup\{O_x : x \in Y\}$ is dense in X. The class $\mathcal{P}'$ will be called *the weak dual of* $\mathcal{P}$ (with respect to neighborhood assignments).

The following two statements are immediate consequences of the relevant definitions.

Proposition 4.1. *The class of H-closed spaces is the weak dual of the class of finite spaces.*

Proposition 4.2. *The class of weakly Lindelöf spaces is the weak dual of the class of countable spaces.*

Theorem 4.3. *If for any neighborhood assignment $\mathcal{O} = \{O_x : x \in X\}$ of a space X there is a hereditarily Lindelöf $Y \subset X$ such that $\mathcal{O}(Y)$ is dense in X, then X is weakly Lindelöf.*

Proof: Take an open cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ of the space X and let $\xi(x) = \min\{\alpha : x \in U_\alpha\}$ for any $x \in X$. This defines a map $\xi : X \to \kappa$; let $O_x = U_{\xi(x)}$ for every $x \in X$. For the neighborhood assignment $\mathcal{O} = \{O_x : x \in X\}$, there exists a hereditarily Lindelöf subspace $Y \subset X$ such that $\mathcal{O}(Y)$ is dense in X.

If $\xi(Y) \subset \kappa$ is uncountable, then choose a set $Z = \{y_\alpha : \alpha < \omega_1\} \subset Y$ such that $\alpha < \beta < \omega_1$ implies $\xi(\alpha) < \xi(\beta)$. For any $\alpha < \omega_1$, let $Z_\alpha = \{y_\beta : \beta < \alpha\}$; it follows from the choice of the map ξ that $O_{y_\beta} \cap Z = U_{\xi(y_\beta)} \cap Z \subset Z_\alpha$ for each $\beta < \alpha$. Therefore,

$\{Z_\alpha : \alpha < \omega_1\}$ is an open cover of Z which has no countable subcover. This contradiction with the hereditary Lindelöf property of the space $Y \supset Z$ shows that $\xi(Y)$ is countable and therefore, we can take a countable $A \subset Y$ such that $\xi(A) = \xi(Y)$.

Given any point $y \in Y$, there exists a point $a \in A$ with $\xi(a) = \xi(y)$ and hence, $O_y = U_{\xi(y)} = U_{\xi(a)} = O_a$ which shows that the set $\mathcal{O}(A) = \mathcal{O}(Y)$ is dense in X, so $\mathcal{U}' = \{O_a : a \in A\}$ is a countable subfamily of $\mathcal{U}$ such that $\bigcup \mathcal{U}'$ is dense in X. Thus, X is weakly Lindelöf. □

Corollary 4.4. *The class of weakly Lindelöf spaces is the weak dual of the class of second countable spaces as well as of the class of spaces with a countable network.*

Theorem 4.5. *Suppose that, for any neighborhood assignment $\{O_x : x \in X\}$ of a space X, there is a Lindelöf subspace $Y \subset X$ such that $\mathcal{O}(Y)$ is dense in X. If additionally $l(X) \leq \omega_1$, then X is weakly Lindelöf.*

Proof: Take an open cover $\mathcal{U}$ of the space X. We can assume, without loss of generality, that $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$; let $\xi(x) = \min\{\alpha : x \in U_\alpha\}$ and $O_x = U_{\xi(x)}$ for every $x \in X$. For the neighborhood assignment $\mathcal{O} = \{O_x : x \in X\}$, there is a Lindelöf subspace $Y \subset X$ such that $\mathcal{O}(Y)$ is dense in X.

Let $A \subset Y$ be a countable set such that $Y \subset \mathcal{O}(A)$; choose $\beta < \omega_1$ with $\xi(a) < \beta$ for any $a \in A$. Given a point $y \in Y$, there is $a \in A$ such that $y \in O_a = U_{\xi(a)}$ and therefore, $\xi(y) \leq \xi(a) < \beta$. This shows that $\mathcal{O}(Y) \subset \bigcup\{U_\alpha : \alpha < \beta\}$ and therefore, for the countable family $\mathcal{U}' = \{U_\alpha : \alpha < \beta\} \subset \mathcal{U}$, the set $\bigcup \mathcal{U}'$ is dense in X. □

Proposition 4.6. *Feeble compactness is weakly self-dual in the class of regular spaces.*

Proof: If X is not feebly compact, then there is a locally finite disjoint family $\mathcal{U} = \{U_n : n \in \omega\}$ of non-empty open subsets of X. By regularity of X, we can choose a non-empty open set V_n such that $\overline{V}_n \subset U_n$ for every $n \in \omega$.

If $x \in X \setminus \bigcup \mathcal{U}$, then let $O_x = X \setminus (\bigcup\{\overline{V}_n : n \in \omega\})$; if $x \in \bigcup \mathcal{U}$, then let $O_x = U_n$ where $n \in \omega$ is the unique number with $x \in U_n$. For the neighborhood assignment $\mathcal{O} = \{O_x : x \in X\}$, suppose that $Y \subset X$ and $\mathcal{O}(Y)$ is dense in X. Then $Y \cap U_n \neq \emptyset$ for any $n \in \omega$

and hence, the family $\{U_n \cap Y : n \in \omega\}$ witnesses that Y is not feebly compact. □

Corollary 4.7. *Pseudocompactness is weakly self-dual in the class of Tychonoff spaces.*

Theorem 4.8. *Suppose that X is Hausdorff and, for every neighborhood assignment $\mathcal{O} = \{O_x : x \in X\}$ of the space X, there is a countably compact $Y \subset X$ such that $\mathcal{O}(Y)$ is dense in X. Then X is feebly compact.*

Proof: If X is not feebly compact then there is a disjoint locally finite family $\mathcal{U} = \{U_n : n \in \omega\}$ of non-empty open subsets of X. Let $W_n = X \setminus \overline{\bigcup\{U_m : m > n\}}$ for any $n \in \omega$. The family $\mathcal{U}$ being locally finite, we have $\bigcup\{W_n : n \in \omega\} = X$. For each $x \in X$, let $\xi(x) = \min\{n \in \omega : x \in W_n\}$ and $O_x = W_{\xi(x)}$.

For the neighborhood assignment $\mathcal{O} = \{O_x : x \in X\}$, there exists a countably compact $Y \subset X$ such that $\mathcal{O}(Y)$ is dense in X. There is $m \in \omega$ with $Y \subset W_m$ and hence, $\xi(y) \leq m$ for any $y \in Y$. As a consequence, $\mathcal{O}(Y) \subset W_m$ and hence, the set W_m is dense in X which is a contradiction with $W_m \cap U_{m+1} = \emptyset$. Therefore, X is feebly compact. □

Theorem 4.9. *There exists a Tychonoff space X which is not countably compact; while for any neighborhood assignment $\{O_x : x \in X\}$, there is a compact $K \subset X$ such that $\mathcal{O}(K)$ is dense in X. In other words, the weak dual of the class of compact spaces contains spaces which are not countably compact.*

Proof: Define a point $a \in \mathbb{D}^{\omega_1}$ by $a(\alpha) = 1$ for all ordinals $\alpha < \omega_1$. It is easy to see that the space $X = \mathbb{D}^{\omega_1} \setminus \{a\}$ is not countably compact. The Σ-product $\Sigma = \{x \in \mathbb{D}^{\omega_1} : |x^{-1}(1)| \leq \omega\}$ is dense in X. It follows from the Souslin property of Σ that, for any neighborhood assignment $\{O_x : x \in X\}$, there is a countable $A \subset \Sigma$ such that $\mathcal{O}(A) = \bigcup\{O_x : x \in A\}$ is dense in X. The set $K = \overline{A}$ is compact and $\mathcal{O}(K) \supset \mathcal{O}(A)$ is dense in X. □

Theorem 4.10. *There exists a Hausdorff space X which is neither feebly compact nor weakly Lindelöf, while for any neighborhood assignment $\mathcal{O} = \{O_x : x \in X\}$, there is an H-closed (and hence feebly compact) subspace $Y \subset X$ with $\mathcal{O}(Y)$ dense in X.*

Proof: Consider the set $I = [0,1] \subset \mathbb{R}$ and denote by ν the topology of I induced from $\mathbb{R}$. We will also need the sets $Q = \mathbb{Q} \cap I$ and $P = I \backslash Q$. Denote by τ the topology generated by $\nu \cup \{Q\}$; it is standard that the space (I, τ) is H-closed. It is straightforward that $P \subset U \in \tau$ implies that $U \cap Q$ is dense in Q.

Let $T = (I \times \{0\}) \cup (Q \times \{1\})$; to introduce a topology μ on T, we declare that both sets $Q_0 = Q \times \{0\}$ and $Q_1 = Q \times \{1\}$ belong to μ and every Q_i carries the Euclidean topology induced from $\mathbb{R} \times \mathbb{R}$. If $x = (t, 0) \in P_0 = P \times \{0\}$, then the local base at x in (T, μ) is the family $\{(V \times \{0,1\}) \cap T : t \in V \in \nu\}$.

Let D be a discrete space with $|D| = \omega_1$ and consider the set $X = (T \times D) \cup \{a\}$ where $a = (0,1) \in \mathbb{R} \times \mathbb{R}$. Then X is the underlying set of our promised space. The set $T_d = T \times \{d\}$ is open in X for each $d \in D$ and the topology on T_d is the copy of the topology of T; i.e., $\{U \times \{d\} : U \in \mu\}$ is the family of all open subsets of T_d. The local base at the point a is the family $\{\{a\} \cup (Q_0 \times (D \backslash E)) : E$ is a finite subset of $D\}$.

Let $I_0 = I \times \{0\}$; we omit an easy proof of the fact that $K = (I_0 \times D) \cup \{a\}$ is an H-closed subspace of X; besides, if $K \subset U \in \tau(X)$, then $U \cap (Q_1 \times D)$ is dense in $Q_1 \times D$ and hence, U is dense in X. This implies that, for any neighborhood assignment $\mathcal{O} = \{O_x : x \in X\}$, the set $\mathcal{O}(K)$ is dense in X, so X belongs to the weak dual of the class of H-closed spaces.

Finally, observe that the space X is not feebly compact because the family $\{Q_1 \times \{d\} : d \in D\} \subset \tau^*(X)$ is infinite (even uncountable) and discrete. Besides, the open cover $\mathcal{U} = \{(I_0 \times D) \cup \{a\}\} \cup \{Q_1 \times \{d\} : d \in D\}$ of the space X has no countable subfamily whose union is dense in X, so X is not weakly Lindelöf. □

5. Open problems

The topic of this paper being new, the unsolved problems are more numerous than the solved ones. We give below the list of questions we could not answer while working on this paper.

Problem 5.1. *Must every linearly Lindelöf space be dually Lindelöf?*

Problem 5.2. *Must every dually Lindelöf space be dually Lindelöf Σ? Must it be dually σ-compact?*

Problem 5.3. *Suppose that a space* X *is dually Lindelöf* Σ*. Must* X *be dually* σ*-compact? What if we assume that* X *is dually* K*-analytic?*

Problem 5.4. *Must a first-countable dually* σ*-compact space* X *be Lindelöf? What if we assume additionally that* X *is locally compact?*

Problem 5.5. *Must every dually Lindelöf space be weakly Lindelöf?*

Problem 5.6. *Must every dually hereditarily separable space be weakly Lindelöf? What happens if we omit "hereditarily"?*

Problem 5.7. *Must every dually* σ*-compact space* X *be weakly Lindelöf? What happens if we assume* X *to be dually Lindelöf* Σ *or dually* K*-analytic?*

Problem 5.8. *Must every dually weakly Lindelöf space be weakly Lindelöf?*

Problem 5.9. *Suppose that for any neighborhood assignment* $\mathcal{O} = \{O_x : x \in X\}$ *of a space* X*, there is a compact* $Y \subset X$ *such that* $\mathcal{O}(Y)$ *is dense in* X*, i.e.,* X *belongs to the weak dual of the class of compact spaces. Must* X *be weakly Lindelöf?*

Problem 5.10. *Suppose that* X *is a first countable space which belongs to the weak dual of the class of compact spaces. Is it true that* $|X| \leq \mathfrak{c}$*?*

Problem 5.11. *Suppose that* X *is a first countable space which belongs to the weak dual of the class of Lindelöf spaces. Is it true that* $|X| \leq \mathfrak{c}$*?*

Problem 5.12. *Suppose that* X *is a normal first countable space which belongs to the weak dual of the class of weakly Lindelöf spaces. Is it true that* $|X| \leq \mathfrak{c}$*?*

Problem 5.13. *Suppose that* X *is a first countable space and for any neighborhood assignment* $\mathcal{O} = \{O_x : x \in X\}$ *there is a set* $Y \subset X$ *such that* $s(Y) \leq \omega$ *and* $\mathcal{O}(Y) = X$*. Is it true that* $|X| \leq \mathfrak{c}$*?*

Problem 5.14. *Suppose that* X *is a first countable space and for any neighborhood assignment* $\mathcal{O} = \{O_x : x \in X\}$ *there is a set* $Y \subset X$ *such that* $s(Y) \leq \omega$ *and* $\mathcal{O}(Y)$ *is dense in* X*. Is it true that* $|X| \leq \mathfrak{c}$*?*

Problem 5.15. *What happens if we iterate the duality? For example, must any dually dually Lindelöf space be dually Lindelöf? Is it true that 2006 times dually Lindelöf space is the same as 2005 times dually Lindelöf space? What about dually dually σ-compact spaces?*

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