ON THE UNIQUENESS OF THE HYPERSPACES $2^X$ AND $C_n(X)$ OF RIM-METRIZABLE CONTINUA

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ABSTRACT. A continuum $X$ has unique hyperspace $\Gamma_X \in \{2^X, C_n(X)\}$ provided that if $Y$ is a continuum and $\Gamma_X$ is homeomorphic to $\Gamma_Y$, then $X$ is homeomorphic to $Y$. In [4], I. Lončar proved that, in the realm of rim-metrizable continua, the following classes of spaces have unique hyperspace $C_1(X)$: hereditarily indecomposable continua, smooth fans and indecomposable continua whose proper and non-degenerate subcontinua are arcs. In this paper, we prove that every rim-metrizable hereditarily indecomposable continuum has unique hyperspace $\Gamma_X \in \{2^X, C_n(X)\}$.

1. INTRODUCTION

In [7], Professor S. Nadler Jr. proved that hereditarily indecomposable metric continua have unique hyperspace $C(X)$, then Professor S. Macías proved that those continua have unique hyperspaces $2^X$ and $C_n(X)$ (see [5, p. 416] and [6, 6.1], respectively). Later, Professor I. Lončar proved that rim-metrizable hereditarily indecomposable continua have unique hyperspace $C(X)$ (see [4, Theorem 2.4]). In this paper, we prove that rim-metrizable hereditarily indecomposable continua have unique hyperspaces $2^X$ and $C_n(X)$. The paper is divided into 2 sections. In section 2, we give the definitions and notation for understanding the paper. In section 3, we present the main result of the paper.

2000 Mathematics Subject Classification. 54B20.

Key words and phrases. Continuum, hereditarily indecomposable continuum, hyperspace, rim-metrizable continuum, unique hyperspace.

* This research is part of the author’s Dissertation at the Universidad Nacional Autónoma de México. This research has been done with the support of a scholarship given by CONACyT.
2. Definitions and notation

By a space we mean a topological space. The closed interval \([0, 1]\) is denoted by \(I\). A Hilbert cube is a space homeomorphic to \(\prod \{I_n : n \in \mathbb{N}\}\), where each \(I_n = I\). By a map we mean a continuous function. A map \(f : X \to Y\) between spaces is monotone provided that all fibers \(f^{-1}(y)\) are connected. The weight of a space \(X\) is denoted by \(\omega(X)\). A continuum is a non-empty Hausdorff compact connected space. A subcontinuum is a continuum contained in a space. A continuum \(X\) is decomposable provided that \(X = A \cup B\), where \(A\) and \(B\) are proper subcontinua of \(X\). A continuum is indecomposable if it is not decomposable. A continuum is hereditarily indecomposable provided that each subcontinuum of it is indecomposable. The symbol \(\mathbb{N}\) denotes the set of the positive integers.

Given a Hausdorff compact space \(X\), we denote by \(2^X\) the family of all non-empty closed subsets of \(X\). Given \(n \in \mathbb{N}\), we denote by \(C_n(X)\) the family of all non-empty closed subsets of \(X\) having at most \(n\) components and by \(\mathcal{F}_n(X)\) the family of all non-empty closed subsets of \(X\) having at most \(n\) points. The topology on \(2^X\) is the Vietoris Topology (see [2, 2.7.20. (a)]) and the spaces \(C_n(X)\) and \(\mathcal{F}_n(X)\) are considered as subspaces of \(2^X\). The spaces \(2^X\), \(C_n(X)\) and \(\mathcal{F}_n(X)\) are called hyperspaces of \(X\). Note that the hyperspace \(\mathcal{F}_1(X)\) is homeomorphic to \(X\).

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Given a map \(f : X \to Y\) between Hausdorff compact spaces, we define the function \(2^f : 2^X \to 2^Y\) by \(2^f(E) = f[E]\) for \(E \in 2^X\). By [2, 3.12.27. (e)], the function \(2^f\) is continuous. Note that \(2^f[C_n(X)] \subseteq C_n(Y)\) and \(2^f[\mathcal{F}_n(X)] \subseteq \mathcal{F}_n(Y)\). The restriction \(2^f|_{C_n(X)}\) is denoted by \(C_n(f)\) and the restriction \(2^f|_{\mathcal{F}_n(X)}\) is denoted by \(\mathcal{F}_n(f)\).

From [2, 3.12.27. (a) and 3.12.27. (b)] we have the following result:

**Theorem 2.1.** If \(X\) is a Hausdorff compact space, then the hyperspace \(2^X\) is a Hausdorff compact space and \(\omega(2^X) = \omega(X)\).

By [3, 14.9 and 15.12], we obtain:

**Theorem 2.2.** If \(X\) is a metrizable continuum, then the hyperspace \(2^X\) and each of the hyperspaces \(C_n(X)\) and \(\mathcal{F}_n(X)\) are metrizable continua.
An inverse system is a family $S = \{X_\alpha, f^\beta_\alpha, \Lambda\}$, where $(\Lambda, \leq)$ is a directed set, $X_\alpha$ is a space for every $\alpha \in \Lambda$, and for any $\alpha, \beta \in \Lambda$ satisfying $\alpha \leq \beta$, $f^\beta_\alpha : X_\beta \to X_\alpha$ is a map such that:

i) $f^\alpha_\alpha$ is the identity map on $X_\alpha$ for every $\alpha \in \Lambda$ and

ii) $f^\gamma_\alpha = f^\beta_\alpha \circ f^{\gamma}_\beta$ for any $\alpha, \beta, \gamma \in \Lambda$ satisfying $\alpha \leq \beta \leq \gamma$.

The maps $f^\beta_\alpha$ are called bonding maps and the spaces $X_\alpha$ are called coordinate spaces. A subset $\Sigma$ of a directed set $\Lambda$ is cofinal provided that for every $\alpha \in \Lambda$ there exists $\beta \in \Sigma$ such that $\alpha \leq \beta$.

Given a point $\hat{x}$ in a product $\prod \{X_\alpha : \alpha \in \Lambda\}$, we write $\hat{x} = (x_\alpha)_{\alpha \in \Lambda}$.

Let $S = \{X_\alpha, f^\beta_\alpha, \Lambda\}$ be an inverse system. The subspace of the product $\prod \{X_\alpha : \alpha \in \Lambda\}$ consisting of all points $\hat{x}$ such that $x_\alpha = f^\beta_\alpha(x_\beta)$ for any $\alpha, \beta \in \Lambda$ satisfying $\alpha \leq \beta$ is called the inverse limit of the inverse system $S$, which is denoted by $\text{Lim}_{\leftarrow} S$ or by $X^\Lambda$.

We define the projection map $f^\Lambda_\alpha : X^\Lambda \to X_\alpha$ by $f^\Lambda_\alpha(\hat{x}) = x_\alpha$.

The following result is well known but we include it for the convenience of the reader since the proof is short.

**Theorem 2.3.** Let $f : X \to Y$ be an onto monotone map between continua. If $X$ is an indecomposable continuum (hereditarily indecomposable continuum), then $Y$ is an indecomposable continuum (hereditarily indecomposable continuum).

**Proof.** Suppose $Y$ is decomposable. Let $E$ and $F$ be two proper subcontinua of $Y$ such that $Y = E \cup F$. By [2, 6.1.29], the sets $f^{-1}[E]$ and $f^{-1}[F]$ are connected, then they are continua. Since $X = f^{-1}[E] \cup f^{-1}[F]$ and $f^{-1}[E]$ and $f^{-1}[F]$ are proper subcontinua of $X$ we conclude that $X$ is decomposable.

If $X$ is hereditarily indecomposable and $Z$ is a subcontinuum of $Y$, then, by [2, 6.1.29], we deduce that the set $f^{-1}[Z]$ is a continuum. Since the map $f \mid_{f^{-1}[Z]} : f^{-1}[Z] \to Z$ is monotone, by the first part of this Theorem, we conclude that $Z$ is indecomposable. Therefore, $Y$ is hereditarily indecomposable. \qed

**Notation 2.4.** Given an inverse system $S = \{X_\alpha, f^\beta_\alpha, \Lambda\}$ of Hausdorff compact spaces, let $2^S = \{2^{X_\alpha}, 2^{f^\beta_\alpha}, \Lambda\}$, let $C_n(\Lambda) = \{C_n(X_\alpha), C_n(f^\beta_\alpha), \Lambda\}$ and let $\mathcal{F}_n(S) = \{\mathcal{F}_n(X_\alpha), \mathcal{F}_n(f^\beta_\alpha), \Lambda\}$. 
Theorem 2.5. Let $S = \{X_\alpha, f^\beta_\alpha, \Lambda\}$ be an inverse system of Hausdorff compact spaces. Then the families $2^S$, $C_n(S)$, and $F_n(S)$ are inverse systems and the map $h : 2^{\lim_S} \to \lim 2^S$ defined by $h(E) = (f^\Lambda_\alpha[E])_{\alpha \in \Lambda}$ is a homeomorphism. Moreover, $h \left[ C_n \left( \lim_S \right) \right] = \lim C_n(S)$ and $h \left[ F_n \left( \lim_S \right) \right] = \lim F_n(S)$.

Proof. It is not difficult to see that the families $2^S$, $C_n(S)$, and $F_n(S)$ are inverse systems. Given $\alpha \in \Lambda$, let $q_\alpha$ be the projection map from $\lim 2^S$ into $2^{X_\alpha}$. Note that, by [2, 3.2.13], $\lim S$ is a Hausdorff compact space. Then, by 2.1, the hyperspace $2^{X_\alpha}$ is a Hausdorff compact space for every $\alpha \in \Lambda \cup \{\Lambda\}$.

Since $2^{f^\Lambda_\alpha} = 2^{f^\beta_\alpha} \circ 2^{f^\beta_\beta}$ for all $\alpha, \beta \in \Lambda$ such that $\alpha \leq \beta$, by [2, 2.5.F], the family $\{2^{f^\Lambda_\alpha} : \alpha \in \Lambda\}$ induces a map $f : 2^{\lim S} \to \lim 2^S$ such that $2^{f^\Lambda_\alpha} = q_\alpha \circ f$ for each $\alpha \in \Lambda$. Then $h = f$.

Now, we define the inverse function of $h$. Given $(E_\alpha)_{\alpha \in \Lambda} \in \lim 2^S$, we have that $E_\alpha = 2^{f^\beta_\alpha}(E_\beta) = f^\beta_\alpha[E_\beta]$. Then, by [2, 3.2.13], the space $E_\Lambda = \lim \{E_\alpha, f^\beta_\alpha|_{E_\beta}, \Lambda\}$ is a non-empty compact space and, by [2, 3.2.15], $E_\alpha = f^\Lambda_\alpha[E_\Lambda]$ for each $\alpha \in \Lambda$. Since $E_\Lambda$ is contained in $X_\Lambda$, we can define $h'(\{(E_\alpha)_{\alpha \in \Lambda}\}) = E_\Lambda$. Moreover, every $E \in 2^{\lim S}$ can be written as $E = \lim \{f^\Lambda_\alpha[E], f^\beta_\alpha|_{f^\beta_\beta[E]}, \Lambda\}$, by [2, 2.5.6]. Then $h'$ is the inverse function of $h$. Thus, the map $h$ is a homeomorphism since the space $2^{\lim S}$ is compact.

In order to see that $h \left[ C_n \left( \lim S \right) \right] = \lim C_n(S)$, let $(E_\alpha)_{\alpha \in \Lambda} \in \lim C_n(S)$. By [8, Lemma 1], we have that $h'(\{(E_\alpha)_{\alpha \in \Lambda}\}) = E_\Lambda \in C_n \left( \lim S \right)$. The other inclusion is clear. In a similar way we get $h \left[ F_n \left( \lim S \right) \right] = \lim F_n(S)$. □

3. Uniqueness of Hyperspaces $2^X$ and $C_n(X)$

Our main result is Theorem 3.20, in which we prove that hereditarily indecomposable rim-metrizable continua $X$ have unique hyperspaces $2^X$ and $C_n(X)$. We begin with a couple of definitions and continue with all the required results to obtain our main Theorem.
**Definition 3.1.** A subset $\Sigma$ of a directed set $\Lambda$ is a *chain* provided that for any $\alpha, \beta \in \Sigma$ we have that $\alpha \leq \beta$ or $\beta \leq \alpha$. A directed set $\Lambda$ is called *$\sigma$-complete* provided that for every sequence $\{\alpha_n : n \in \mathbb{N}\}$ in $\Lambda$ there exists $\sup\{\alpha_n : n \in \mathbb{N}\} \in \Lambda$.

Let $S = \{X_\alpha, f^\beta_\alpha, \Lambda\}$ be an inverse system and let $\Sigma \subseteq \Lambda$ be a chain with $\gamma = \sup\Sigma \in \Lambda$. By [2, 2.5.F], the family $\{f^\gamma_\alpha : \alpha \in \Sigma\}$ induces a map $h_\gamma : X_\gamma \to \lim\{X_\alpha, f^\beta_\alpha, \Sigma\}$ such that $f^\gamma_\alpha = f^\Sigma_\alpha \circ h_\gamma$ for every $\alpha \in \Sigma$. Note that $h_\gamma$ is defined by $h_\gamma(x_\gamma) = (f^\gamma_\alpha(x_\gamma))_{\alpha \in \Sigma}$.

**Definition 3.2.** An inverse system $\{X_\alpha, f^\beta_\alpha, \Lambda\}$ is *continuous* provided that for each chain $\Sigma \subseteq \Lambda$, with $\gamma = \sup\Sigma \in \Lambda$, the induced map, $h_\gamma : X_\gamma \to \lim\{X_\alpha, f^\beta_\alpha, \Sigma\}$, by the family $\{f^\gamma_\alpha : \alpha \in \Sigma\}$ is a homeomorphism.

**Theorem 3.3.** Let $S = \{X_\alpha, f^\beta_\alpha, \Lambda\}$ be a continuous inverse system of Hausdorff compact spaces. If $A$ is a closed subset of $X_\Lambda$, then the inverse system $\{f^A_\alpha \in \Lambda, f^\beta_\alpha | f^A_\alpha, \Lambda\}$ is continuous.

**Proof.** Let $\Sigma$ be a chain contained in $\Lambda$, with $\gamma = \sup\Sigma \in \Lambda$, and let $h_\gamma : X_\gamma \to \lim\{X_\alpha, f^\beta_\alpha, \Sigma\}$ be the induced map by the family $\{f^\gamma_\alpha : \alpha \in \Sigma\}$. Since $S$ is continuous, the map $h_\gamma$ is a homeomorphism.

Note that, by [2, 3.2.13], $X_\Lambda$ is a Hausdorff compact space. Then, by [2, 2.5.6], we have that:

$$h_\gamma [f^A_\alpha | A] = \lim \left\{ f^\Sigma_\alpha [h_\gamma [f^A_\alpha | A]], f^\beta_\alpha | f^\Sigma_\alpha [h_\gamma [f^A_\alpha | A]], \Sigma \right\}$$

$$= \lim \left\{ f^\gamma_\alpha [f^A_\alpha | A], f^\beta_\alpha | f^\gamma_\alpha [f^A_\alpha | A], \Sigma \right\}$$

$$= \lim \left\{ f^A_\alpha [A], f^\beta_\alpha | f^A_\alpha [A], \Sigma \right\}.$$

Note that the homeomorphism

$$h_\gamma | f^A_\alpha [A] : f^A_\alpha [A] \to \lim \left\{ f^A_\alpha [A], f^\beta_\alpha | f^A_\alpha [A], \Sigma \right\}$$

satisfies $f^\gamma_\alpha | f^A_\alpha [A] = f^\Sigma_\alpha \circ h_\gamma | f^A_\alpha [A]$ for each $\alpha \in \Sigma$. Then, by [2, 2.5.F], the map $h_\gamma | f^A_\alpha [A]$ is the induced map by $\{f^\gamma_\alpha | f^A_\alpha [A] : \alpha \in \Sigma\}$. Hence, the inverse system $\{f^A_\alpha [A], f^\beta_\alpha | f^A_\alpha [A], \Lambda\}$ is continuous. \qed
The next theorem tells us that the induced hyperspace inverse system of a continuous inverse system is continuous.

**Theorem 3.4.** Let \( S = \{X_\alpha, f^\beta_\alpha, \Lambda\} \) be a continuous inverse system of Hausdorff compact spaces. Then the inverse systems \( 2^S = \{2^{X_\alpha}, 2^{f^\beta_\alpha}, \Lambda\} \) and \( C_n(S) = \{C_n(X_\alpha), C_n(f^\beta_\alpha), \Lambda\} \) are continuous.

**Proof.** Let \( \Sigma \) be a chain contained in \( \Lambda \), with \( \gamma = \sup \Sigma \in \Lambda \), and let \( h_\gamma : X_\gamma \to \lim_{\leftarrow} \{X_\alpha, f^\beta_\alpha, \Sigma\} \) be the induced map by \( \{f^\gamma_\alpha : \alpha \in \Sigma\} \) (\( f^\gamma_\alpha = f^\Sigma_\alpha \circ h_\gamma \)).

Since \( S \) is continuous, the map \( h_\gamma \) is a homeomorphism. Moreover, \( 2f^\gamma_\alpha = 2f^\Sigma_\alpha \circ 2h_\gamma \) since \( f^\gamma_\alpha = f^\Sigma_\alpha \circ h_\gamma \).

Let \( q^\Sigma_\alpha \) denote the projection map from \( \lim_{\leftarrow} \{2^{X_\alpha}, 2^{f^\beta_\alpha}, \Sigma\} \) into \( 2^{X_\alpha} \).

By 2.5, the map \( h : 2^{X_\Sigma} \to \lim_{\leftarrow} \{2^{X_\alpha}, 2^{f^\beta_\alpha}, \Sigma\} \) defined by \( h(E) = (f^\Sigma_\alpha(E))_{\alpha \in \Sigma} \) is a homeomorphism. Since \( 2f^\Sigma_\alpha = q^\Sigma_\alpha \circ h \) for every \( \alpha \in \Sigma \), we have that \( 2f^\gamma_\alpha = 2f^\Sigma_\alpha \circ 2h_\gamma = q^\Sigma_\alpha \circ h \circ 2h_\gamma \) for each \( \alpha \in \Sigma \). Then, by [2, 2.5.F], the homeomorphism \( h \circ 2h_\gamma : 2^{X_\gamma} \to \lim_{\leftarrow} \{2^{X_\alpha}, 2^{f^\beta_\alpha}, \Sigma\} \) is the induced map by the family \( \{2f^\gamma_\alpha : \alpha \in \Sigma\} \). Hence the inverse system \( 2^S \) is continuous.

In a similar way, we can prove that the inverse system \( C_n(S) \) is continuous.

**Definition 3.5.** An inverse system \( S = \{X_\alpha, f^\beta_\alpha, \Lambda\} \) is \( \sigma \)-complete if \( S \) is continuous and \( \Lambda \) is \( \sigma \)-complete.

From 3.4 and the definition of a \( \sigma \)-complete inverse system, we obtain the following result:

**Theorem 3.6.** Let \( S = \{X_\alpha, f^\beta_\alpha, \Lambda\} \) be an inverse system of Hausdorff compact spaces. If \( S \) is \( \sigma \)-complete, then the inverse systems \( 2^S \) and \( C_n(S) \) are \( \sigma \)-complete.

**Definition 3.7.** An inverse system \( S = \{X_\alpha, f^\beta_\alpha, \Lambda\} \) is an inverse \( \sigma \)-system if \( S \) is \( \sigma \)-complete and \( \omega(X_\alpha) \leq \aleph_0 \) for each \( \alpha \in \Lambda \).

**Theorem 3.8.** Let \( S = \{X_\alpha, f^\beta_\alpha, \Lambda\} \) be an inverse \( \sigma \)-system of Hausdorff compact spaces. Then the inverse systems \( 2^S = \{2^{X_\alpha}, 2^{f^\beta_\alpha}, \Lambda\} \) and \( C_n(S) = \{C_n(X_\alpha), C_n(f^\beta_\alpha), \Lambda\} \) are inverse \( \sigma \)-systems.
Proof. By 2.1, we have that \( \omega(2^{X_\alpha}) = \omega(X_\alpha) \leq \aleph_0 \). Then, by 3.6, we deduce that \( 2^S \) and \( C_\alpha(S) \) are inverse \( \sigma \)-systems.

From [9, Theorem 15], we have the following result:

**Theorem 3.9.** Let \( \{X_\alpha, f_\alpha^\beta, \Lambda\} \) and \( \{Y_\alpha, g_\alpha^\beta, \Lambda\} \) be two inverse \( \sigma \)-systems of Hausdorff compact spaces with onto bonding maps. If \( l : X_\Lambda \to Y_\Lambda \) is a map, then there exist a cofinal subset \( \Sigma \) of \( \Lambda \) and maps \( l_\alpha : X_\alpha \to Y_\alpha \) for every \( \alpha \in \Sigma \), such that \( l_\alpha \circ f_\alpha^\beta = g_\alpha^\beta \circ l \) and \( l_\alpha \circ f_\alpha^\beta = g_\alpha^\beta \circ l_\beta \) for any \( \alpha, \beta \in \Sigma \) satisfying \( \alpha \leq \beta \). Moreover, if \( l : X_\Lambda \to Y_\Lambda \) is a homeomorphism, then each \( l_\alpha \) is a homeomorphism.

**Remark 3.10.** Let \( \{X_\alpha, f_\alpha^\beta, \Lambda\} \) and \( \{Y_\alpha, g_\alpha^\beta, \Lambda\} \) be a pair of inverse \( \sigma \)-systems of Hausdorff compact spaces with onto bonding maps and let \( l : X_\Lambda \to Y_\Lambda \) be a map. Let \( \Sigma \) be the cofinal subset of \( \Lambda \) and let \( l_\alpha : X_\alpha \to Y_\alpha \) be the maps satisfying 3.9.

Let \( X_\Sigma = \lim\{X_\alpha, f_\alpha^\beta, \Sigma\} \) and let \( Y_\Sigma = \lim\{Y_\alpha, g_\alpha^\beta, \Sigma\} \). Define the maps \( g : X_\Lambda \to X_\Sigma \) and \( g' : Y_\Lambda \to Y_\Sigma \) by \( g((x_\alpha)_{\alpha \in \Lambda}) = (x_\alpha)_{\alpha \in \Sigma} \) and \( g'((y_\alpha)_{\alpha \in \Lambda}) = (y_\alpha)_{\alpha \in \Sigma} \). By [2, 2.5.11], the maps \( g \) and \( g' \) are homeomorphisms. Note that \( f_\alpha^\Lambda = f_\alpha^\Sigma \circ g \) and \( g_\alpha^\Lambda = g_\alpha^\Sigma \circ g' \) for each \( \alpha \in \Sigma \).

The family \( \{l_\alpha : \alpha \in \Sigma\} \) induces a map \( l_\Sigma : X_\Sigma \to Y_\Sigma \) such that \( l_\alpha \circ f_\alpha^\Sigma = g_\alpha^\Sigma \circ l_\Sigma \).

Since \( g_\alpha^\Sigma \circ g' \circ l \circ g^{-1} = g_\alpha^\Lambda \circ l \circ g^{-1} = l_\alpha \circ f_\alpha^\Lambda \circ g^{-1} = l_\alpha \circ f_\alpha^\Sigma \), we deduce that \( l_\Sigma = g' \circ l \circ g^{-1} \).

**Theorem 3.11.** Let \( \{X_\alpha, f_\alpha^\beta, \Lambda\} \) and \( \{Y_\alpha, g_\alpha^\beta, \Lambda\} \) be a pair of inverse \( \sigma \)-systems of metrizable continua with onto bonding maps. If each \( f_\alpha^\beta \) is monotone and \( C_\alpha(X_\Lambda) \) is homeomorphic to \( C_\alpha(Y_\Lambda) \), then there exists a cofinal subset \( \Sigma \) of \( \Lambda \) such that the maps \( g_\alpha^\beta \) are monotone for any \( \alpha, \beta \in \Sigma \) satisfying \( \alpha \leq \beta \).

**Proof.** By 2.5, the hyperspace \( C_\alpha(X_\Lambda) \) is homeomorphic to the space \( X = \lim\{C_\alpha(X_\alpha), C_\alpha(f_\alpha^\beta), \Lambda\} \) and the hyperspace \( C_\alpha(Y_\Lambda) \) is homeomorphic to the space \( Y = \lim\{C_\alpha(Y_\alpha), C_\alpha(g_\alpha^\beta), \Lambda\} \). Let \( l : X \to Y \) be a homeomorphism.

By 2.2, the hyperspaces \( C_\alpha(X_\alpha) \) and \( C_\alpha(Y_\alpha) \) are metrizable continua. Then, by [2, 6.1.20], \( X \) and \( Y \) are continua. By 3.8, the inverse systems \( \{C_\alpha(X_\alpha), C_\alpha(f_\alpha^\beta), \Lambda\} \) and \( \{C_\alpha(Y_\alpha), C_\alpha(g_\alpha^\beta), \Lambda\} \) are inverse \( \sigma \)-systems.
Given $\alpha \in \Lambda$, let $p^\Lambda_\alpha$ be the projection map from $X$ into $C_n(X_\alpha)$, let $q^\Lambda_\alpha$ be the projection map from $Y$ into $C_n(Y_\alpha)$ and let $Z_\alpha = q^\Lambda_\alpha[Y]$. Then, by [2, 2.5.6], $Y = \lim\downarrow \{Z_\alpha, C_n(g^\beta_\alpha)_{|Z_\beta}, \Lambda\}$.

By 3.3, the inverse system $\{Z_\alpha, C_n(g^\beta_\alpha)_{|Z_\beta}, \Lambda\}$ is continuous, then it is an inverse $\sigma$-system with onto bonding maps. Moreover, each $C_n(f^\beta_\alpha)$ is onto since every $f^\beta_\alpha$ is monotone (see [1, Proposition 1]).

By 3.9, there exist a cofinal subset $\Sigma$ of $\Lambda$ and homeomorphisms $l_\alpha : C_n(X_\alpha) \to Z_\alpha$ for every $\alpha \in \Sigma$, such that $l_\alpha \circ p^\Lambda_\alpha = q^\Lambda_\alpha \circ l$ and $l_\alpha \circ C_n(f^\beta_\alpha) = C_n(g^\beta_\alpha)_{|Z_\beta} \circ l_\beta$ for any $\alpha, \beta \in \Sigma$ satisfying $\alpha \leq \beta$.

By [1, Theorem 4], the maps $C_n(f^\beta_\alpha)$ are monotone, then each map $C_n(g^\beta_\alpha)_{|Z_\beta} = l_\alpha \circ C_n(f^\beta_\alpha) \circ l_\beta^{-1}$ is monotone.

By [2, 3.2.15], the projections $g^\Lambda_\alpha$ are onto, then $F_1(Y_\alpha) \subseteq Z_\alpha$. Let $\alpha, \beta \in \Sigma$ with $\alpha \leq \beta$ and let $y_\alpha \in Y_\alpha$. By [3, 15.9 (2)], the set $\bigcup(C_n(g^\beta_\alpha)_{|Z_\beta}^{-1}(\{y_\alpha\})$ is connected. It is not difficult to see that $(g^\beta_\alpha)^{-1}(y_\alpha) = \bigcup(C_n(g^\beta_\alpha)_{|Z_\beta})^{-1}(\{y_\alpha\})$. Then $g^\beta_\alpha$ is monotone. □

Let us recall the following result due to I. Lončar

**Theorem 3.12.** [4, THEOREM 3.4] Let $X$ be a Hausdorff compact space with $\omega(X) \geq \aleph_1$. Then there exists an inverse $\sigma$-system $\{X_\alpha, f^\beta_\alpha, \Lambda\}$ such that $X$ is homeomorphic to $X_\Lambda$.

**Remark 3.13.** In the previous Theorem, the directed set $\Lambda$ only depends on $\omega(X)$ and each space $X_\alpha$ is contained in a Hilbert cube (see [4, THEOREM 3.3]). So, we can assume that the spaces $X_\alpha$ are compact and metrizable. Moreover, if two Hausdorff compact spaces have the same weight, then the inverse $\sigma$-systems satisfying 3.12, for those two spaces, can be chosen with the same directed set.

**Definition 3.14.** A space $X$ is rim-metrizable if it has a basis $\mathcal{B}$ such that every $U \in \mathcal{B}$ has metrizable boundary.

The following result is used in the proof of Theorem 3.20.
Theorem 3.15. [4, THEOREM 3.7] Let $S = \{X_\alpha, f^\beta_{\alpha}, \Lambda\}$ be an inverse system of Hausdorff compact spaces with onto bonding maps. Then:

1. There exists an inverse system $M(S) = \{M_\alpha, m^\beta_{\alpha}, \Lambda\}$ of Hausdorff compact spaces such that the bonding maps $m^\beta_{\alpha}$ are monotone surjections and the space $\lim \leftarrow S$ is homeomorphic to the space $\lim \leftarrow M(S)$,
2. If $S$ is $\sigma$-directed, then the inverse system $M(S)$ is $\sigma$-directed,
3. If $S$ is $\sigma$-complete, then the inverse system $M(S)$ is $\sigma$-complete,
4. If every $X_\alpha$ is a metric space and $\lim \leftarrow S$ is locally connected (a rim-metrizable continuum) then every $M_\alpha$ is metrizable.

Definition 3.16. Let $X$ be a continuum and let $\Gamma_X \in \{2^X, \mathcal{C}_n(X)\}$, where $n \in \mathbb{N}$. The continuum $X$ has unique hyperspace $\Gamma_X$ provided that:

- if $Y$ is a continuum and $\Gamma_Y$ is homeomorphic to $\Gamma_X$, then $Y$ is homeomorphic to $X$.

In the previous definition, the hyperspace $\Gamma_Y = 2^Y$ when $\Gamma_X = 2^X$ and $\Gamma_Y = \mathcal{C}_n(Y)$ when $\Gamma_X = \mathcal{C}_n(X)$.

Given a map $f : X \to Y$ between Hausdorff compact spaces, let $\Gamma_f$ denote the induced map between the hyperspaces $\Gamma_X$ and $\Gamma_Y$.

From [7, (0.60) and (1.61)] we have:

Theorem 3.17. Hereditarily indecomposable metrizable continua have unique hyperspace $\mathcal{C}_1(X)$. In fact, if $f : \mathcal{C}_1(X) \to \mathcal{C}_1(Y)$ is a homeomorphism, where $X$ is a hereditarily indecomposable metrizable continuum and $Y$ is a metrizable continuum, then $f[\mathcal{F}_1(X)] = \mathcal{F}_1(Y)$.

From [5, p. 416] and [6, 6.1] we obtain:

Theorem 3.18. Hereditarily indecomposable metrizable continua have unique hyperspace $\Gamma_X$, where $\Gamma_X \in \{2^X, \mathcal{C}_n(X)\}$ and $n \in \mathbb{N}$. In fact, if $f : \Gamma_X \to \Gamma_Y$ is a homeomorphism, where $X$ is a hereditarily indecomposable metrizable continuum and $Y$ is a metrizable continuum, then $f[\mathcal{F}_1(X)] = \mathcal{F}_1(Y)$. 
Theorem 3.19. [4, Theorem 2.4] Hereditarily indecomposable rim-metrizable continua have unique hyperspace $\mathcal{C}_1(X)$, i.e., if $X$ is a hereditarily indecomposable non-metric rim-metrizable continuum and $Y$ is a continuum such that $\mathcal{C}_1(X)$ is homeomorphic to $\mathcal{C}_1(Y)$, then $X$ is homeomorphic to $Y$. In fact, if $f : \mathcal{C}_1(X) \rightarrow \mathcal{C}_1(Y)$ is a homeomorphism, then $f[\mathcal{F}_1(X)] = \mathcal{F}_1(Y)$.

The following Theorem is the main result of this paper.

Theorem 3.20. Hereditarily indecomposable rim-metrizable continua have unique hyperspace $\Gamma_X$, where $\Gamma_X \in \{2^X, \mathcal{C}_n(X)\}$ and $n \in \mathbb{N}$. In fact, if $f : \Gamma_X \rightarrow \Gamma_Y$ is a homeomorphism, where $X$ is a hereditarily indecomposable rim-metrizable continuum and $Y$ is a continuum then $f[\mathcal{F}_1(X)] = \mathcal{F}_1(Y)$.

Proof. Suppose $\Gamma_X$ is homeomorphic to $\Gamma_Y$ and let $f : \Gamma_X \rightarrow \Gamma_Y$ be a homeomorphism. Since $\mathcal{F}_1(X) \subseteq \Gamma_X \subseteq 2^X$, by 2.1, we have that $\omega(\Gamma_X) = \omega(X)$. In a similar way, we obtain that $\omega(\Gamma_Y) = \omega(Y)$. Then $\omega(Y) = \omega(X)$. If $\omega(X) \leq \aleph_0$, then $X$ and $Y$ are metrizable. Hence, it follows by 3.18.

Suppose $\omega(X) \geq \aleph_1$. By 3.12 and 3.13, there exist two inverse $\sigma$-systems $\{X_\alpha, f^\beta_\alpha, \Lambda\}$ and $\{Y_\alpha, g^\beta_\alpha, \Lambda\}$ of metrizable continua such that $X$ is homeomorphic to $X_\Lambda$ and $Y$ is homeomorphic to $Y_\Lambda$.

Note that, if $g : X \rightarrow X_\Lambda$ is a homeomorphism, then the homeomorphism $\Gamma_g : \Gamma_X \rightarrow \Gamma_X$ satisfies $\Gamma_g[\mathcal{F}_1(X)] = \mathcal{F}_1(X_\Lambda)$. Hence, we can assume that $X = X_\Lambda$ and $Y = Y_\Lambda$.

By [2, 2.5.6], we also assume that each of the bonding maps $f^\beta_\alpha$ and $g^\beta_\alpha$ are onto. Since every $X_\alpha$ is metrizable and $X_\Lambda$ is rim-metrizable, by 3.15, we assume that the bonding maps $f^\beta_\alpha$ are monotone. By [2, 3.2.15], every projection map $f^\Lambda_\alpha$ is onto and, by [2, 6.3.16.(a)], they are monotone. Then, by 2.3, every metrizable continuum $X_\alpha$ is hereditarily indecomposable.

If $\Gamma_X = 2^{X_\Lambda}$, each of the maps $\Gamma_f^\beta_\alpha$ and $\Gamma_{g^\beta_\alpha}$ are onto.

If $\Gamma_X = \mathcal{C}_n(X_\Lambda)$, by 3.11, we may assume that the maps $g^\beta_\alpha$ are monotone. Then, by [2, 6.1.29], we deduce that each of the maps $\Gamma_f^\beta_\alpha$ are $\Gamma_{g^\beta_\alpha}$ are onto.

In both cases, the inverse systems $\{\Gamma_X, \Gamma_f^\beta_\alpha, \Lambda\}$ and $\{\Gamma_Y, \Gamma_{g^\beta_\alpha}, \Lambda\}$ have onto bonding maps.

By 2.2 and 3.8, the inverse systems $\{\Gamma_X, \Gamma_f^\beta_\alpha, \Lambda\}$ and $\{\Gamma_Y, \Gamma_{g^\beta_\alpha}, \Lambda\}$ are inverse $\sigma$-systems of metrizable continua.
By 2.5, the map \( h : \Gamma_{X_\alpha} \to \lim \{ \Gamma_{X_\alpha}, \Gamma_{f_\beta}^{\Lambda}, \Lambda \} \) defined by \( h(C) = (f^{\Lambda}_\alpha[C])_{\alpha \in \Lambda} \) and the map \( h' : \Gamma_{Y_\Lambda} \to \lim \{ \Gamma_{Y_\alpha}, \Gamma_{g_\beta}^{\Lambda}, \Lambda \} \) defined by \( h'(C) = (g^{\Lambda}_\alpha[C])_{\alpha \in \Lambda} \) are homeomorphisms. Moreover, \( h \left[ F_1(X_\Lambda) \right] = \lim \{ F_1(X_\alpha), F_1(f_\beta^{\Lambda}), \Lambda \} \) and \( h' \left[ F_1(Y_{\Lambda}) \right] = \lim \{ F_1(Y_\alpha), F_1(g_\beta^{\Lambda}), \Lambda \} \).

Let \( l = h' \circ f \circ h^{-1} : \lim \{ \Gamma_{X_\alpha}, \Gamma_{f_\beta}^{\Lambda}, \Lambda \} \to \lim \{ \Gamma_{Y_\alpha}, \Gamma_{g_\beta}^{\Lambda}, \Lambda \} \). Then, by 3.9, there exist a cofinal subset \( \Sigma \) of \( \Lambda \) and homeomorphisms \( l_\alpha : \Gamma_{X_\alpha} \to \Gamma_{Y_\alpha} \) for every \( \alpha \in \Sigma \), such that \( l_\alpha \circ p^{\Lambda}_\alpha = q^{\Lambda}_\alpha \circ l \) and \( l_\alpha \circ \Gamma_{f_\beta}^{\Lambda} = \Gamma_{g_\beta}^{\Lambda} \circ l_\alpha \) for any \( \alpha, \beta \in \Sigma \) satisfying \( \alpha \leq \beta \), where \( p^{\Lambda}_\alpha : \lim \{ \Gamma_{X_\alpha}, \Gamma_{f_\beta}^{\Lambda}, \Lambda \} \to \Gamma_{X_\alpha} \) and \( q^{\Lambda}_\alpha : \lim \{ \Gamma_{Y_\alpha}, \Gamma_{g_\beta}^{\Lambda}, \Lambda \} \to \Gamma_{Y_\alpha} \) are the projection maps. By 3.10, we may assume that \( \Sigma = \Lambda \).

By 3.18, the spaces \( X_\alpha \) and \( Y_\alpha \) are homeomorphic and \( l_\alpha \left[ F_1(X_\alpha) \right] = F_1(Y_\alpha) \). Thus, the homeomorphism

\[
l_\alpha \mid_{F_1(X_\alpha)} : F_1(X_\alpha) \to F_1(Y_\alpha)
\]

induces a homeomorphism:

\[
l' : \lim \{ F_1(X_\alpha), F_1(f_\beta^{\Lambda}), \Lambda \} \to \lim \{ F_1(Y_\alpha), F_1(g_\beta^{\Lambda}), \Lambda \}
\]

such that \( l \left|_{\lim \{ F_1(X_\alpha), F_1(f_\beta^{\Lambda}), \Lambda \}} \right. = l' \). Hence:

\[
f[F_1(X_\Lambda)] = (h')^{-1} \circ h' \circ f \circ h^{-1} \left[ \lim \{ F_1(X_\alpha), F_1(f_\beta^{\Lambda}), \Lambda \} \right]
\]

\[
= (h')^{-1} \circ l \left[ \lim \{ F_1(X_\alpha), F_1(f_\beta^{\Lambda}), \Lambda \} \right]
\]

\[
= (h')^{-1} \left[ \lim \{ F_1(Y_\alpha), F_1(g_\beta^{\Lambda}), \Lambda \} \right]
\]

\[
= F_1(Y_\Lambda).
\]

Since \( F_1(X_\Lambda) \) and \( X_\Lambda \) are homeomorphic, and \( F_1(Y_\Lambda) \) is homeomorphic to \( Y_\Lambda \), we conclude that \( X_\Lambda \) and \( Y_\Lambda \) are homeomorphic. \( \square \)

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