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REFLECTION THEOREMS FOR SOME CARDINAL FUNCTIONS

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Abstract. In this paper we have two goals. The first one is to present some reflection theorems for cardinal functions in the class of topological groups. In *Reflection theorems for cardinal functions*, [Topology Appl. **100** (2000), no. 1, 47–66], R. E. Hodel and J. E. Vaughan proved that, assuming GCH, the pseudocharacter and the point separating weight reflect all infinite cardinals for the class of compact Hausdorff spaces. We show that the situation is different in the class of compact topological groups. The second goal is to establish reflection theorems for the cardinal functions wl, aql, ql, ac, and lc.

1. Introduction

In topology, the term *reflection theorem* often means a theorem which states that *if a space X does not have a property $\mathcal{P}$, then some small* (in some sense) *subspace of X does not have the property $\mathcal{P}$*. It is clear that this is equivalent to *if X is a space such that every small subspace of X has $\mathcal{P}$, then X has $\mathcal{P}$*.

Recently, several articles related to reflection theorems in topology have appeared in the literature. For example, in [1], the authors study when a given property $\mathcal{P}$ for the closures of discrete subspaces of a space X implies that X has $\mathcal{P}$. For them, the "small" subspaces of X are the closures of discrete subspaces of X. Another kind of

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"small" subspaces are those of cardinality $\leq \kappa$. In this case, the reflection question is *If κ is an infinite cardinal and X is a space such that every subspace of X of cardinality at most κ has $\mathcal{P}$, does this guarantee that X has $\mathcal{P}$?*

The first systematic study of reflection theorems for cardinal functions is due to R. E. Hodel and J. E. Vaughan [8].

In this paper, we have two goals. Our first one is to present some reflection theorems for cardinal functions in the class of topological groups. In [8], Hodel and Vaughan proved that, under GCH, the cardinal functions ψ and psw reflect all infinite cardinals for the class of compact Hausdorff spaces. We will show that we can omit the GCH if we work in the class of compact groups. Our second goal is to establish some reflection theorems for the cardinal functions wl, aql, ql, ac, and lc.

2. Notation and terminology

For any set X and any cardinal number κ, $[X]^{\leq\kappa}$ denotes the collection of all subsets of X with the cardinality $\leq \kappa$; the collection $[X]^{<\kappa}$ is defined similarly. $\overline{A}$ is the closure of A in X. If Y is a subspace of X, the closure of a subset A with respect to Y is denoted $cl_Y(A)$.

We refer the reader to [7] and [9] for definitions and terminology on cardinal functions not explicitly given below. The symbols w, πw, psw, L, wl, d, χ, and ψ denote the weight, the π-weight, the point separating weight, the Lindelöf degree, the weak Lindelöf degree, the density, the character, and the pseudo-character, respectively. If ϕ is a cardinal function, then $h\phi$ is the hereditary version of ϕ, that is, $h\phi(X) = \sup\{\phi(Y) : Y \subseteq X\}$. Clearly, ϕ is monotone if and only if $\phi = h\phi$.

The following cardinal functions are due to Alessandro Fedeli [4].

Definition 2.1. Let X be a topological space.

(1) $ac(X)$ is the smallest infinite cardinal κ such that there is a subset S of X such that $|S| \leq 2^\kappa$ and for every open collection $\mathcal{U}$ in X, there is a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$, with $\bigcup\mathcal{U} \subseteq S \cup \overline{\bigcup\{V : V \in \mathcal{V}\}}$.

(2) $lc(X)$ is the smallest infinite cardinal κ such that there is a closed subset F of X such that $|F| \leq 2^\kappa$ and for every

open collection $\mathcal{U}$ in X, there is a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$, with $\bigcup\mathcal{U} \subseteq F \cup \overline{\bigcup\{V : V \in \mathcal{V}\}}$.

(3) $aql(X)$ is the smallest infinite cardinal κ such that there is a subset S of X such that $|S| \leq 2^\kappa$ and for every open cover $\mathcal{U}$ of X, there is a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ with $X = S \cup (\bigcup\mathcal{V})$.

Clearly, $ac(X) \leq lc(X) \leq c(X)$ and $aql(X) \leq L(X)$ for every topological space.

Following the definition of aql, we introduce the next definition.

Definition 2.2. Let X be a topological space. The cardinal number $qwl(X)$ is the smallest infinite cardinal κ such that there is a subset S of X such that $|S| \leq 2^\kappa$, and for every open cover $\mathcal{U}$ of X, there is a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ with $X = \overline{S \cup \bigcup\mathcal{V}}$.

It is obvious that $qwl(X) \leq wl(X)$, $qwl(X) \leq aql(X)$, and $qwl(X) \leq d(X)$ for every topological space X.

Recall that a set A in X is *κ-quasi-dense* if $|A| \leq 2^\kappa$ and for every open cover $\mathcal{U}$ of X, there are $B \in [A]^{\leq\kappa}$ and $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $X = \overline{B} \cup \bigcup\mathcal{V}$. In [11], Shu-Hao Sun introduced the following cardinal invariant.

Definition 2.3. Let X be a topological space. The cardinal number $ql(X)$ is the smallest infinite cardinal κ such that X has a κ-quasi-dense subset.

Clearly, $ql(X) \leq d(X)$ and $wl(X) \leq ql(X) \leq L(X)$ for every topological space X.

In this paper, all topological groups under consideration are assumed to be Hausdorff. By a *subgroup* of a topological group G, we mean an algebraic subgroup H of G endowed with the subspace topology. We will write $H \leq G$ when H is subgroup of G.

Let κ be an infinite cardinal. A topological group G is called *κ-bounded* if for every open neighborhood U of the identity in G, there is a set $K \subseteq G$ with $|K| \leq \kappa$ and $K \cdot U = G$.

An important and useful cardinal function is the *index of boundedness* of a topological group G, denoted by $ib(G)$. By definition, $ib(G)$ is the minimal cardinal $\kappa \geq \omega$ such that G is κ-bounded. It is easy to see that every subgroup of a κ-bounded topological group is κ-bounded; hence, $ib(H) \leq ib(G)$ for all subgroups H of G. Thus, we can say that the cardinal function ib is monotone in the class of topological groups. Besides, it is clear that $ib(G) \leq |G|$ for all topological groups G. For a detailed discussion about this topic, we refer the reader to [12].

Recall that a compact Hausdorff space X is a *dyadic* space if X is a continuous image of the Cantor cube D^κ for some cardinal number κ (the symbol D stands for the discrete two-points set).

For the reader's convenience, we give here the formulations of a few results of reflection theory we use in this article. (See [8].)

Definition 2.4. Let ϕ be a cardinal function and $\kappa \geq \omega$ a cardinal number.

(1) ϕ *reflects* κ means if $\phi(X) \geq \kappa$, then there exists $Y \in [X]^{\leq \kappa}$ such that $\phi(Y) \geq \kappa$.
(2) ϕ *strongly reflects* κ means if $\phi(X) \geq \kappa$, then there exists $Y \in [X]^{\leq \kappa}$ such that for each $Y \subseteq Z \subseteq X$, $\phi(Z) \geq \kappa$.
(3) ϕ *satisfies the increasing union property in* κ ($IU(\kappa)$) if the following holds. If X is the union of an increasing family of its subspaces $\{X_\alpha : \alpha \in \lambda\}$, where $\kappa < \lambda$, λ is regular, and $\phi(X_\alpha) < \lambda$ for all $\alpha \in \lambda$, then $\phi(X) < \lambda$.

Remark 2.5. For some cardinal functions it is necessary to restrict the class of spaces under consideration in order to obtain a reflection theorem. The appropriate definition in this case is ϕ *reflects* κ *for a class* $\mathcal{C}$ if, given $X \in \mathcal{C}$ with $\phi(X) \geq \kappa$, there exists $Y \subseteq X$ with $|Y| \leq \kappa$ and $\phi(X) \geq \kappa$.

The next lemma summarizes the relation between reflection, strong reflection, and $IU(\kappa)$ (see [8]).

Lemma 2.6. *Suppose ϕ is a cardinal function and $\kappa \geq \omega$.*

(1) *If ϕ strongly reflects κ, then ϕ reflects κ.*
(2) *If ϕ is monotone and ϕ reflects κ, then ϕ strongly reflects κ.*
(3) *If ϕ strongly reflects κ, then ϕ satisfies $IU(\kappa)$.*
(4) *If ϕ strongly reflects all successor cardinals, then ϕ strongly reflects all infinite cardinals. In particular, if ϕ is monotone and reflects all successor cardinals, then ϕ strongly reflects all infinite cardinals.*

Remark 2.7. Every cardinal function ϕ is defined so that $\phi(X) \geq \omega$ always holds; it easily follows that ϕ reflects ω. So in our discussion of reflection theorems, we assume that $\kappa > \omega$.

The following remarkable theorem, due to Hajnal and Juhász, is the key to our results.

Theorem 2.8 (Hajnal-Juhász reflection theorem [6])**.** *If $w(X) \geq \kappa$, then there exists $Y \subseteq X$ such that $|Y| \leq \kappa$ and $w(Y) \geq \kappa$.*

3. Reflection theorems in topological groups

In [6], Hajnal and Juhász proved that πw reflects every regular cardinal. On the other hand, Hodel and Vaughan [8] proved that assuming GCH, πw reflects every cardinal for the class of Hausdorff spaces. Our first result shows that the GCH may be omitted if we restrict ourselves to the class of topological groups.

Theorem 3.1. *The cardinal function πw reflects every infinite cardinal in the class of topological groups.*

Proof: Suppose that $\pi w(G) \geq \kappa$, then $w(G) \geq \kappa$. By Theorem 2.8, there exists a subset Y of G with $|Y| \leq \kappa$ and $w(Y) \geq \kappa$. Let H be the subgroup of G generated by Y. Then $|H| \leq \kappa$ and $w(H) \geq \kappa$. Thus, since for topological groups $\pi w = w$, we have $\pi w(H) \geq \kappa$. □

The cardinal functions d and πw are very similar with respect to their reflection properties (see [8]); hence, it is natural to ask if d reflects every infinite cardinal for the class of topological groups. At the moment, we do not know the answer to this question; nevertheless, we show that d and L reflect every infinite cardinal for the class of metrizable topological groups. We will deduce this from a stronger result. First, we need the following theorem (see [12]).

Theorem 3.2. *Let G be a topological group. Then $w(G) = \phi(G)\chi(G)$, where $\phi \in \{ib, d, L, wl\}$.*

Remark 3.3. Since the cardinal function ib is defined only for topological groups, the definition of reflection for this cardinal function must be modified.

Let G be a topological group and let κ be an infinite cardinal number. The cardinal function ib *reflects* κ if whenever $ib(G) \geq \kappa$, there is a subgroup H of G with $|H| \leq \kappa$ and $ib(H) \geq \kappa$. Also, we will say that ib *strongly reflects* κ if whenever $ib(G) \geq \kappa$, there is a subgroup H of G with $|H| \leq \kappa$ and $ib(K) \geq \kappa$ for all topological subgroups K such that $H \subseteq K \subseteq G$. Note that for this cardinal function the notions of reflection and strong reflection are equivalent (because ib is monotone in the class of topological groups).

Theorem 3.4. *Let G be a topological group and let $\kappa > \omega$. For $\phi \in \{ib, d, L, wl\}$, if $\chi(G) < \kappa$ and for all $Y \in [G]^{\leq\kappa}$, $\phi(Y) < \kappa$, then $\phi(G) < \kappa$. (In the case of the cardinal function ib we also assume that Y is a topological subgroup of G).*

Proof: Suppose that $\phi(G) \geq \kappa$, then $w(G) \geq \kappa$; hence, by the Hajnal-Juhasz theorem, there is $Y \in [G]^{\leq\kappa}$ such that $w(Y) \geq \kappa$. Denote by H the subgroup of G generated by Y. It is clear that $|H| \leq \kappa$ and $w(H) \geq \kappa$. By Theorem 3.2, $\phi(H)\chi(H) \geq \kappa$. Hence, $\phi(H) \geq \kappa$, a contradiction. □

From Theorem 3.4, it follows that ib reflects every infinite cardinal in the class of metrizable topological groups. Therefore, we can conclude that ib strongly reflects every infinite cardinal in the class of metrizable topological groups. However, it is possible to improve this result. We will show that ib strongly reflects every infinite cardinal for the class of topological groups.

Theorem 3.5. *For the class of topological groups, ib strongly reflects every infinite cardinal.*

Proof: It suffices to prove the statement of the theorem for successor cardinals. Indeed, suppose that ib reflects every successor cardinal. Let λ be a limit cardinal and suppose that $ib(G) \geq \lambda$. Then $ib(G) > \kappa^+$ for every $\kappa < \lambda$. Fix $\kappa < \lambda$. Since ib reflects κ^+, there is a topological subgroup G_κ of G with $|G_\kappa| \leq \kappa^+$ and $ib(G_\kappa) \geq \kappa^+$. Define $S = \bigcup_{\kappa<\lambda} G_\kappa$. Clearly, $|S| \leq \lambda$. Let H be the topological subgroup of G generated by S. Then $|H| \leq \lambda$ and $ib(H) \geq \lambda$. Therefore, ib reflects λ. This proves that ib reflects every limit cardinal.

Thus, we need only to check that ib reflects every successor cardinal. Let κ be an infinite cardinal and suppose that $ib(G) > \kappa$, that is, $ib(G) \geq \kappa^+$. Then there exists an open neighborhood U of the identity of G such that $G \neq KU$ for every K subset of G with $|K| \leq \kappa$. Construct a sequence $\{p_\alpha : 0 \leq \alpha < \kappa^+\}$ of points of G such that for all $\beta < \kappa$,

$$p_\beta \in G \setminus \big(\{p_\rho : \rho < \beta\} \cdot U\big).$$

Put $Y = \{p_\alpha : 0 \leq \alpha < \kappa^+\}$, and denote by H the subgroup of G generated by Y. It is clear that $|H| \leq \kappa^+$. The proof is complete if $ib(H) = \kappa^+$. To see this, suppose that $ib(H) = \mu \leq \kappa$ and fix an open neighborhood V of the identity in H such that $V^{-1}V \subseteq U$. Then there exists $F \subseteq H$ with $|F| \leq \mu$ such that $H = F(V \cap H)$. Now for each $\alpha < \kappa^+$, there exist $g_\alpha \in F$ and $v_\alpha \in V$ such that $p_\alpha = g_\alpha v_\alpha$. Since $|F| < \kappa^+$, by the regularity of κ^+, we can find $\beta < \alpha < \kappa^+$ such that $g_\beta = g_\alpha = g$; then $p_\beta = gv_\beta$ and $p_\alpha = gv_\alpha$, and $p_\alpha = p_\beta \cdot v_\beta^{-1} \cdot v_\alpha \in p_\beta V^{-1}V \subseteq (\{p_\beta : \beta < \alpha\}) \cdot U$, a contradiction. □

Corollary 3.6. *Let G be a topological group and let κ be an infinite cardinal. If G is the union of an increasing family of its subgroups $\{G_\alpha : \alpha \in \lambda\}$, where $\kappa < \lambda$, λ is regular, and $ib(G_\alpha) < \kappa$ for all $\alpha \in \lambda$, then $ib(G) < \kappa$.*

Proof: Assume for contradiction that $ib(G) \geq \kappa$. Since ib strongly reflects κ, there is a topological subgroup $H \leq G$ with $|H| \leq \kappa$ and $ib(K) \geq \kappa$ for all $H \leq K \leq G$. Take now a G_β ($\beta < \lambda$) with $H \subseteq G_\beta$. Then $ib(G_\beta) \geq \kappa$, a contradiction. □

Corollary 3.7. *If G is a topological group, then for each infinite cardinal $\lambda < ib(G)$, there exists a subgroup $H \leq G$ such that $|H| \leq \lambda = ib(H)$.*

Proof: Suppose that $ib(G) > \lambda$. Since ib reflects λ, there is a topological subgroup H of G such that $|H| \leq \lambda$ and $ib(H) \geq \lambda$. But $ib(H) \leq |H| \leq \lambda$. □

We now turn to the monotone cardinal function χ. In [8], Hodel and Vaughan proved that χ strongly reflects every infinite cardinal for the class of compact Hausdorff spaces. In relation to this we will prove, as a consequence of Proposition 3.8, that *for the class of $\aleph_0$-bounded groups, χ strongly reflects every infinite cardinal.* The proof of 3.8 is just a slight modification of the one given for Theorem 3.4.

Proposition 3.8. *Let G be a topological group and let κ be an infinite cardinal, and suppose that $\phi(G) \leq \kappa$, where $\phi \in \{ib, d, L, wl\}$. If $\chi(H) < \kappa^+$ for all $H \leq G$ with $|H| \leq \kappa^+$, then $\chi(G) < \kappa^+$. Moreover, $w(G) < \kappa^+$.*

Corollary 3.9. *For the class of $\aleph_0$-bounded groups, χ strongly reflects every infinite cardinal.*

Since $\chi = \pi\chi$ for topological groups (see [12]), it is not difficult to check that Proposition 3.8 is true if χ is replaced by $\pi\chi$.

On the other hand, it was proved in [13] that if $G = \bigcup\{X_\alpha : \alpha \in \kappa\}$ is a σ-precompact topological group, where $cf(\kappa) > \omega_1$, $X_\alpha \subseteq X_\beta$ whenever $\alpha < \beta < \kappa$, and every X_α is first countable, then G is second countable. This result is now a consequence of the following, more general result.

Corollary 3.10. *Let G be a topological group with $ib(G) = \lambda$, and $G = \bigcup\{X_\alpha : \alpha \in \kappa\}$ where $cf(\kappa) > \lambda^+$ and $X_\alpha \subseteq X_\beta$ whenever $\alpha < \beta < \kappa$. If $\chi(X_\alpha) \leq \lambda$ for each $\alpha \in \kappa$, then $w(G) \leq \kappa$.*

Proposition 3.8 generates the following question naturally.

Question 3.11. Let $\phi \in \{d, L, wl, hd\}$, $\kappa \geq \omega$, and $\mathcal{C}_{\phi,\kappa}$ be the class of all topological spaces X such that $\phi(X) \leq \kappa$. Does χ reflect κ^+ in the class $\mathcal{C}_{\phi,\kappa}$?

We will give positive answers in the cases of the cardinal functions d and hd (see 3.12, 3.14, and 3.15, below).

Proposition 3.12. *The cardinal function χ reflects κ^+ in the class of all regular topological spaces X such that $d(X) \leq \kappa$.*

Proof: Let D be a dense subset of X with $|D| \leq \kappa$ and suppose that $\chi(Y) < \kappa^+$ for all $Y \in [X]^{\leq\kappa^+}$. Let x be any point of X. It is clear that $Y = \{x\} \cup D$ is a dense subset of X and $|Y| \leq \kappa$. Since X is regular, $\chi(x, X) = \chi(x, Y) \leq \kappa$. Therefore, $\chi(X) < \kappa^+$. □

For an arbitrary set X, a family $\mathcal{S}$ of subsets of X and $Y \subseteq X$, we put $\mathcal{S}\,|Y$, the trace of $\mathcal{S}$ on Y. The proofs of Lemma 3.13 and Theorem 3.14 below follow the same pattern of the proof of Lemma and Theorem in [6].

Lemma 3.13. *Let κ be an uncountable regular cardinal and let X be a space with $hd(X) < \kappa$ and let p be an arbitrary point of X . Let $\{Y_\alpha : \alpha \in \kappa\}$ be an increasing sequence of subspaces of X with $p \in Y_0$ and let $\{\mathcal{L}_\alpha : \alpha \in \kappa\}$ be a sequence of open collections in X such that for each $\alpha \in \kappa$, $\mathcal{L}_\alpha\,|Y_\alpha$ is a local base of $p \in Y_\alpha$. Then $\mathcal{L}\,|Y$ is a local base of $p \in Y = \bigcup\{Y_\alpha : \alpha \in \kappa\}$, where $\mathcal{L} = \bigcup\{\mathcal{L}_\alpha : \alpha \in \kappa\}$.*

Proof: Suppose that there is an open neighborhood U of p in X such that for each $L \in \mathcal{L}$, we have $L \cap Y \not\subseteq U$. Let D be a dense subset of $Y \setminus U$ of cardinality $< \kappa$ and let $C = \{p\} \cup D$.

Since $\{Y_\alpha : \alpha \in \kappa\}$ is increasing, $C \subseteq Y$, $|C| < \kappa$, and κ is regular, there exists $\alpha < \kappa$ such that $C \subseteq Y_\alpha$. Since $\mathcal{L}\,|Y_\alpha$ is a local base of p in Y_α and $U \cap Y_\alpha \neq \emptyset$, because $p \in U \cap Y_\alpha$, there exists $L \in \mathcal{L}_\alpha$ such that $L \cap Y_\alpha \subseteq U$. But $(L \cap Y) \not\subseteq U$; hence, $L \cap Y \setminus U \neq \emptyset$. Now D is dense in $Y \setminus U$; there exists $d \in D \cap L$. But $L \cap D \subseteq L \cap Y_\alpha \subseteq U$; hence, $d \in U$; this contradicts $D \subseteq Y \setminus U$. □

Theorem 3.14. *The cardinal function χ reflects κ^+ in the class $\mathcal{C}_{hd,\kappa}$ of all topological spaces X with $hd(X) \leq \kappa$.*

Proof: Suppose that $\chi(Y) < \kappa^+$, for all $Y \in [X]^{\leq \kappa^+}$. Since $\chi(X) \geq \kappa^+$, there is $p \in X$ such that $\chi(p, X) \geq \kappa^+$. Construct an increasing sequence $\{Y_\alpha : \alpha \in \kappa^+\}$ of subsets of X and a sequence $\{\mathcal{B}_\alpha : \alpha \in \kappa^+\}$ of collections of open neighborhoods of p in X such that

(1) $|Y_\alpha| \leq \kappa^+$, for each $\alpha < \kappa^+$ and $p \in Y_0$;
(2) $|\mathcal{B}_\alpha| \leq \kappa$, for each $\alpha < \kappa^+$;
(3) for each $\alpha < \kappa$, $\mathcal{B}_\alpha\,|Y_\alpha$ is a local base of p in Y_α;
(4) if $\alpha = \gamma + 1$, then $\mathcal{B}_\gamma\,|Y_\alpha$ is not a local base of p in Y_α.

Since $hd(X) \leq \kappa$, by Lemma 3.13, $\mathcal{B}\,|Y$ is a local base of p in $Y = \bigcup_{\alpha<\kappa} Y_\alpha$ where $\mathcal{B} = \bigcup\{\mathcal{B}_\alpha : \alpha < \kappa^+\}$. As $|Y| \leq \kappa^+$, there exists $\mathcal{C} \subseteq \mathcal{B}$ such that $|\mathcal{C}| = \chi(Y) < \kappa$ and $\mathcal{C}$ is a local base of p in Y. Hence, by regularity of κ^+, there exists $\alpha < \kappa^+$ such that $\mathcal{C} \subseteq \mathcal{B}_\alpha$. But, by (4), $\mathcal{B}_\alpha\,|Y_{\alpha+1}$ is not a local base of p in $Y_{\alpha+1}$. Thus, $\mathcal{B}_\alpha\,|Y$ is not a local base of p in Y, a contradiction. □

Corollary 3.15. *The cardinal function χ strongly reflects every infinite cardinal in the class of hereditarily separable spaces.*

To finish this section, we will show that the cardinal functions ψ and psw strongly reflect every infinite cardinal in the class of compact topological groups.

Assuming GCH, Hodel and Vaughan proved in [8] that, for the class of compact Hausdorff spaces, the cardinal functions ψ and psw reflect every infinite cardinal. The question of whether these results can be proved without any additional set-theoretical assumptions seems to be open. However, we will prove that ψ and psw

strongly reflect all infinite cardinals for the class of compact topological groups. These results are a consequence of the following more general theorem.

Theorem 3.16. *For the class of dyadic spaces, the cardinal function ψ strongly reflects all infinite cardinals.*

Proof: Since ψ is monotone, it is sufficient to prove the result for a successor cardinal κ^+. Suppose that $\psi(X) \geq \kappa^+$. Then $w(X) \geq \kappa^+$. Now, by Efimov-Gerlits-Hagler's theorem [3], X contains a topological copy of D^{κ^+}. Let $p \in D^{\kappa^+}$ and note that $\chi(p, D^{\kappa^+}) = \kappa^+$. By Efimov's theorem [2], there exists $M \subseteq D^{\kappa^+}$ such that M is discrete, $|M| = \kappa^+$ and $M \cup \{p\}$ is homeomorphic to $A(\kappa^+)$, where $A(\kappa^+)$ denotes the one-point of compactification of a discrete space of cardinality κ^+. Clearly, we can conclude that $\psi(M \cup \{p\}) \geq \kappa^+$. □

From the proof of Theorem 3.16, we have the following proposition.

Proposition 3.17. *For the class of dyadic spaces, the cardinal function psw strongly reflects all infinite cardinals.*

Since every compact topological group is dyadic [10], Corollary 3.18 follows.

Corollary 3.18. *For the class of compact topological groups, the cardinal functions ψ and psw strongly reflect all infinite cardinals.*

4. Reflection theorems for wl, qwl, ac, lc, aql, and ql

It is well known that the following inequalities are the most important cardinal inequalities.

(1) (Hajnal-Juhász) If $X \in \mathcal{T}_2$, then $|X| \leq 2^{c(X)\cdot\chi(X)}$.
(2) (Šapirovskiĭ) For all $X \in \mathcal{T}_3$, we have $|X| \leq \pi\chi(X)^{c(X)\cdot\psi(X)}$.
(3) (Arhangel'skiĭ) Let X be a Hausdorff space, then $|X| \leq 2^{L(X)\cdot\chi(X)}$.
(4) (Bell-Ginsburg-Woods) If $X \in \mathcal{T}_4$, then $|X| \leq 2^{wl(X)\cdot\chi(X)}$.

There have recently been important improvements to these inequalities; for example, in [4], the author introduces three new cardinal invariants–*aql*, *ac*, and *lc*–and obtains a generalization of inequalities (1) and (3). Another generalization of (3) is due to Sun in [11], where he introduces the cardinal function *ql*.

The purpose of this section is to establish some reflection theorems for the cardinal functions *wl*, *qwl*, *ac*, *lc*, *aql*, and *ql*. In the proofs of the reflection theorems for some of these cardinal functions, we make use of the following well-known result.

Theorem 4.1 ([5]). *If κ is a singular strong limit cardinal, then every Hausdorff space X with $|X| \geq \kappa$ contains a discrete subspace of cardinality κ.*

Also we need the following auxiliary result.

Lemma 4.2. *Let $\phi \in \{wl, qwl, aql, ac, lc, ql\}$ and let κ be a singular strong limit cardinal. Then $\phi(D(\kappa)) \geq \kappa$, where $D(\kappa)$ is the discrete space of cardinality κ.*

Proof: In the cases of the cardinal functions *wl* and *qwl*, the proofs are immediate. On the other hand, for the cardinal functions *aql*, *ac*, *lc*, and *ql*, the proofs are similar; for this reason, we will give only the proof for *aql*.

If $aql(D(\kappa)) = \gamma < \kappa$, then there exists $S \in [D(\kappa)]^{<2^\gamma}$, and $\mathcal{V} \subseteq \{\{y\} : y \in D(\kappa)\}$, with $|\mathcal{V}| \leq \gamma$ such that $D(\kappa) \subseteq S \cup \bigcup \mathcal{V}$; hence, $|D(\kappa)| < \kappa$, a contradiction. □

The following result shows that *wl* and *ql* have very similar reflection properties to L (see [8, Theorem 2.3]). The examples in part (4) of the following theorem and in part (3) of Theorem 4.6 are constructed by Hodel and Vaughan in [8].

Theorem 4.3. *Let $\phi \in \{wl, ql\}$. The following hold.*

(1) *ϕ reflects every successor cardinal;*
(2) *ϕ reflects every singular strong limit cardinal for the class of Hausdorff spaces;*
(3) *assuming* GCH +(no inaccessible cardinals)*, ϕ reflects all cardinals for the class of Hausdorff spaces;*
(4) *for each uncountable cardinal κ, ϕ need not satisfy $IU(\kappa)$ and need not strongly reflect κ for the class of completely normal Hausdorff spaces.*

Proof: (1) The proof for wl is a now standard technique due to Hajnal and Juhász. Suppose that $wl(X) \geq \kappa^+$. Then there is an open cover $\mathcal{U}$ of X such that for all subcollections of $\mathcal{U}$ of cardinality $\leq \kappa$, its union is not a dense subspace of X. Construct $\{U_\alpha : \alpha \in \kappa^+\} \subseteq \mathcal{U}$ and $Y = \{x_\alpha : \alpha \in \kappa^+\} \subseteq X$ with $x_\alpha \in U_\alpha \setminus cl_X(\bigcup_{\beta\in\alpha} U_\beta)$, for all $\alpha < \kappa^+$. So we have that $wl(Y) \geq \kappa^+$.

Consider now the cardinal function ql. Suppose that $ql(X) \geq \kappa^+$. Clearly, we have that $s(X) \geq \kappa^+$. Hence, there is a discrete subset Y of X with $|Y| = \kappa^+$. Note that $ql(Y) \geq \kappa^+$. To see this, assume that $ql(Y) = \gamma < \kappa^+$. Let $A \in [Y]^{\leq 2^\gamma}$ be a subset that witnesses that $ql(Y) = \gamma$ and consider $\mathcal{U} = \{\{y\} : y \in Y\}$. Then there exists $B \in [A]^{\leq\gamma}$ and $\mathcal{V} \in [\mathcal{U}]^{\leq\gamma}$ such that $Y = \overline{B} \cup \bigcup \mathcal{V}$. Hence, $|Y| \leq \gamma$, a contradiction.

(2) Let $wl(X) \geq \kappa$, where κ is a singular strong limit cardinal. Since $wl(X) \leq |X|$, we have that $|X| \geq \kappa$. By 4.1, there is a discrete subspace Y of X such that $|Y| = \kappa$. Applying Lemma 4.2, we can conclude that $wl(Y) \geq \kappa$.

In the case of ql, note that if $ql(X) \geq \kappa$, then $s(X) \geq \kappa$. So there is a subspace Y of X such that $|Y| = \kappa$. By Lemma 4.2, we have that $ql(Y) \geq \kappa$.

(4) Let κ be an uncountable cardinal. The required space is $X = [0, \kappa^+)$ with the order topology. Let $X_\alpha = [0, \alpha]$ for each $\alpha \in \kappa^+$. Clearly $X = \bigcup\{X_\alpha : \alpha < \kappa\}$ and $X_\alpha \subseteq X_\beta$ for all $\alpha < \beta < \kappa$. Besides $wl(X_\alpha) = ql(X_\alpha) = L(X_\alpha) = \omega$. Also note that $wl(X) = \kappa^+$ and $ql(X) \geq \kappa$. Indeed, suppose $ql(X) = \gamma < \kappa$ and let A be an element of $[X]^{\leq 2^\gamma}$ that witness that $ql(X) = \gamma$. Since $L(X) = \kappa^+$, there is an open cover $\mathcal{U}$ of X such that no subcollection of $\mathcal{U}$ of cardinality $\leq \kappa$ cover of X. For this $\mathcal{U}$, there is $\mathcal{V} \in [\mathcal{U}]^{\leq\gamma}$ and $B \in [A]^{\leq\gamma}$ such that $X = \overline{B} \cup \bigcup \mathcal{V}$. Let $\alpha_0 = \sup B < \kappa^+$. It is clear that $\overline{B} \subseteq X_{\alpha_0}$. On the other hand, there exists $U_1, ..., U_n \in \mathcal{U}$ such that $X_{\alpha_0} \subseteq \bigcup\{U_1, ..., U_n\}$. Hence, $X = \bigcup\{U_1, ..., U_n\} \cup \bigcup \mathcal{V}$, a contradiction. □

Corollary 4.4. *The cardinal functions hwl and hql strongly reflect every infinite cardinal.*

Lemma 4.5. *Assume that $2^\kappa < 2^{\kappa^+}$ and let $\phi \in \{qwl, aql, qc, lc, ql\}$. Then $\phi(X) \geq \kappa^+$, where X is the discrete space of cardinality 2^{κ^+}.*

Proof: We make the proof only for the case of the cardinal function aql. The arguments for the other cases are similar.

Assume that $aql(X) = \gamma \leq \kappa$, and let $S \subseteq X$ with $|S| \leq 2^\gamma$ witness $aql(X) = \gamma$. Clearly, $\mathcal{U} = \{\{x\} : x \in X\}$ cover X. Since $aql(X) = \gamma$, there exists $\mathcal{V} \in [\mathcal{U}]^{\leq\gamma}$ such that $X = S \cup \bigcup \mathcal{V}$. Therefore, we have $|X| \leq 2^\kappa$, a contradiction. □

Theorem 4.6. *Let $\phi \in \{qwl, aql\}$. The following hold.*

(1) *If $2^\kappa < 2^{\kappa^+}$, then ϕ does not reflect the cardinal κ^+;*
(2) *ϕ reflects every singular strong limit cardinal for the class of Hausdorff spaces;*
(3) *for each uncountable cardinal κ, ϕ need not satisfy $IU(\kappa)$ and need not strongly reflect κ for the class of completely normal Hausdorff spaces.*

Proof: (1) Consider $X = D(2^{\kappa^+})$. By Lemma 4.5, $aql(X) \geq \kappa^+$. Now note that for each $Y \in [X]^{<\kappa^+}$, we have that $aql(Y) < \kappa^+$.

The proof for the case $\phi = qwl$ is similar.

(2) Let κ be a singular strong limit cardinal. Suppose $qwl(X) \geq \kappa$. Since $qwl(X) \leq |X|$, we have $|X| \geq \kappa$. Hence, by Theorem 4.1, there exists $Y \in [X]^\kappa$ such that Y is discrete in X. Applying Lemma 4.2, we can conclude that $qwl(Y) \geq \kappa$.

The proof for the case $\phi = aql$ is similar.

(3) Let $\kappa > \omega$ and let $\lambda = (2^\kappa)^+$. The required space is $X = [0, \lambda)$ with the order topology. For each $\alpha \in \lambda$, we put $X_\alpha = [0, \alpha]$. Note that $X = \bigcup\{X_\alpha : \alpha \in \lambda\}$, $X_\alpha \subseteq X_\beta$ for all $\alpha < \beta < \kappa$, and for each $\alpha \in \lambda$, $aql(X_\alpha) = qwl(X_\alpha) = \omega < \kappa$. But $aql(X) \geq qwl(X) \geq \kappa$. □

Corollary 4.7. *Assuming GCH, the cardinal functions qwl and aql don't reflect successor cardinals.*

We now turn to ac and lc. Recall that $ac(X) \leq lc(X) \leq c(X)$ for every topological space. It was proved in [8] that c strongly reflects every infinite cardinal. The situation changes drastically for ac and lc.

Theorem 4.8. *Let $\phi \in \{ac, lc\}$. The following hold.*

(1) *If $2^\kappa < 2^{\kappa^+}$, then ϕ does not reflect κ^+;*

(2) *ϕ reflects every singular strong limit cardinal for the class of Hausdorff spaces.*

Proof: (1) Consider $X = D(2^{\kappa^+})$ and apply Lemma 4.5.

(2) Let κ be a strong limit cardinal and suppose that $ac(X) \geq \kappa$ ($lc(X) \geq \kappa$, respectively). Clearly, we have $|X| \geq \kappa$. Hence, there is a discrete subset Y of X with $|Y| = \kappa$ (see Theorem 4.1). By Lemma 4.2, we have $ac(Y) \geq \kappa$ ($lc(Y) \geq \kappa$, respectively). □

Corollary 4.9. *Assuming GCH, the cardinal functions ac and lc don't reflect successor cardinals.*

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