AN $m$-DIMENSIONAL
HEREDITARILY INDECOMPOSABLE CONTINUUM
WITH EXACTLY $n$ CONTINUOUS MAPPINGS
ONTO ITSELF

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Abstract. We show that for every $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{\infty\}$, there exists a hereditarily indecomposable $m$-dimensional continuum $X$ which has exactly $n$ continuous surjections onto itself (each one being a homeomorphism).

Moreover, we construct a family of cardinality $2^{\aleph_0}$ of continua of this type such that no two different continua from this family are comparable either by continuous mappings or by embeddings.

1. Introduction

Our terminology follows [5] and [8]. We assume that all our spaces are separable metrizable. By dimension, we mean the covering dimension $\dim$ and by a continuum, we mean a compact connected space. A continuum $X$ is hereditarily indecomposable, abbreviated HI, if for any two intersecting subcontinua $K$, $L$ of $X$, either $K \subset L$ or $L \subset K$.

The first HI continuum, now called the pseudo-arc, was constructed by Bronislaw Knaster [7]. The pseudo-arc, which will be denoted by $P$, is an HI one-dimensional chainable continuum.

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(unique, up to a homeomorphism), and every non-trivial subcontinent of \( P \) is homeomorphic to \( P \). (For more information and references concerning the pseudo-arc see [12].)

The first examples of HI continua of dimension \( m \), where \( m = 2, 3, \ldots, \infty \), were constructed by R. H. Bing [3].

We say that two continua are comparable by continuous mappings (by embeddings, respectively) if there exists a continuous mapping (an embedding, respectively) of one of those continua onto (into, respectively) the other. By a Cook continuum, we understand a non-trivial continuum \( X \) such that no two different nondegenerate subcontinua of \( X \) are comparable by continuous mappings. The first example of a hereditarily indecomposable Cook continuum was constructed in [4]. In the same paper, H. Cook constructed for every \( n \in \mathbb{N} = \{1, 2, \ldots\} \), a continuum \( H_n \) which has exactly \( n \) continuous mappings onto itself, each one being a homeomorphism. The continuum \( H_n \) is decomposable and admits an atomic mapping onto a simple closed curve. Applying the ideas from [17], [19], and [10], we will prove the following theorem.

**Theorem 1.1.** For each \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \cup \{\infty\} \), there exists a hereditarily indecomposable continuum \( X_{nm} \) of dimension \( m \) which has exactly \( n \) continuous mappings onto itself, each one being a homeomorphism. Moreover, \( X_{nm} \) admits an atomic mapping onto the pseudo-arc \( P \) and the group of autohomeomorphisms of \( X_{nm} \) onto \( X_{nm} \) is the cyclic group of order \( n \).

In the special cases when \( m = 1 \) or \( n = 1 \), these results were obtained in [19]. Any 1-dimensional HI Cook continuum satisfies the condition of Theorem 1.1 for \( m = n = 1 \).

Moreover, we will prove the following theorem.

**Theorem 1.2.** For every \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \cup \{\infty\} \), there exists a family \( \{X_{nm}(s) : s \in S\} \), where \( S \) is a set of cardinality \( 2^{\aleph_0} \) of topologically different HI \( m \)-dimensional continua such that every \( X_{nm}(s) \) has exactly \( n \) continuous surjections onto itself and admits an atomic mapping \( p_s \) onto the pseudo-arc \( P \). Moreover,

(i) if \( s \neq t \), then there is no continuous mapping of \( X_{nm}(s) \) onto \( X_{nm}(t) \);

(ii) if \( s \neq t \), then \( X_{nm}(s) \) does not embed into \( X_{nm}(t) \).
Our construction is a modification of the ones given in [17] and [19] and applies a method of condensation of singularities. As before, we exploit an HI Cook continuum and we use a theorem of Wayne Lewis stating that for each \( n \in \mathbb{N} \) there exists an embedding of the pseudo-arc \( P \) in the plane such that the restriction \( r \) of a period \( n \) rotation of the plane around \((0,0)\) to \( P \) is a homeomorphism of \( P \) onto \( P \) of period \( n \) [11]. To raise the dimension of the space obtained in [19], we construct our space in such a way that it contains a certain \( m \)-dimensional continuum \( Y_1 \). The new idea in the proof lies in Lemma 2.2 below. Roughly speaking, this lemma states that one can “replace” one point of a given continuum \( X \) by a special continuum in such a way that the resulting space can be mapped onto any given Waraszkiewicz spiral. In this way, we can “improve” a given continuum \( X \) so that a given continuum \( Y_1 \) does not map onto the whole \( X \).

2. Preliminaries

A continuum \( Y \) is a common model for a family of continua \( W \), if every member of \( W \) is a continuous image of \( Y \) (we do not assume that \( Y \in W \)).

By the ray, we will understand a space homeomorphic to the half-line \([0, +\infty)\). In [22], Z. Waraszkiewicz constructed a family \( W \) of planar continua without a common model. By a Waraszkiewicz spiral, we mean a member of this family. Every Waraszkiewicz spiral \( W \) is a compactification of the ray \( L \) with the remainder \( S \) homeomorphic to the circle. We have

(1) for every continuum \( A \) there exists a Waraszkiewicz spiral \( W \) such that \( A \) cannot be mapped onto \( W \).

The composant of a point \( x \) in a continuum \( X \) is the union of all proper subcontinua of \( X \) containing \( x \). If \( X \) is a non-degenerate HI continuum, then \( X \) has \( 2^{\aleph_0} \) different composants, which are pairwise disjoint and are connected \( F_\sigma \)-subsets of \( X \), both dense and a boundary set in \( X \) (see [8, §48, VI]).

A mapping \( f : X \rightarrow Y \) between continua is confluent (weakly confluent, respectively), if for each subcontinuum \( Q \) of \( Y \) each (some, respectively) component of \( f^{-1}(Q) \) is mapped by \( f \) onto \( Q \). As proved by Cook in [4], each mapping of a continuum onto an HI continuum is confluent.
A subcontinuum $K$ of a continuum $X$ is terminal if every subcontinuum of $X$ which intersects both $K$ and its complement must contain $K$. A continuous mapping from a continuum $X$ onto $Y$ is atomic if every fiber of $f$ is a terminal subcontinuum of $X$.

**Lemma 2.1** ([1], cf. [14] and [20]). Let $X$ and $Y$ be two continua and $a \in X$. Then there exists a continuum $M(X,Y,a)$ and an atomic mapping $p : M(X,Y,a) \to X$ onto $X$ such that $p^{-1}(a)$ is homeomorphic to $Y$ and $p | p^{-1}(X \setminus \{a\}) : p^{-1}(X \setminus \{a\}) \to X \setminus \{a\}$ is a homeomorphism.

Every continuum $M(X,Y,a)$ with the properties described in this lemma will be called a pseudosuspension of $Y$ over $X$ at the point $a$ (cf. [14, 1.13]) and the mapping $p$ will be called a natural projection from $M(X,Y,a)$ onto $X$.

Since $p^{-1}(a)$ is a terminal continuum in $M(X,Y,a)$, then (see [13, Proposition 11])

(2) if $X$ and $Y$ are HI, then so is $M(X,Y,a)$.

By the countable sum theorem (see [5], Theorem 1.5.3), we get

(3) $\dim M(X,Y,a) = \max\{\dim X, \dim Y\}$.

The following lemma, which was suggested by the referee of [16] (see Remark 5.2), was proved in detail in [10, Lemma 5.1].

**Lemma 2.2.** Let $X$ be any continuum, let $a$ be any point of $X$, and let $W = L \cup S$ be a Waraszkiewicz spiral, being a compactification of the ray $L$ with the remainder $S$ homeomorphic to the circle. Let $Y$ be a continuum satisfying the following condition:

(4) There exists a mapping $f : Y \to W$ of $Y$ onto $W$ and a sequence $M_1 \subset M_2 \subset \ldots$ of subcontinua of $Y$ contained in $f^{-1}(L)$ such that the union $\bigcup_{i=1}^{\infty} M_i$ is dense in $Y$.

Then there exists a pseudosuspension $M(X,Y,a)$ which admits a mapping $\tilde{f} : M(X,Y,a) \to W$ onto $W$.

**Lemma 2.3.** For every Waraszkiewicz spiral $W$ there exists an HI continuum $Y$ of dimension $\leq 2$ which satisfies condition (4) of Lemma 2.2. Moreover, $Y$ can be chosen as a subcontinuum of any given HI continuum $Z$ with $2 \leq \dim Z < \infty$.

**Proof:** Let $Z$ be any given HI continuum of finite dimension $\geq 2$ and $W$ be a Waraszkiewicz spiral. Let $Z' \subset Z$ be a $2$-dimensional
subcontinuum of $Z$. By a theorem of Mazurkiewicz [15], there exists a weakly confluent mapping of $Z'$ onto the square $I^2$. Since $W \subset I^2$, there exists a subcontinuum $X \subset Z'$ which is mapped by $f$ onto $W$. By Lemma 5.2 of [10], $X$ contains a subcontinuum $Y$, which satisfies (4). □

**Lemma 2.4** ([4]). There exists a one-dimensional HI continuum $H$ such that for any two different non-degenerate subcontinua of $H$, there is no mapping from one onto the other.

**Lemma 2.5** (see [4] and [21], cf. [19], Lemma 2.2). If $f : P \to H$ is a continuous mapping of the pseudo-arc into a Cook continuum $H$, then $f$ is a constant mapping.

**Lemma 2.6** (see Lemma 5.1 of [9] and its proof). For any proper subcontinuum $M$ of a 1-dimensional HI Cook continuum $H$ and for every $m = 1, 2, \ldots, \infty$, there exists an $m$-dimensional HI continuum $M_m$ such that every map from a subcontinuum of $M$ into $M_m$ is constant.

3. **Proofs**

**Proof of Theorem 1.1:** For every $n \in \mathbb{N}$, a 1-dimensional continuum $X_n$ with the required properties was constructed in [17] and [19]. We shall modify this construction in order to raise the dimension of such a space. Fix $n \in \mathbb{N}$ and $m \in \{2, 3, \ldots, \infty\}$. Inductively, let us define a sequence $Y_1, Y_2, \ldots$ of HI continua and a sequence $W_1, W_2, \ldots$ of Waraszkiewicz spirals such that

1. $\dim Y_1 = m$ and $\dim Y_l \leq 2$ for $l = 2, 3, \ldots$;
2. condition (4) is satisfied for $Y = Y_l$ and $W = W_{l-1}$, for every $l = 1, 2, \ldots$;
3. $Y_l$ cannot be mapped onto $W_l$ for $l = 1, 2, \ldots$.

Let $Y_1$ be any HI $m$-dimensional continuum. By (1), there exists a Waraszkiewicz spiral $W_1$ such that $Y_1$ cannot be mapped onto $W_1$. Suppose now that $Y_1, Y_2, \ldots, Y_{l-1}$ and $W_1, W_2, \ldots, W_{l-1}$ are already defined for some $l \geq 2$. For $Y_l$, we take a continuum $Y$ of dimension $\leq 2$ from Lemma 2.3, where we put $W = W_{l-1}$. Thus, $Y_l$ can be mapped onto $W_{l-1}$ and satisfies (4) for $W = W_{l-1}$. Again by condition (1), there exists a Waraszkiewicz spiral $W_l$ such that $Y_l$ does not map onto $W_l$. 
By a theorem of Lewis [11], there exists a pseudo-arc $P$ in the Euclidean plane and a homeomorphism $r : P \to P$ of period $n$, which is the restriction of the rotation of the plane about the point $(0,0)$ through the angle $\frac{2\pi}{n}$. Note that $(0,0) \in P$, since the pseudo-arc $P$ has the fixed point property (see [6]). Let $P_0 = \{(x_1, x_2) \in P : x_1 = \lambda \cos \alpha$ and $x_2 = \lambda \sin \alpha$ for some $0 < \lambda < \infty$ and $0 < \alpha < 2\pi\}$, and $P_k = r^k(P_0)$ for $k = 0, 1, \ldots, n - 1$. Let $\{b_1, b_2, \ldots\}$ be a countable dense subset of $P_0$ such that $b_i$ and $b_j$ are in the same composant of $P$ if and only if $i = j$.

There exists a composant $C$ in $P$ which does not contain any $b_i$. In $C \cap P_0$, we choose a point $c_0$, a sequence $Q_i$ of continua containing $c_0$ and converging to $\{c_0\}$, and a sequence $c_1, c_2, \ldots$ of points such that $c_i \in Q_i$ and $c_i \neq c_j$ for $i \neq j$.

Now, let $\{a_1, a_2, \ldots\}$ be a sequence such that $a_{2l-1} = c_l$ and $a_{2l} = b_l$ for $l = 1, 2, \ldots$. Put $B_0 = \bigcup\{b_l\}_{l=1}^{\infty}$, $C_0 = \bigcup\{c_l\}_{l=1}^{\infty}$, and $A_0 = B_0 \cup C_0$.

Then

(8) the set $B_0 \setminus F$, where $F$ is any finite subset of $B_0$, is dense in $P_0$.

Let $B = \bigcup_{k=0}^{n-1} r^k(B_0)$, $C = \bigcup_{k=0}^{n-1} r^k(C_0)$, and $A = B \cup C$. Since a homeomorphic image of a composant of $P$ is a composant of $P$, then

(9) every composant of $P$ contains at most $n$ points from $B$, and

(10) $C$ intersects at most $n$ composants of $P$.

Finally, let $K_1, K_2, \ldots$ be a sequence of disjoint non-degenerate subcontinua of the hereditarily indecomposable Cook continuum $H$ from Lemma 2.4. Thus,

(11) for every $j \neq i$, every continuous mapping from a subcontinuum of $K_j$ into $K_i$ is constant.

Let us define an inverse sequence $\{L_i, p^i_j, \{0\} \cup \mathbb{N}\}$ in the following way. Put $L_0 = P$. Let $L_1 = M(P, Y_1, a_1)$ be a pseudosuspension of an $m$-dimensional HI continuum $Y_1$ over $P$ at $a_1 = c_1$ and let $p^1_0$ be the natural projection. Suppose that $L_i$ and $p^i_j$ are already defined for $j \leq i \leq s$, where $s \in \mathbb{N}$. If $s = 2l$ for $l \geq 1$, then let $L_s = L_{2l} = M(L_{s-1}, K_1, (p^s_0)^{-1}(a_s))$ be a pseudosuspension of a
Cook continuum $K_l$ over $L_{2l-1}$ at $(p_0^{s-1})^{-1}(a_s) = (p_0^{s-1})^{-1}(b_l)$. If $s = 2l - 1$ for some $l \geq 2$, then $a_s = c_l$, and by (6), the conditions of Lemma 2.2 are satisfied for $W = W_{l-1}$ and $Y = Y_l$, so there exists a pseudosuspension $L_s = L_{2l-1} = M(L_{s-1}, Y_l, (p_0^{s-1})^{-1}(a_s))$ of $Y_l$ over $L_{s-1} at (p_0^{s-1})^{-1}(a_s)$ which admits a mapping onto $W_{l-1}$.

Now, let $p_s^{s-1}$ be the natural projection and $p_j^s = p_{j+1}^s \circ \ldots \circ p_{s-1}^s$ for $j < s$. Let $L$ be an inverse limit of this inverse sequence and let $p_s : L \rightarrow L_s$ be the projection. In particular, let $p = p_0$ be the projection of the limit space onto $L_0 = P$.

Let us note that for every $s \in \mathbb{N}$, $L_s$ is the union of an open subset homeomorphic to $P \setminus \bigcup_{i=0}^s \{a_i\}$, of a copy of the $m$-dimensional continuum $Y_1$, and of finitely many copies of at most 2-dimensional continua from the family $\{K_1, Y_2, K_2, Y_3, \ldots\}$. Thus, by the countable sum theorem, $\dim L_s = m$ for every $s \in \mathbb{N}$. By the theorem on the dimension of the limit of an inverse sequence (see [5, Theorem 1.13.4]) and since $L$ contains a topological copy of $Y_1$, it follows that

(12) the dimension of the limit space $L$ is equal to $m$.

Since $L_{2l-1}$ can be mapped onto $W_{l-1}$ for $l \geq 2$, and $L$ projects onto $L_{2l-1}$, then

(13) $L$ can be mapped onto every $W_l$, for $l = 1, 2, \ldots$.

Since the projection $p_j^s : L_i \rightarrow L_j$ is a composition of finitely many atomic mappings, then it is atomic (see [13, (1.4)]). Hence, $p$ is atomic (see [2, Theorem II]).

Let us note also that by (8) and from the definition of topology of the inverse limit,

(14) for every finite subset $F$ of $B_0$, every open subset of $p^{-1}(P_0)$ contains some set $p^{-1}(b)$, where $b \in B_0 \setminus F$.

We can assume additionally that $L \subset P \times \mathbb{I}^\infty$, where $\mathbb{I} = [0, 1]$, and that $p$ is the restriction of the projection of $P \times \mathbb{I}^\infty$ onto $P$. Moreover, we can assume that $p^{-1}(y) = (y, (0, 0, \ldots))$ for every $y \in P \setminus P_0$.

Indeed, assume that $L \subset \mathbb{I}^\infty$ and for $x, y \in \mathbb{R}^2$ let $\rho(x, y) = \min(\rho_s(x, y), 1)$, where $\rho_s$ is the Euclidean metric in the plane. If $f(x) = (p(x), \rho(p(x), \mathbb{R}^2 \setminus P_0), x)$ for $x \in L$, then $f$ is continuous and one-to-one; hence, it is a homeomorphism of $L$ onto $f(L) \subset P \times \mathbb{I}^\infty$. 
Thus, we can replace $L$ by $f(L)$ and $p$ by the restriction of the projection of $P \times \mathbb{I}^\infty$ onto $P$.

From the construction, it follows that for $l \in \mathbb{N}$, $p^{-1}(a_{2l-1})$ is homeomorphic to the continuum $Y_l$, $p^{-1}(a_{2l})$ is homeomorphic to the Cook continuum $K_l$, and $p \mid p^{-1}(P \setminus A_0) : p^{-1}(P \setminus A_0) \to (P \setminus A_0)$ is a homeomorphism.

Let $\overline{\mathbf{r}}(y, t) = (r(y), t)$ for $(y, t) \in P \times \mathbb{I}^\infty$. Let $\overline{P}_0$ be the closure of $P_0$ in $P$. For $k = 0, 1, \ldots, n - 1$, let $\overline{P}_k = \mathbf{r}^k(p^{-1}(\overline{P}_0))$ and $X_{nm} = \bigcup_{k=0}^{n-1} \overline{P}_k$.

Note that $X_{nm}$ admits a continuous mapping $g$ onto $L$, being the identity on $\overline{P}_0$, such that $g \mid \bigcup_{k=1}^{n-1} \overline{P}_k$ is the restriction of the projection of $P \times \mathbb{I}^\infty$ onto $P$. Thus, by (13),

(15) $X_{nm}$ can be mapped onto $W_l$ for every $l = 1, 2, \ldots$.

Let $\overline{p} : X_{nm} \to P$ be the restriction of the projection of $P \times \mathbb{I}^\infty$ onto the first axis. The mapping $\overline{r} = \mathbf{r} \mid X_{nm}$ is a period $n$ homeomorphism of $X_{nm}$ onto $X_{nm}$, such that

(16) $\overline{p} \circ r^k = r^k \circ \overline{p}$ for every $k = 0, 1, \ldots, n - 1,$

and

(17) $p(x) = \overline{p}(x)$ for $x \in \overline{P}_0$.

As in [17], we check that $\overline{p}$ is atomic (cf. [17, Lemma 2.7]).

Note that

(18) $\overline{p} \mid \overline{p}^{-1}(P \setminus A) : \overline{p}^{-1}(P \setminus A) \to P \setminus A$ is a homeomorphism.

Moreover, for $k \in \{0, 1, \ldots, n - 1\}$, we have that

(19) if $x = r^k(a_1) = r^k(c_1)$, then $\overline{p}^{-1}(x)$ is a copy of the $m$-dimensional HI continuum $Y_1$;

(20) if $x = r^k(a_{2l-1}) = r^k(c_l)$ for some $l \geq 2$, then the set $Y_l^k = \overline{p}^{-1}(x)$ is a copy of the HI continuum $Y_l$ with $\dim Y_l \leq 2$;

and

(21) if $x = r^k(a_{2l}) = r^k(b_l)$ for some $l \geq 1$, then $K_l^k = \overline{p}^{-1}(x)$ is a copy of the HI Cook continuum $K_l$.

From (14), it follows that

(22) if $F$ is a finite subset of $B$, and $x(b) \in \overline{p}^{-1}(b)$ for $b \in B \setminus F$, then the set $\{x(b) : b \in B \setminus F\}$ is dense in $X_{nm}$.
Since \( X_{nm} \) is the union of \( n \) closed subspaces which embed into \( L \), then, by (12) and (19), \( \dim X_{nm} = m \).

The space \( X_{nm} \) is HI, since it is the preimage of an HI continuum \( P \) under the atomic mapping \( \tilde{p} \) with HI fibers.

Since \( \tilde{p} \) is an atomic mapping, every composant of \( X_{nm} \) is equal to \( \tilde{p}^{-1}(L) \) for some composant \( L \) of \( P \) (see [20, Lemma 2.8]). By (9),

(23) every composant of \( X_{nm} \) contains at most \( n \) copies of continua from the family \( \mathcal{K} = \{ K_i^l : l \in \mathbb{N}, k = 0, 1, \ldots, n - 1 \} = \{ \tilde{p}^{-1}(b) : b \in B \} \).

Let \( f : X_{nm} \to X_{nm} \) be an arbitrary continuous mapping of \( X_{nm} \) onto \( X_{nm} \). We will show that \( f = \tilde{r}^k \) for some \( k \in \{ 0, 1, \ldots, n - 1 \} \).

For every \( k \) and \( l \), \( Y_l^k \) cannot be mapped onto \( W_l \), while \( X_{nm} \) admits a mapping onto \( W_l \) by (15), so \( f(Y_l^k) \neq X_{nm} \). Thus,

(24) for every \( k \in \{ 0, 1, \ldots, n - 1 \} \) and \( l \in \mathbb{N} \), \( f(Y_l^k) \) is contained in one of the composants of \( X_{nm} \).

Recall that \( Q_i \) is a sequence of continua in \( P \) containing \( c_i \), with diameters tending to 0. In every \( L \), the sequence of continua \( \{ (p_{0}^n)^{-1}(Q_i) \}_{i=1}^{\infty} \) converges to the point \( (p_{0}^n)^{-1}(c_0) \), so in the inverse limit space \( L \), the sequence of continua \( \{ p^{-1}(Q_i) \}_{i=1}^{\infty} \) converges to the point \( p^{-1}(c_0) \).

It follows that for every \( k \in \{ 0, 1, \ldots, n - 1 \} \), the sequence of continua \( \{ \tilde{p}^{-1}(r^k(Q_i)) \}_{i=1}^{\infty} \) converges to the one-point set \( \{ \tilde{p}^{-1}(r^k(c_0)) \} \), so the sequence of continua \( \{ f(\tilde{p}^{-1}(r^k(Q_i))) \}_{i=1}^{\infty} \) converges to the one-point set \( \{ f(\tilde{p}^{-1}(r^k(c_0))) \} \). Thus, for a fixed \( k \), almost all continua \( f(\tilde{p}^{-1}(r^k(Q_i))) \), where \( i \in \mathbb{N} \), are contained in the same composant of \( X_{nm} \) and thus, almost all continua \( f(Y_l^k) \), where \( l \in \mathbb{N} \), are contained in the same composant of \( X_{nm} \). From this and (24), it follows that the union of all sets \( f(Y_l^k) \), for \( k = 0, 1, \ldots, n - 1 \) and \( l = 0, 1, \ldots \), is contained in finitely many composants of \( X_{nm} \).

Thus, by (23), only finitely many continua from the family \( \{ K_i^l : l \in \mathbb{N}, k = 0, 1, \ldots, n - 1 \} \) can intersect the image under \( f \) of the union \( \bigcup \{ Y_l^k : l \in \mathbb{N}, k = 0, 1, \ldots, n - 1 \} \) of a composant \( C \). It follows that

(25) there exists \( l_0 \) such that for \( l \geq l_0 \) and every \( k \), \( K_i^l \cap f(\tilde{p}^{-1}(C)) = \emptyset \).

Let \( B' = \{ r^k(b_l) : l \geq l_0, k = 0, 1, \ldots, n - 1 \} \). We will show that
for every \( b \in B' \), there is a nontrivial subcontinuum \( Q \) of \( \tilde{p}^{-1}(b) \) and \( t \in \{1, 2, \ldots, n-1\} \) such that \( f(\tilde{t}^t(x)) = x \) for every \( x \in Q \).

Fix \( b \in B' \). Then \( \tilde{p}^{-1}(b) \) is equal to the Cook continuum \( K^k_i \) for some \( l \geq l_0 \) and \( k \in \{1, 2, \ldots, n-1\} \). Since \( X_{nm} \) is III, then \( f \) is confluent and thus, there exists a proper subcontinuum \( T \) of \( X_{nm} \) such that \( f(T) = K^k_i \). Then \( T \) is disjoint with \( \tilde{p}^{-1}(C) \) by (25) and is contained in some composant of \( X_{nm} \). From this and (23), it follows that either \( T \) is contained in \( \tilde{p}^{-1}(b') \) for some \( b' \in B \), or \( T \) is the union of a non-empty subset of \( \tilde{p}^{-1}(P \setminus A) \) and of finitely many continua \( \tilde{p}^{-1}(b(1)), \ldots, \tilde{p}^{-1}(b(r)) \), where \( b(i) \in B \). In the second case, there exists \( i \) such that \( f(\tilde{p}^{-1}(b(i))) \) is a nondegenerate subcontinuum of \( \tilde{p}^{-1}(b) \). For otherwise, the set \( K^k_i \setminus \bigcup_{j=1}^{r} f(\tilde{p}^{-1}(b(i))) \) would contain a non-degenerate subcontinuum, which is the image of a subcontinuum \( T' \subset \tilde{p}^{-1}(P \setminus A) \cap T \). However, by (18), each non-degenerate subcontinuum \( T' \) of \( X_{nm} \) contained in \( \tilde{p}^{-1}(P \setminus A) \) is homeomorphic to \( P \); therefore, by Lemma 2.5, \( T' \) admits only constant mappings into the Cook continuum \( K^k_i \), which gives a contradiction.

Therefore, in both cases, there exists \( b' \in B \) such that \( f(T \cap \tilde{p}^{-1}(b')) \) is a nondegenerate subcontinuum of \( K^k_i \). By (11), \( \tilde{p}^{-1}(b') \) must be equal to \( K^k_i \) for some \( t \), and, for \( Q = T \cap \tilde{p}^{-1}(b) \), condition (26) is satisfied, because \( Q \) and \( f(Q) \) must be topological copies of the same nondegenerate subcontinuum of the Cook continuum \( K^k_i \). By choosing a point \( x(b) \in \tilde{p}^{-1}(b) \cap K \), we get the result that

(27) for every \( b \in B' \), there is a point \( x(b) \in \tilde{p}^{-1}(b) \) such that \( f(\tilde{t}^t(x(b))) = x(b) \) for some \( t \in \{1, 2, \ldots, n-1\} \).

By (22), the set \( Y = \{x(b) : b \in B'\} \) is dense in \( X_{nm} \).

The remaining part of the proof repeats the arguments from the proof of Theorem 3.1 in [19]. First, let us note that

(28) for every \( x \in X_{nm} \), there is \( t \in \{0, 1, \ldots, n-1\} \) such that \( f(\tilde{t}^t(x)) = x \).

Indeed, one can find a sequence \( \{x(b_j)\}_{j=1}^\infty \), where \( b_j \in B' \), converging to \( x \), such that for some \( t \in \{0, 1, \ldots, n-1\} \), \( f(\tilde{t}^t(x(b_j))) = x(b_j) \) for every \( j \). Thus, \( f(\tilde{t}^t(x(b_j))) \rightarrow f(\tilde{t}^t(x)) \), so \( f(\tilde{t}^t(x)) = x \).

For every \( x \neq \tilde{p}^{-1}((0,0)) \), the set \( Y(x) = \bigcup_{k=1}^{n} x^k(x) \) has \( n \) elements and every point of \( Y(x) \) is the image of a point in \( Y(x) \);
hence, \( f(Y(x)) = Y(x) \) and \( f \mid Y(x) \to Y(x) \) is one-to-one. In particular, for every \( x \neq \tilde{p}^{-1}((0,0)) \), there exists \( k \in \{0,1,\ldots,n-1\} \) such that \( f(x) = \tilde{r}^k(x) \).

For \( k \in \{0,1,\ldots,n-1\} \), let \( X(k) = \{ x \in X_{nm} : f(x) = \tilde{r}^k(x) \} \).

It is easy to see that every \( X(k) \) is closed in \( X_{nm} \) and \( X(k) \cap X(l) = \{ \tilde{p}^{-1}((0,0)) \} \) for \( k \neq l, k,l \in \{0,1,\ldots,n-1\} \). It follows that every \( X(k) \) is a continuum. Indeed, if \( X(k) \) were the union of two disjoint closed subsets \( F_1 \) and \( F_2 \) with \( \tilde{p}^{-1}((0,0)) \in F_2 \), then \( X_{nm} \) would be the union of two sets \( F_1 \) and \( F_2 \cup \bigcup_{l \neq k} X(l) \), disjoint and closed in \( X_{nm} \). Since \( X_{nm} \) is hereditarily indecomposable, then \( X_{nm} = X(k) \) for some \( k \in \{0,1,\ldots,n-1\} \), and thus, \( f = \tilde{r}^k \) and \( f \) is a homeomorphism.

This ends the proof that the set of all continuous mappings from \( X_{nm} \) onto \( X_{nm} \) is equal to the set \( \{ \tilde{r}^0, \tilde{r}^1, \ldots, \tilde{r}^{n-1} \} \) and forms the cyclic group of order \( n \).

\[ \square \]

**Proof of Theorem 1.2:** Let \( S \) be a set of cardinality \( 2^{\aleph_0} \), \( H \) be the 1-dimensional HI Cook continuum (see Lemma 2.4), and \( K \) be a proper non-degenerate subcontinuum of \( H \). For every \( s \in S \), let us choose a sequence \( \{K_1(s), K_2(s), \ldots\} \) of non-degenerate subcontinua of \( K \) in such a way that \( K_i(s) \cap K_j(t) = \emptyset \) if \( s \neq t \) or \( i \neq j \). Such a family \( \{K_i(s) : i \in \mathbb{N}, s \in S\} \) exists, because \( K \) has \( 2^{\aleph_0} \) composants which are pairwise disjoint. Thus,

\[ (29) \text{ every mapping from a subcontinuum of } K_i(s) \text{ into } K_j(t) \text{ is constant.} \]

If, in the proof of Theorem 1.1, we replace in the construction of \( X_{nm} \), the sequence \( K_1, K_2, \ldots \) by the sequence \( \{K_1(s), K_2(s), \ldots\} \), then we obtain an HI continuum \( X_{nm}(s) \) with exactly \( n \) continuous surjections onto itself, which admits an atomic mapping \( \tilde{p}_s : X_{nm}(s) \to P \) onto \( P \). As we will prove below, the family \( \{X_{nm}(s) : s \in S\} \) satisfies condition (i) of Theorem 1.2. In order to obtain such a family also satisfying condition (ii), we assume additionally that \( Y_1 \) is a space \( M_m \) constructed in Lemma 2.6 for \( M = K \), and \( Y_l \) for \( l \geq 2 \) is a space \( Y \) of dimension \( \leq 2 \) constructed in Lemma 2.3 for \( W = W_{l-1} \), which is contained in the 2-dimensional continuum \( M_2 \) from Lemma 2.6 (where we put \( M = K \)).

Thus,
(30) every mapping from a subcontinuum of \( K \) into \( Y_t \), for \( l = 1, 2, \ldots \), is constant.

Let us show condition (i). From the construction, it follows that \( X_{nm}(s) \) is the union of the set \( \tilde{p}_s^{-1}(P \setminus A) \) homeomorphic to a subset of \( P \), of continua from the family \( K(s) = \{ \tilde{p}_s^{-1}(b) : b \in B \} \), and of continua from the family \( \mathcal{Y} = \{ \tilde{p}_s^{-1}(c) : c \in C \} \). Note that the family \( K(s) \) contains exactly \( n \) copies of every continuum \( K_i(s) \), for every \( i \in \mathbb{N} \), and the family \( \mathcal{Y} \) contains \( n \) copies of every continuum \( Y_l \), for \( l \in \mathbb{N} \).

Let \( f : X_{nm}(s) \rightarrow X_{nm}(t) \) be an arbitrary continuous surjection. Suppose that \( t \neq s \). Similar to the proof of Theorem 1.1, one shows that the set \( f(\bigcup \mathcal{Y}) \) intersects only finitely many components of \( X_{nm}(t) \), so it intersects only finitely many continua from the family \( K(t) \). Hence, there exists \( \tilde{p}_t^{-1}(b) \in K(t) \), being a copy of some \( K_i(t) \), which is disjoint with \( f(\bigcup \mathcal{Y}) \). Since \( f \) is confluent, there exists a nontrivial subcontinuum \( T \) of \( X_{nm}(s) \), disjoint with \( \bigcup \mathcal{Y} = \tilde{p}_s^{-1}(C) \), such that \( f(T) = \tilde{p}_t^{-1}(b) \). Since \( T \) is a proper subcontinuum of \( X_{nm}(s) \), it is contained in some component of \( X_{nm}(s) \). It follows that either \( T \) is contained in some \( \tilde{p}_s^{-1}(b') \) for some \( b' \in B \), or \( T \) is the union of a non-empty subset of \( \tilde{p}_s^{-1}(P \setminus A) \) and of finitely many continua \( \tilde{p}_s^{-1}(b(1)), \ldots, \tilde{p}_s^{-1}(b(r)) \), where \( b(i) \in B \). In the second case, there exists \( i \) such that \( f(\tilde{p}_s^{-1}(b(i))) \) is a nondegenerate subcontinuum of \( \tilde{p}_t^{-1}(b) \). For otherwise, the set \( \tilde{p}_s^{-1}(b) \setminus \bigcup_{i=1}^{r} f(\tilde{p}_s^{-1}(b(i))) \) would contain a non-degenerate subcontinuum, which is the image of a subcontinuum \( T' \subset \tilde{p}_s^{-1}(P \setminus A) \cap T \). However, each non-degenerate subcontinuum of \( X_{nm}(s) \) contained in \( \tilde{p}_s^{-1}(P \setminus A) \) is homeomorphic to \( P \); therefore, by Lemma 2.5, it admits only constant mappings into the Cook continuum \( \tilde{p}_t^{-1}(b) \), which gives a contradiction.

Therefore, in both cases, there exists \( b' \in B \) such that \( f(T \cap \tilde{p}_s^{-1}(b')) \) is a nondegenerate subcontinuum of \( \tilde{p}_t^{-1}(b) \). But \( \tilde{p}_s^{-1}(b') \) and \( \tilde{p}_t^{-1}(b) \) are homeomorphic to two disjoint subcontinua of the Cook continuum \( H \), which yields a contradiction. Thus, \( s = t \).

To prove (ii), suppose that \( s \neq t \), and \( h : X_{nm}(s) \rightarrow X_{nm}(t) \) is an embedding. Let \( K_1(s)' \) be a copy of \( K_1(s) \) in \( X_{nm}(s) \). Then \( h(K_1(s)') \) is a copy of \( K_1(s) \) in \( X_{nm}(t) \), so it does not embed in \( \tilde{p}_t^{-1}(P \setminus A) \) by Lemma 2.5. Thus, \( h(K_1(s)') \) intersects some \( \tilde{p}_t^{-1}(a_i) \). By (29) and (30), \( h(K_1(s)') \) is not contained in \( \tilde{p}_t^{-1}(a_i) \) for any
\[ i = 1, 2, \ldots \]; hence, \( \tilde{p}_t^{-1}(a_i) \subset h(K_1(s')) \) for some \( i \). However, if \( i = 2l \), then \( \tilde{p}_t^{-1}(a_i) \) is a copy of \( K_l(t) \), which gives a contradiction by (29). If \( i = 2l - 1 \), then \( \tilde{p}_t^{-1}(a_i) \) is a copy of a continuum \( Y_l \). Since \( h : K_1(s) \to h(K_1(s')) \) is a homeomorphism, then \( Z = h^{-1}(\tilde{p}_t^{-1}(a_i)) \) is a subcontinuum of \( K_1(s') \) such that \( h(Z) = \tilde{p}_t^{-1}(a_i) \), which contradicts (30). This shows that \( s = t \). \( \Box \)

References


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